# Module MA3428: Annual Examination 2015
# Worked solutions

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## Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3428/`

## Module Contents


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Candidates will be informed that, in relation to the basic material on modules, and particularly free modules, the proofs are non-examinable. They will need to know however basic definitions: in particular, they should know what is meant by saying that a subset of a left module over a unital ring *freely generates* the module. They should also know that, given an arbitrary set, it is possible to construct a free module with a free basis whose elements are in bijective correspondence with elements of the given set (i.e., they should be aware of the universal property that characterizes free modules on a given set). Also candidates only need to know basic definitions and the statement of certain basic results from section 4 on the definition of the simplicial chain groups. Also proofs of certain specific propositions will be explicitly flagged as non-examinable, including an elementary but lengthy proof, in subsection 7.5, that a triangulated polygonal region bounded by a single simple polygon with more than one triangle can be obtained from a smaller such polygonal region by attaching a single triangle, where the intersection of the attached triangle with the smaller polygonal region consists either of one edge or else of two edges of the attached triangle. In relation to applications of the Mayer-Vietoris sequence for calculating the homology groups of the torus, Klein bottle, and real projective plane, candidates will be informed that they do not need to know, as bookwork, details of all the triangulations etc. that are necessary for logical completeness in the notes. Instead they should focus on how the Mayer-Vietoris exact sequence is applied, and sample problems resembling potential examination questions, and actual examination questions from past years up to 2005 (when the Mayer-Vietoris sequence was last examined) should be provided and discussed in the weeks following study week.

In summary, candidates will be informed that, for examination purposes, they should be familiar with methods for calculating homology groups from first principles (as in the examples of Section 6 and on all recent examination papers set by the present examiner going back over two decades), and they should also be familiar with techniques for calculating homology groups of simplicial complexes using the Mayer-Vietoris Exact Sequence, given information on the homology of appropriate subcomplexes and the induced homomorphisms relating them (as candidates were regularly required to do in examinations in the predecessor module up to 2007). Also lemmas, propositions, theorems and corollaries in specified sections of the lecture course, excluding certain results that are necessary for logical completeness but are not suited to to formal examination.

1. (a) [Bookwork.]

$$\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

$$\begin{aligned}
&\partial_{q-1}\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) \\
&\qquad = \sum_{j=0}^{q} (-1)^j \partial_{q-1}\left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle\right) \\
&\qquad = \sum_{j=1}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad\quad + \sum_{j=0}^{q-1} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\
&\qquad = 0
\end{aligned}$$

(b) [Bookwork.] Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ be the vertices of the simplicial complex $K$. Every 0-chain of $K$ with coefficients in $R$ can be expressed uniquely as a formal sum of the form

$$r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + \cdots + r_s\langle \mathbf{v}_s \rangle$$

for some $r_1, r_2, \ldots, r_s \in R$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K;R) \to R$ defined such that

$$\varepsilon\left(r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + \cdots + r_s\langle \mathbf{v}_s \rangle\right) = r_1 + r_2 + \cdots + r_s.$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$.

Now $\varepsilon(\partial_1(\langle \mathbf{u}, \mathbf{v} \rangle)) = \varepsilon(\langle \mathbf{v} \rangle - \langle \mathbf{u} \rangle) = 0$ whenever $\mathbf{u}$ and $\mathbf{v}$ are endpoints of an edge of $K$. It follows that $\varepsilon \circ \partial_1 = 0$, and therefore $B_0(K;R) \subset \ker \varepsilon$.

Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ be vertices of $K$ determining an edge path. Then $\mathbf{w}_{j-1}\,\mathbf{w}_j$ is an edge of $K$ for $j = 1, 2, \ldots, m$, and

$$\langle \mathbf{w}_m \rangle - \langle \mathbf{w}_0 \rangle = \sum_{j=1}^{m} \Big( \langle \mathbf{w}_j \rangle - \langle \mathbf{w}_{j-1} \rangle \Big) = \partial_1 \left( \sum_{j=1}^{m} \langle \mathbf{w}_{j-1}, \mathbf{w}_j \rangle \right) \in B_0(K;R).$$

Now $|K|$ is connected, and therefore any pair of vertices of $K$ can be joined by an edge path. We deduce that $\langle \mathbf{v} \rangle - \langle \mathbf{u} \rangle \in B_0(K;R)$ for all vertices $\mathbf{u}$ and $\mathbf{v}$ of $K$.

Choose a vertex $\mathbf{u} \in K$. Then

$$\begin{aligned}\sum_{j=1}^{s} r_j \langle \mathbf{v}_j \rangle &= \sum_{j=1}^{s} r_j (\langle \mathbf{v}_j \rangle - \langle \mathbf{u} \rangle) \\ &\qquad + \left( \sum_{j=1}^{s} r_j \right) \langle \mathbf{u} \rangle \in B_0(K;R) + \left( \sum_{j=1}^{s} r_j \right) \langle \mathbf{u} \rangle\end{aligned}$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$, and therefore

$$z - \varepsilon(z)\langle \mathbf{u} \rangle \in B_0(K;R)$$

for all $z \in C_0(K;R)$. It follows that $\ker \varepsilon \in B_0(K;R)$. But we have already shown that $B_0(K;R) \subset \ker \varepsilon$. It follows that $\ker \varepsilon = B_0(K;R)$.

Now the homomorphism $\varepsilon\colon C_0(K;R) \to R$ is surjective and its kernel is $B_0(K;R)$. Moreover $Z_0(K;R) = C_0(K;R)$ (because $\partial_0\colon C_0(K;R) \to C_{-1}(K;R)$ is defined to be the zero homomorphism from $C_0(K;R)$ to the zero module $C_{-1}(K;R)$), and therefore

$$H_0(K;R) = Z_0(K;R)/B_0(K;R) = C_0(K;R)/B_0(K;R).$$

It follows that the homomorphism $\varepsilon$ induces an isomorphism from $H_0(K;R)$ to $R$, and therefore $H_0(K;R) \cong R$, as required.

2. (a) [Seen similar.] Calculating the images of the generators of $C_2(K;\mathbb{Z})$ under the boundary homomorphism according to the usual rule, we find that

$$\begin{aligned}
\partial_2\left(\sum_{j=1}^{7} n_j\gamma_j\right) &= n_1\left(\langle \mathbf{v}_2\,\mathbf{v}_3\rangle - \langle \mathbf{v}_1\,\mathbf{v}_3\rangle + \langle \mathbf{v}_1\,\mathbf{v}_2\rangle\right)\\
&\quad + n_2\left(\langle \mathbf{v}_2\,\mathbf{v}_5\rangle - \langle \mathbf{v}_1\,\mathbf{v}_5\rangle + \langle \mathbf{v}_1\,\mathbf{v}_2\rangle\right)\\
&\quad + n_3\left(\langle \mathbf{v}_3\,\mathbf{v}_4\rangle - \langle \mathbf{v}_1\,\mathbf{v}_4\rangle + \langle \mathbf{v}_1\,\mathbf{v}_3\rangle\right)\\
&\quad + n_4\left(\langle \mathbf{v}_3\,\mathbf{v}_5\rangle - \langle \mathbf{v}_1\,\mathbf{v}_5\rangle + \langle \mathbf{v}_1\,\mathbf{v}_3\rangle\right)\\
&\quad + n_5\left(\langle \mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_1\,\mathbf{v}_5\rangle + \langle \mathbf{v}_1\,\mathbf{v}_4\rangle\right)\\
&\quad + n_6\left(\langle \mathbf{v}_3\,\mathbf{v}_5\rangle - \langle \mathbf{v}_2\,\mathbf{v}_5\rangle + \langle \mathbf{v}_2\,\mathbf{v}_3\rangle\right)\\
&\quad + n_7\left(\langle \mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_3\,\mathbf{v}_5\rangle + \langle \mathbf{v}_3\,\mathbf{v}_4\rangle\right)\\
&= (n_1+n_2)\rho_1 + (n_3+n_4-n_1)\rho_2\\
&\quad + (n_5-n_3)\rho_3 + (-n_2-n_4-n_5)\rho_4\\
&\quad + (n_1+n_6)\rho_5 + (n_2-n_6)\rho_6\\
&\quad + (n_3+n_7)\rho_7 + (n_4+n_6-n_7)\rho_8\\
&\quad + (n_5+n_7)\rho_9
\end{aligned}$$

Thus

$$\begin{gathered}
m_1 = n_1+n_2, \quad m_2 = n_3+n_4-n_1, \quad m_3 = n_5-n_3,\\
m_4 = -n_2-n_4-n_5, \quad m_5 = n_1+n_6, \quad m_6 = n_2-n_6,\\
m_7 = n_3+n_7, \quad m_8 = n_4+n_6-n_7, \quad m_9 = n_5+n_7.
\end{gathered}$$

(b) [Seen similar.] The quantities $m_k$ determined by $n_j$ by the equations derived in (a) must be zero when $\sum_{j=1}^{7} n_j\gamma_j = 0$. It follows that

$$n_2 = -n_1, \quad n_4 = n_1 - n_3, \quad n_5 = n_3, \quad n_6 = -n_1, \quad n_7 = -n_3.$$

Therefore the group $Z_2(K;\mathbb{Z})$ of 2-cycles of $K$ is as follows:

$$Z_2(K;\mathbb{Z}) = \{n_1 z_1 + n_3 z_3 : n_1, n_2 \in \mathbb{Z}\},$$

where

$$z_1 = \gamma_1 - \gamma_2 + \gamma_4 - \gamma_6, \quad z_2 = \gamma_3 - \gamma_4 + \gamma_5 - \gamma_7.$$

Also there are no non-zero 2-boundaries of $K$, because there are no non-zero 3-chains, and thus $B_2(K;\mathbb{Z}) = 0$. It follows that $H_2(K;\mathbb{Z}) = Z_2(K;\mathbb{Z}) \cong \mathbb{Z}^2$. Indeed there is an isomorphism from $Z_2(K;\mathbb{Z})$ to $\mathbb{Z}^2$ that sends $n_1z_1+n_3z_3$ to $(n_1, n_3)$ for all $n_1, n_3 \in \mathbb{Z}$.

(c) [Seen similar.] Setting $n_1 = 0$ and $n_3 = 0$ in the equations derived in (a) we need to solve the equations

$$m_1 = n_2, \quad m_2 = n_4, \quad m_3 = n_5,$$

$$m_4 = -n_2 - n_4 - n_5, \quad m_5 = n_6, \quad m_6 = n_2 - n_6,$$

$$m_7 = n_7, \quad m_8 = n_4 + n_6 - n_7, \quad m_9 = n_5 + n_7.$$

for the $n_j$ in terms of the $m_k$. Clearly we require

$$n_2 = m_1, \quad n_4 = m_2, \quad n_5 = m_3, \quad n_6 = m_5, \quad n_7 = m_7.$$

But for the full overdetermined system of equations to be solvable we require

$$m_4 = -m_1 - m_2 - m_3, \quad m_6 = m_1 - m_5,$$

$$m_8 = m_2 + m_5 - m_7, \quad m_9 = m_3 + m_7.$$

The homomorphism sending $\sum_{k=1}^{9} m_k \rho_k$ to $(m_1, m_2, m_3, m_5, m_7)$ is then an isomorphism from $B_1(K, \mathbb{Z})$ to $\mathbb{Z}^5$.

3. (a) [Definitions. From printed lecture notes.] A *chain complex* $C_*$ is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of $R$-modules, together with homomorphisms $\partial_i: C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers $i$.

The $i$th *homology group* $H_i(C_*)$ of the complex $C_*$ is defined to be the quotient module $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i: C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}: C_{i+1} \to C_i$.

Let $C_*$ and $D_*$ be chain complexes. A *chain map* $f: C_* \to D_*$ is a sequence $f_i: C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes $A_*$, $B_*$ and $C_*$ and chain maps $p_*: A_* \to B_*$ and $q_*: B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer $i$.

(b) [Bookwork.] First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_* \circ p_* = 0$. Thus the image of $p_*: H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_*: H_i(B_*) \to H_i(C_*)$. Let $x$ be an element of $Z_i(B_*)$ for which $[x] \in \ker q_*$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}: B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x - \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus $a$ represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x - \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now

$$\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w],$$

where $w$ is the unique element of $Z_i(A_*)$ satisfying $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let $z$ be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*([b - p_i(a)]) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$. Now let $w$ be an element of $Z_{i-1}(A_*)$ for which $[w] \in \ker p_*$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_*)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_*)$, as required.

4. (a) [Seen similar.] The sequence

$$H_3(L;\mathbb{Z})\oplus H_3(M;\mathbb{Z})\xrightarrow{w_*}H_3(K;\mathbb{Z})\xrightarrow{\alpha_3}H_2(L\cap M;\mathbb{Z})\xrightarrow{k_*}H_2(L;\mathbb{Z})\oplus H_2(M;\mathbb{Z})$$

is exact and

$$H_3(L;\mathbb{Z})=0, \quad H_3(M;\mathbb{Z})=0, \quad H_2(L;\mathbb{Z})=0 \quad \text{and} \quad H_2(M;\mathbb{Z})=0.$$

It follows that $H_3(K;\mathbb{Z})\cong H_2(L\cap M;\mathbb{Z})\cong\mathbb{Z}$.

(b) [Seen similar.] The homomorphism $k_*\colon H_1(L\cap M;\mathbb{Z})\to H_1(L;\mathbb{Z})\oplus H_1(M;\mathbb{Z})$ is injective because

$$k_*(n_1\alpha+n_2\beta)=(n_1 i_*(\alpha),-n_2 j_*(\beta))$$

for all $n_1,n_2\in\mathbb{Z}$, where $i_*(\alpha)$ and $j_*(\beta)$ are non-zero elements that generate the infinite cyclic groups $H_1(L;\mathbb{Z})$ and $H_1(M;\mathbb{Z})$. It follows from the exactness of the Mayer-Vietoris sequence that $\alpha_2\colon H_2(K;\mathbb{Z})\to H_1(L\cap M;\mathbb{Z})$ is the zero homomorphism. Therefore the sequence

$$H_2(L;\mathbb{Z})\oplus H_2(M;\mathbb{Z})\xrightarrow{w_*}H_2(K;\mathbb{Z})\to 0$$

is exact. But $H_2(L;\mathbb{Z})=0$ and $H_2(M;\mathbb{Z})=0$. It follows that $H_2(K;\mathbb{Z})=0$, as required.

(c) [Seen similar.] The homomorphism $k_*\colon H_1(L\cap M;\mathbb{Z})\to H_1(L;\mathbb{Z})\oplus H_1(M;\mathbb{Z})$ is surjective, because

$$k_*(n_1\alpha+n_2\beta)=(n_1 i_*(\alpha),-n_2 j_*(\beta))$$

for all $n_1,n_2\in\mathbb{Z}$, and $i_*(\alpha)$ and $j_*(\beta)$ generate $H_1(L;\mathbb{Z})$ and $H_1(M;\mathbb{Z})$ respectively. Therefore the kernel of $w_*\colon H_1(L;\mathbb{Z})\oplus H_1(M;\mathbb{Z})\to H_1(K)$ is the whole of its domain, and thus this homomorphism $w_*$ is zero homomorphism. It follows from the exactness of the Mayer-Vietoris sequence that

$$0\to H_1(K;\mathbb{Z})\xrightarrow{\alpha_1}\to H_0(L\cap M;\mathbb{Z})\xrightarrow{k_*}H_0(L;\mathbb{Z})\oplus H_0(M;\mathbb{Z}),$$

and thus

$$H_1(K;\mathbb{Z})\cong\ker(k_*\colon H_0(L\cap M;\mathbb{Z})\to H_0(L;\mathbb{Z})\oplus H_0(M;\mathbb{Z})).$$

But the components of $k_*$ are isomorphisms (as stated in the question) and therefore $k_*\colon H_0(L\cap M;\mathbb{Z})\to H_0(L;\mathbb{Z})\oplus H_0(M;\mathbb{Z})$ is injective. It follows that $H_1(K;\mathbb{Z})=0$.

(d) [Seen similar.] Let $\rho$ be a generator of $H_0(L \cap M; \mathbb{Z})$. Then $i_*(\rho)$ generates $H_0(L; \mathbb{Z})$ and $j_*(\rho)$ generates $H_0(M; \mathbb{Z})$, because (as stated in the question) $i_*\colon H_0(L \cap M; \mathbb{Z}) \to H_0(L; \mathbb{Z})$ and $j_*\colon H_0(L \cap M; \mathbb{Z}) \to H_0(M; \mathbb{Z})$ as isomorphism. Therefore

$$H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z}) = \{(s_1 i_*(\gamma), s_2 j_*(\gamma)) : s_1, s_2 \in \mathbb{Z}\}.$$

Moreover $(s_1 i_*(\gamma), s_2 j_*(\gamma)) = 0$ if and only if $s_1 = s_2 = 0$. There is thus a well-defined surjective homomorphism $\theta\colon H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z}) \to \mathbb{Z}$ defined such that $\theta(s_1 i_*(\gamma), s_2 j_*(\gamma) = 0) = s_1 + s_2$ for all integers $s_1$ and $s_2$. Then

$$\begin{aligned} \ker\theta &= \{(s i_*(\gamma), -s j_*(\gamma) = 0) : s \in \mathbb{Z}\} = \{i_*(s\gamma), -j_*(s\gamma) : s \in \mathbb{Z}\} \\ &= k_*(H_0(L \cap M; \mathbb{Z}). \end{aligned}$$

It follows from the exactness of the Mayer-Vietoris sequence that $\ker\theta = \ker w_*$. Therefore

$$\begin{aligned} H_0(K; \mathbb{Z}) &\cong (H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z}))/\ker w_* \\ &= (H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z}))/\ker\theta \\ &\cong \theta(H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z})) = \mathbb{Z}, \end{aligned}$$

as required.