Module MA3428: Algebraic Topology II
Hilary Term 2015
Part II (Sections 9 and 10)
Preliminary Draft

D. R. Wilkins

Contents

9 Introduction to Homological Algebra

9.1 Exact Sequences

In homological algebra we consider sequences

$$\cdots \longrightarrow F \xrightarrow{p} G \xrightarrow{q} H \xrightarrow{\cdots}$$

where F, G, H etc. are modules over some unital ring R and p, q etc. are R-module homomorphisms. We denote the trivial module $\{0\}$ by 0, and we denote by $0 \longrightarrow G$ and $G \longrightarrow 0$ the zero homomorphisms from 0 to G and from G to 0 respectively. (These zero homomorphisms are of course the only homomorphisms mapping out of and into the trivial module 0.)

Unless otherwise stated, all modules are considered to be left modules.

Definition Let R be a unital ring, let F, G and H be R-modules, and let $p: F \to G$ and $q: G \to H$ be R-module homomorphisms. The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of modules and homomorphisms is said to be *exact* at G if and only if $\text{image}(p: F \to G) = \ker(q: G \to H)$. A sequence of modules and homomorphisms is said to be *exact* if it is exact at each module occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *monomorphism* is an injective homomorphism. An *epimorphism* is a surjective homomorphism. An *isomorphism* is a bijective homomorphism.

The following result follows directly from the relevant definitions.

Lemma 9.1 *let R be a unital ring, and let $h: G \to H$ be a homomorphism of R-modules. Then*

- *$h: G \to H$ is a monomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H$ is an exact sequence;*
- *$h: G \to H$ is an epimorphism if and only if $G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence;*
- *$h: G \to H$ is an isomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*

Let R be a unital ring, and let F be a submodule of an R-module G. Then the sequence

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} G/F \longrightarrow 0,$$

is exact, where G/F is the quotient module, $i\colon F \hookrightarrow G$ is the inclusion homomorphism, and $q\colon G \to G/F$ is the quotient homomorphism. Conversely, given any exact sequence of the form

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} H \longrightarrow 0,$$

we can regard F as a submodule of G (on identifying F with $i(F)$), and then H is isomorphic to the quotient module G/F. Exact sequences of this type are referred to as *short exact sequences.*

We now introduce the concept of a *commutative diagram.* This is a diagram depicting a collection of homomorphisms between various modules occurring on the diagram. The diagram is said to *commute* if, whenever there are two routes through the diagram from a module G to a module H, the homomorphism from G to H obtained by forming the composition of the homomorphisms along one route in the diagram agrees with that obtained by composing the homomorphisms along the other route. Thus, for example, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ D & \xrightarrow{h} & E & \xrightarrow{k} & F \end{array}$$

commutes if and only if $q \circ f = h \circ p$ and $r \circ g = k \circ q$.

Proposition 9.2 *Let R be a unital ring. Suppose that the following diagram of R-modules and R-module homomorphisms*

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_4} & & \downarrow{\scriptstyle \psi_5} \\ H_1 & \xrightarrow{\phi_1} & H_2 & \xrightarrow{\phi_2} & H_3 & \xrightarrow{\phi_3} & H_4 & \xrightarrow{\phi_4} & H_5 \end{array}$$

commutes and that both rows are exact sequences. Then the following results follow:

(i) *if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism,*

(ii) *if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism.*

Proof First we prove (i). Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2\left(\psi_2(y)\right) = \psi_3\left(\theta_2(y)\right) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2\left(\theta_1(w)\right) = \phi_1\left(\psi_1(w)\right) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2\left(\theta_1(w)\right) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove (ii). Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5\left(\theta_4(b)\right) = \phi_4\left(\psi_4(b)\right) = \phi_4\left(\phi_3(a)\right) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3\left(\psi_3(c)\right) = \psi_4\left(\theta_3(c)\right) = \psi_4(b),$$

hence $\phi_3\left(a - \psi_3(c)\right) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3\left(\theta_2(e)\right) = \phi_2\left(\psi_2(e)\right) = a - \psi_3(c).$$

Hence $a = \psi_3\left(c + \theta_2(e)\right)$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required. ∎

The following result is an immediate corollary of Proposition 9.2.

Lemma 9.3 (Five-Lemma) *Suppose that the rows of the commutative diagram of Proposition 9.2 are exact sequences and that ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms. Then ψ_3 is also an isomorphism.*

9.2 Chain Complexes

Definition A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of modules over some unital ring, together with homomorphisms $\partial_i: C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i: C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}: C_{i+1} \to C_i$.

Note that if the modules C_* occuring in a chain complex C_* are modules over some unital ring R then the homology groups of the complex are also modules over this ring R.

Definition Let C_* and D_* be chain complexes. A *chain map* $f: C_* \to D_*$ is a sequence $f_i: C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

Note that a collection of homomorphisms $f_i: C_i \to D_i$ defines a chain map $f_*: C_* \to D_*$ if and only if the diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow \cdots \\ & \big\downarrow f_{i+1} & & \big\downarrow f_i & & \big\downarrow f_{i-1} & \\ \cdots \longrightarrow & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \longrightarrow \cdots \end{array}$$

is commutative.

Let C_* and D_* be chain complexes, and let $f_*: C_* \to D_*$ be a chain map. Then $f_i(Z_i(C_*)) \subset Z_i(D_*)$ and $f_i(B_i(C_*)) \subset B_i(D_*)$ for all i. It follows from this that $f_i: C_i \to D_i$ induces a homomorphism $f_*: H_i(C_*) \to H_i(D_*)$ of homology groups sending $[z]$ to $[f_i(z)]$ for all $z \in Z_i(C_*)$, where $[z] = z + B_i(C_*)$, and $[f_i(z)] = f_i(z) + B_i(D_*)$.

Definition A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*: A_* \to B_*$ and $q_*: B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

We see that $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \partial_{i+2} & & \downarrow \partial_{i+2} & & \downarrow \partial_{i+2} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{p_{i+1}} & B_{i+1} & \xrightarrow{q_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \\
0 & \longrightarrow & A_{i} & \xrightarrow{p_{i}} & B_{i} & \xrightarrow{q_{i}} & C_{i} & \longrightarrow & 0\,. \\
 & & \downarrow \partial_{i} & & \downarrow \partial_{i} & & \downarrow \partial_{i} & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{p_{i-1}} & B_{i-1} & \xrightarrow{q_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{i-1} & & \downarrow \partial_{i-1} & & \downarrow \partial_{i-1} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

Lemma 9.4 *Given any short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes, there is a well-defined homomorphism*

$$\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$$

which sends the homology class $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i \colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2} \colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) = \partial_i(b') = p_{i-1}(w'),$$

and $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i: Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}: B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i: Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$, as required. ∎

Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ and $0 \longrightarrow A'_* \xrightarrow{p'_*} B'_* \xrightarrow{q'_*} C'_* \longrightarrow 0$ be short exact sequences of chain complexes, and let $\lambda_*: A_* \to A'_*$, $\mu_*: B_* \to B'_*$ and $\nu_*: C_* \to C'_*$ be chain maps. For each integer i, let $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$ and $\alpha'_i: H_i(C'_*) \to H_{i-1}(A'_*)$ be the homomorphisms defined as described in Lemma 9.4. Suppose that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{p_*} & B_* & \xrightarrow{q_*} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\lambda_*} & & \downarrow{\mu_*} & & \downarrow{\nu_*} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{p'_*} & B'_* & \xrightarrow{q'_*} & C'_* & \longrightarrow & 0
\end{array}$$

commutes (i.e., $p'_i \circ \lambda_i = \mu_i \circ p_i$ and $q'_i \circ \mu_i = \nu_i \circ q_i$ for all i). Then the square

$$\begin{array}{ccc}
H_i(C_*) & \xrightarrow{\alpha_i} & H_{i-1}(A_*) \\
\downarrow{\nu_*} & & \downarrow{\lambda_*} \\
H_i(C'_*) & \xrightarrow{\alpha'_i} & H_{i-1}(A'_*)
\end{array}$$

commutes for all $i \in \mathbb{Z}$ (i.e., $\lambda_* \circ \alpha_i = \alpha'_i \circ \nu_*$).

Proposition 9.5 *Let* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *be a short exact sequence of chain complexes. Then the (infinite) sequence*

$$\cdots \xrightarrow{\alpha_{i+1}} H_i(A_*) \xrightarrow{p_*} H_i(B_*) \xrightarrow{q_*} H_i(C_*) \xrightarrow{\alpha_i} H_{i-1}(A_*) \xrightarrow{p_*} H_{i-1}(B_*) \xrightarrow{q_*} \cdots$$

of homology groups is exact, where $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$ *is the well-defined homomorphism that sends the homology class* $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_* \circ p_* = 0$. Thus the image of $p_*: H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_*: H_i(B_*) \to H_i(C_*)$. Let x be an element of $Z_i(B_*)$ for which $[x] \in \ker q_*$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}: B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x - \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus a represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x - \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now

$$\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w],$$

where w is the unique element of $Z_i(A_*)$ satisfying $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let z be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*([b - p_i(a)]) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$.

Now let w be an element of $Z_{i-1}(A_*)$ for which $[w] \in \ker p_*$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_*)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_*)$, as required. ∎

10 The Mayer-Vietoris Exact Sequence

10.1 The Mayer Vietoris Sequence of Homology Groups

Proposition 10.1 (The Mayer Vietoris Exact Sequence) *Let K be a simplicial complex, let L and M be subcomplexes of K such that $K = L \cup M$, and let R be an unital ring. Let*

$$i_q\colon C_q(L \cap M; R) \to C_q(L; R), \qquad j_q\colon C_q(L \cap M; R) \to C_q(M; R),$$
$$u_q\colon C_q(L; R) \to C_q(K; R), \qquad v_q\colon C_q(M; R) \to C_q(K; R)$$

be the inclusion homomorphisms induced by the inclusion maps $i\colon L \cap M \hookrightarrow L$, $j\colon L \cap M \hookrightarrow M$, $u\colon L \hookrightarrow K$ and $v\colon M \hookrightarrow K$, and let

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M; R)$, $c' \in C_q(L; R)$ and $c'' \in C_q(M; R)$. Then there is a well-defined homomorphism $\alpha_q\colon H_q(K; R) \to H_{q-1}(L \cap M; R)$ such that $\alpha_q([z]) = [\partial_q(c')] = -[\partial_q(c'')]$ for any $z \in Z_q(K; R)$, where c' and c'' are any q-chains of L and M respectively satisfying $z = c' + c''$. The resulting infinite sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M; R) \xrightarrow{k_*} H_q(L; R) \oplus H_q(M; R) \xrightarrow{w_*} H_q(K; R)$$
$$\xrightarrow{\alpha_q} H_{q-1}(L \cap M; R) \xrightarrow{k_*} \cdots,$$

of homology groups is then exact.

Proof The sequence

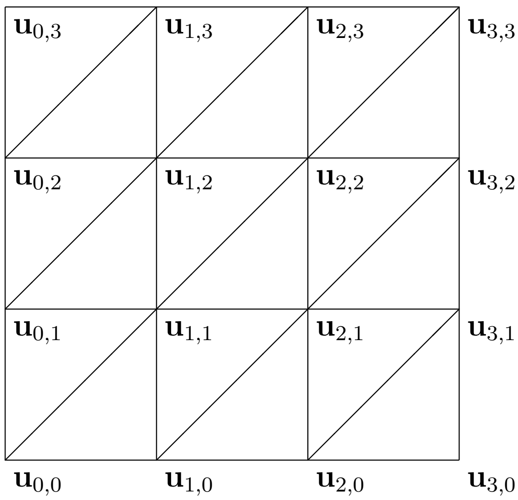
$$0 \longrightarrow C_*(L \cap M; R) \xrightarrow{k_*} C_*(L; R) \oplus C_*(M; R) \xrightarrow{w_*} C_*(K; R) \longrightarrow 0$$

is a short exact sequence of chain complexes. The existence and basic properties of the homomorphism $\alpha_q\colon H_q(K; R) \to H_{q-1}(L \cap M; R)$ then follow on applying Lemma 9.4. Indeed if c' and c'' are q-chains of L and M respectively, and if $c' + c'' \in Z_q(K; R)$ then $\partial_q(c') = -\partial_q(c'')$. But $\partial_q(c') \in Z_{q-1}(L; R)$ and $\partial_q(c'') \in Z_{q-1}(M; R)$ and $Z_{q-1}(L; R) \cap Z_{q-1}(M; R) = Z_{q-1}(L \cap M; R)$. Therefore $\partial_q(c') \in Z_{q-1}(L \cap M; R)$. Lemma 9.4 then ensures that the homology class of $\partial_q(c')$ in $H_{q-1}(L \cap M; R)$ is determined by the homology class of $c + c''$ in $Z_q(K; R)$. The exactness of the resulting infinite sequence of homology groups then follows on applying Proposition 9.5. ∎

The long exact sequence of homology groups Proposition 10.1 is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.

10.2 The Homology Groups of a Torus

We construct a simplicial complex K_{Sq} in the plane whose polyhedron is the square $[0,3] \times [0,3]$. We let $\mathbf{u}_{i,j} = (i,j)$ for $i = 0,1,2,3$ and $j = 0,1,2,3$. Then the simplicial complex K_{Sq} consists of the triangles $\mathbf{u}_{i,j}\,\mathbf{u}_{i+1,j}\,\mathbf{u}_{i+1,j+1}$ and $\mathbf{u}_{i,j}\,\mathbf{u}_{i+1,j+1}\,\mathbf{u}_{i,j+1}$ for $i = 0,1,2$ and $j = 0,1,2$, together with all the vertices and edges of those triangles. This simplicial complex is depicted in the following diagram:—



The simplicial complex K_{Sq} has 24 vertices, 33 edges and 18 triangles.

One can construct a simplicial map $s\colon K_{\mathrm{Sq}} \to K_{\mathrm{Torus}}$ mapping the simplicial complex K_{Sq} onto a simplicial complex K_{Torus} whose polyhedron is homeomorphic to a torus. One way of achieving this is to determine points $\mathbf{v}_{i,j}$ of $\mathbb{R}^3$ for $i = 0,1,2$ and $j = 0,1,2$ such that

$$\mathbf{v}_{0,0} = (1,-1,0),\;\; \mathbf{v}_{0,1} = (3,-1,1),\;\; \mathbf{v}_{0,2} = (1,-3,-1),$$

$$\mathbf{v}_{1,0} = (0,1,-1),\;\; \mathbf{v}_{1,1} = (1,3,-1),\;\; \mathbf{v}_{1,2} = (-1,1,-3),$$

$$\mathbf{v}_{2,0} = (-1,0,1),\;\; \mathbf{v}_{2,1} = (-1,1,3),\;\; \mathbf{v}_{2,2} = (-3,-1,1).$$

One can verify that these nine points are vertices of a simplicial complex K_{Torus} in $\mathbb{R}^3$ which consists of the 18 triangles

$$\mathbf{v}_{0,0}\,\mathbf{v}_{1,0}\,\mathbf{v}_{1,1},\quad \mathbf{v}_{0,0}\,\mathbf{v}_{1,1}\,\mathbf{v}_{0,1},\quad \mathbf{v}_{1,0}\,\mathbf{v}_{2,0}\,\mathbf{v}_{2,1},$$

$$\mathbf{v}_{1,0}\,\mathbf{v}_{2,1}\,\mathbf{v}_{1,1},\quad \mathbf{v}_{2,0}\,\mathbf{v}_{0,0}\,\mathbf{v}_{0,1},\quad \mathbf{v}_{2,0}\,\mathbf{v}_{0,1}\,\mathbf{v}_{2,1},$$

$$
\begin{array}{lll}
\mathbf{v}_{0,1}\,\mathbf{v}_{1,1}\,\mathbf{v}_{1,2}, & \mathbf{v}_{0,1}\,\mathbf{v}_{1,2}\,\mathbf{v}_{0,2}, & \mathbf{v}_{1,1}\,\mathbf{v}_{2,1}\,\mathbf{v}_{2,2}, \\
\mathbf{v}_{1,1}\,\mathbf{v}_{2,2}\,\mathbf{v}_{1,2}, & \mathbf{v}_{2,1}\,\mathbf{v}_{0,1}\,\mathbf{v}_{0,2}, & \mathbf{v}_{2,1}\,\mathbf{v}_{0,2}\,\mathbf{v}_{2,2}, \\
\mathbf{v}_{0,2}\,\mathbf{v}_{1,2}\,\mathbf{v}_{1,0}, & \mathbf{v}_{0,2}\,\mathbf{v}_{1,0}\,\mathbf{v}_{0,0}, & \mathbf{v}_{1,2}\,\mathbf{v}_{2,2}\,\mathbf{v}_{2,0}, \\
\mathbf{v}_{1,2}\,\mathbf{v}_{2,0}\,\mathbf{v}_{1,0}, & \mathbf{v}_{2,2}\,\mathbf{v}_{0,2}\,\mathbf{v}_{0,0}, & \mathbf{v}_{2,2}\,\mathbf{v}_{0,0}\,\mathbf{v}_{2,0},
\end{array}
$$

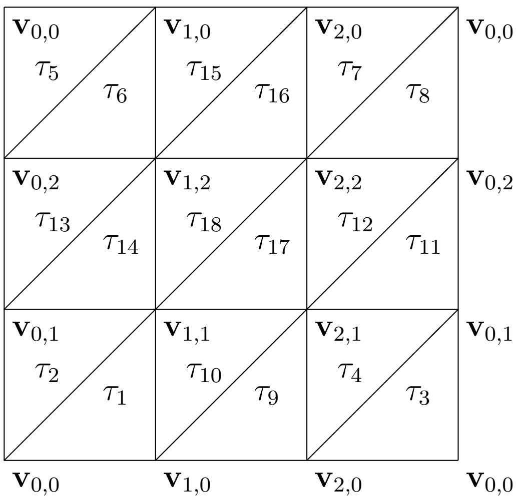
together with all the vertices and edges of these triangles. This simplicial complex K_{Torus} has 9 vertices, 27 edges and 18 triangles.

There is then a well-defined simplicial map $s\colon K_{\mathrm{Sq}} \to K_{\mathrm{Torus}}$ defined such that

$$
\begin{array}{rcll}
s_{\mathrm{Vert}}(\mathbf{u}_{i,j}) & = & \mathbf{v}_{i,j} & \text{for } i = 0,1,2 \text{ and } j = 0,1,2; \\
s_{\mathrm{Vert}}(\mathbf{u}_{i,3}) & = & \mathbf{v}_{i,0} & \text{for } i = 0,1,2; \\
s_{\mathrm{Vert}}(\mathbf{u}_{3,j}) & = & \mathbf{v}_{0,j} & \text{for } j = 0,1,2; \\
s_{\mathrm{Vert}}(\mathbf{u}_{3,3}) & = & \mathbf{v}_{0,0}. &
\end{array}
$$

Each triangle of K_{Torus} is then the image under this simplicial map of exactly one triangle of K_{Sq}.

The following diagram represents the simplicial complex K_{Torus}. The 18 triangles in this diagram represent the 18 triangles of K_{Torus} and are labelled $\tau_1, \tau_2, \ldots, \tau_{18}$. Moreover the vertices of each triangle in the diagram are labelled by the vertices of the corresponding triangle of the simplicial complex K_{Torus}.



These 18 triangles $\tau_1, \tau_2, \ldots, \tau_{18}$ are determined by their vertices as follows:

$$\tau_1 = \mathbf{v}_{0,0}\,\mathbf{v}_{1,0}\,\mathbf{v}_{1,1}, \quad \tau_2 = \mathbf{v}_{0,0}\,\mathbf{v}_{1,1}\,\mathbf{v}_{0,1}, \quad \tau_3 = \mathbf{v}_{2,0}\,\mathbf{v}_{0,0}\,\mathbf{v}_{0,1},$$

$$\tau_4 = \mathbf{v}_{2,0}\,\mathbf{v}_{0,1}\,\mathbf{v}_{2,1}, \quad \tau_5 = \mathbf{v}_{0,2}\,\mathbf{v}_{1,0}\,\mathbf{v}_{0,0}, \quad \tau_6 = \mathbf{v}_{0,2}\,\mathbf{v}_{1,2}\,\mathbf{v}_{1,0},$$

$$\tau_7 = \mathbf{v}_{2,2}\,\mathbf{v}_{0,0}\,\mathbf{v}_{2,0}, \quad \tau_8 = \mathbf{v}_{2,2}\,\mathbf{v}_{0,2}\,\mathbf{v}_{0,0}, \quad \tau_9 = \mathbf{v}_{1,0}\,\mathbf{v}_{2,0}\,\mathbf{v}_{2,1},$$

$$\tau_{10} = \mathbf{v}_{1,0}\,\mathbf{v}_{2,1}\,\mathbf{v}_{1,1}, \quad \tau_{11} = \mathbf{v}_{2,1}\,\mathbf{v}_{0,1}\,\mathbf{v}_{0,2}, \quad \tau_{12} = \mathbf{v}_{2,1}\,\mathbf{v}_{0,2}\,\mathbf{v}_{2,2},$$

$$\tau_{13} = \mathbf{v}_{0,1}\,\mathbf{v}_{1,2}\,\mathbf{v}_{0,2}, \quad \tau_{14} = \mathbf{v}_{0,1}\,\mathbf{v}_{1,1}\,\mathbf{v}_{1,2}, \quad \tau_{15} = \mathbf{v}_{1,2}\,\mathbf{v}_{2,0}\,\mathbf{v}_{1,0},$$

$$\tau_{16} = \mathbf{v}_{1,2}\,\mathbf{v}_{2,2}\,\mathbf{v}_{2,0}, \quad \tau_{17} = \mathbf{v}_{1,1}\,\mathbf{v}_{2,1}\,\mathbf{v}_{2,2}, \quad \tau_{18} = \mathbf{v}_{1,1}\,\mathbf{v}_{2,2}\,\mathbf{v}_{1,2}.$$

Let L_0 be the subcomplex of K_{Torus} consisting of the five vertices

$$\mathbf{v}_{0,0}, \ \mathbf{v}_{1,0}, \ \mathbf{v}_{2,0}, \ \mathbf{v}_{0,1} \ \text{and} \ \mathbf{v}_{0,2}$$

and the six edges

$$\mathbf{v}_{0,0}\,\mathbf{v}_{1,0}, \ \mathbf{v}_{1,0}\,\mathbf{v}_{2,0}, \ \mathbf{v}_{2,0}\,\mathbf{v}_{0,0}, \ \mathbf{v}_{0,0}\,\mathbf{v}_{0,1}, \ \mathbf{v}_{0,1}\,\mathbf{v}_{0,2} \ \text{and} \ \mathbf{v}_{0,2}\,\mathbf{v}_{0,0},$$

and let L be the subcomplex of K_{Torus} consisting of the vertices and edges of L_0 together with the 16 triangles τ_i for $0 \leq i \leq 16$ and all the vertices and edges of those triangles. This subcomplex L is the subcomplex of K_{Torus} obtained from removing from K_{Torus} the two triangles τ_{17} and τ_{18} together with the edge $\mathbf{v}_{1,1}\,\mathbf{v}_{2,2}$ of K_{Torus} that is common to τ_{17} and τ_{18}.

We claim that the inclusion map $i_0\colon L_0 \hookrightarrow L$ induces isomorphisms

$$i_{0*}\colon H_q(L_0;\mathbb{Z}) \to H_q(L;\mathbb{Z})$$

of homology groups for all non-negative integers q. To see this note that there is a finite sequence $L_0, L_1, L_2, \ldots, L_{16}$ of subcomplexes of K, where, for each integer k between 1 and 16, the subcomplex L_k is obtained by adding to L_{k-1} the triangle τ_k together with all its vertices and faces. The order in which the triangles $\tau_1, \tau_2, \ldots, \tau_{16}$ have been listed then ensures that the intersection $\tau_k \cap |L_{k-1}|$ of the triangle τ_k with the polyhedron of the subcomplex L_{k-1} is either a single edge of τ_k or else is the union of two edges of τ_k. Lemma 7.4 and Lemma 7.5 then ensure that the inclusion of the subcomplex L_{k-1} in L_k induces isomorphisms of homology groups for $k = 1, 2, \ldots, 16$. It follows that $i_{0*}\colon H_q(L_0;\mathbb{Z}) \to H_q(L;\mathbb{Z})$ is an isomorphism for $q = 0, 1, 2$.

Let z_1 and z_2 be the 1-cycles of L_0 with integer coefficients defined such that

$$\begin{aligned} z_1 &= \langle \mathbf{v}_{0,0}\,\mathbf{v}_{1,0}\rangle + \langle \mathbf{v}_{1,0}\,\mathbf{v}_{2,0}\rangle + \langle \mathbf{v}_{2,0}\,\mathbf{v}_{0,0}\rangle \\ z_2 &= \langle \mathbf{v}_{0,0}\,\mathbf{v}_{0,1}\rangle + \langle \mathbf{v}_{0,1}\,\mathbf{v}_{0,2}\rangle + \langle \mathbf{v}_{0,2}\,\mathbf{v}_{0,0}\rangle. \end{aligned}$$

A simple calculation shows that $Z_2(L_0;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and moreover, given any 1-cycle z of L_0, there exist uniquely-determined integers r_1 and r_2 such that $z = r_1 z_1 + r_2 z_2$. Moreover $H_1(L_0;\mathbb{Z}) = Z_1(L;\mathbb{Z})$, because $B_1(L_0;\mathbb{Z}) = 0$. (The subcomplex L_0 has no 2-simplices, and therefore it has no non-zero 1-boundaries.) The inclusion map $i_0\colon L_0 \to L$ induces isomorphisms of homology groups, and therefore $H_1(L;\mathbb{Z})$ must also be freely generated by the homology classes of the cycles z_1 and z_2. Therefore, given any 2-cycle z of L, there exist uniquely determined integers r_1 and r_2 such that $[z]_L = r_1[z_1]_L + r_2[z_2]_L$, where $[z]_L$, $[z_1]_L$ and $[z_2]_L$ denote the homology classes of the 1-cycles z, z_1 and z_2 in $H_1(L;\mathbb{Z})$. In consequence, given any 1-cycle z of L, there exist uniquely-determined integers r_1 and r_2 such that $z - r_1 z_1 - r_2 z_2 \in B_1(L;\mathbb{Z})$.

Let

$$z_3 = \langle \mathbf{v}_{1,1}\,\mathbf{v}_{1,2}\rangle + \langle \mathbf{v}_{1,2}\,\mathbf{v}_{2,2}\rangle + \langle \mathbf{v}_{2,2}\,\mathbf{v}_{2,1}\rangle + \langle \mathbf{v}_{2,1}\,\mathbf{v}_{1,1}\rangle.$$

Then $[z_3]_L = 0$. Indeed each triangle τ_i determines a corresponding generator γ_i of $C_2(L;\mathbb{Z})$ for $i = 1, 2, \ldots, 16$ that is determined by an anti-clockwise ordering of the vertices of τ_i, so that

$$\gamma_1 = \langle \mathbf{v}_{0,0}\,\mathbf{v}_{1,0}\,\mathbf{v}_{1,1}\rangle, \quad \gamma_2 = \langle \mathbf{v}_{0,0}\,\mathbf{v}_{1,1}\,\mathbf{v}_{0,1}\rangle, \quad \gamma_3 = \langle \mathbf{v}_{2,0}\,\mathbf{v}_{0,0}\,\mathbf{v}_{0,1}\rangle \text{ etc.},$$

and direct computation shows that if $c \in C_2(L;\mathbb{Z})$ is the 2-chain of L defined such that

$$c = \gamma_1 + \gamma_2 + \cdots + \gamma_{16},$$

then $\partial_2 c = -z_3$. Indeed terms corresponding to the edges

$$\mathbf{v}_{0,0}\,\mathbf{v}_{1,1}, \quad \mathbf{v}_{1,0}\,\mathbf{v}_{1,1}, \quad \mathbf{v}_{1,0}\,\mathbf{v}_{2,1}, \quad \mathbf{v}_{2,0}\,\mathbf{v}_{2,1}, \quad \mathbf{v}_{0,2}\,\mathbf{v}_{0,1}, \quad \mathbf{v}_{2,1}\,\mathbf{v}_{0,1},$$

$$\mathbf{v}_{2,1}\,\mathbf{v}_{0,2}, \quad \mathbf{v}_{2,2}\,\mathbf{v}_{0,2}, \quad \mathbf{v}_{2,2}\,\mathbf{v}_{0,0}, \quad \mathbf{v}_{2,2}\,\mathbf{v}_{2,0}, \quad \mathbf{v}_{2,2}\,\mathbf{v}_{2,0}, \quad \mathbf{v}_{1,2}\,\mathbf{v}_{2,0},$$

$$\mathbf{v}_{1,2}\,\mathbf{v}_{1,0}, \quad \mathbf{v}_{0,2}\,\mathbf{v}_{1,0}, \quad \mathbf{v}_{0,2}\,\mathbf{v}_{1,2}, \quad \mathbf{v}_{0,1}\,\mathbf{v}_{1,2} \quad \text{and} \quad \mathbf{v}_{0,1}\,\mathbf{v}_{1,1}$$

cancel off in pairs, with the result that

$$\begin{aligned}
\partial_2 c &= \langle \mathbf{v}_{0,0}\,\mathbf{v}_{1,0}\rangle + \langle \mathbf{v}_{1,0}\,\mathbf{v}_{2,0}\rangle + \langle \mathbf{v}_{2,0}\,\mathbf{v}_{0,0}\rangle \\
&\qquad + \langle \mathbf{v}_{0,0}\,\mathbf{v}_{0,1}\rangle + \langle \mathbf{v}_{0,2}\,\mathbf{v}_{0,2}\rangle + \langle \mathbf{v}_{0,2}\,\mathbf{v}_{0,0}\rangle \\
&\qquad + \langle \mathbf{v}_{0,0}\,\mathbf{v}_{2,0}\rangle + \langle \mathbf{v}_{2,0}\,\mathbf{v}_{1,0}\rangle + \langle \mathbf{v}_{1,0}\,\mathbf{v}_{0,0}\rangle \\
&\qquad + \langle \mathbf{v}_{0,0}\,\mathbf{v}_{0,2}\rangle + \langle \mathbf{v}_{0,2}\,\mathbf{v}_{0,1}\rangle + \langle \mathbf{v}_{0,1}\,\mathbf{v}_{0,0}\rangle \\
&\qquad - \langle \mathbf{v}_{1,1}\,\mathbf{v}_{1,2}\rangle - \langle \mathbf{v}_{1,2}\,\mathbf{v}_{2,2}\rangle - \langle \mathbf{v}_{2,2}\,\mathbf{v}_{1,2}\rangle - \langle \mathbf{v}_{1,2}\,\mathbf{v}_{1,1}\rangle \\
&= z_1 + z_2 - z_1 - z_2 - z_3 \\
&= = -z_3
\end{aligned}$$

(The contributing edges may be identified by working round the outer boundary of the large square in the diagram above depicting the structure of the simplicial complex K_{Torus} in an anticlockwise direction, starting at the bottom left hand corner of the large square, and then subtracting off terms corresponding to the edges of the small inner square.)

It follows from this computation that $z_3 \in B_1(L;\mathbb{Z})$, and thus $[z_3]_L = 0$ in $H_1(L;\mathbb{Z})$. The subcomplex L_0 is connected, and therefore $H_0(L_0,\mathbb{Z}) \cong \mathbb{Z}$. Indeed $H_0(L_0,\mathbb{Z})$ is generated by $[\langle \mathbf{v}_{0,0}\rangle]_{L_0}$. It follows that $H_0(L;\mathbb{Z}) \cong \mathbb{Z}$, and indeed the homology class $[\langle \mathbf{v}_{i,j}\rangle]$ of any vertex of K_{Torus} in $H_0(L;\mathbb{Z})$ generates $H_0(L;\mathbb{Z})$.

Let M be the subcomplex of K_{Torus} consisting of the union of the two triangles τ_{17} and τ_{18}, together with the vertices and edges of those triangles. Then M has 4 vertices, 5 edges and 2 triangles. The vertices of M are $\mathbf{v}_{1,1}$ $\mathbf{v}_{2,1}$, $\mathbf{v}_{2,2}$ and $\mathbf{v}_{1,2}$, the edges of M are

$$\mathbf{v}_{1,1}\,\mathbf{v}_{2,1}, \quad \mathbf{v}_{2,1}\,\mathbf{v}_{2,2}, \quad \mathbf{v}_{2,2}\,\mathbf{v}_{1,2}, \quad \mathbf{v}_{1,2}\,\mathbf{v}_{1,1} \quad \text{and} \quad \mathbf{v}_{1,1}\,\mathbf{v}_{2,2},$$

and the triangles of M are

$$\mathbf{v}_{1,1}\,\mathbf{v}_{2,1}\,\mathbf{v}_{2,2} \quad \text{and} \quad \mathbf{v}_{1,1}\,\mathbf{v}_{2,2}\,\mathbf{v}_{1,2}.$$

Then $H_0(M,\mathbb{Z}) \cong \mathbb{Z}$, and $H_q(M,\mathbb{Z}) = 0$ for all integers q satisfying $q > 0$.

The intersection $L \cap M$ of the subcomplexes L and M of K_{Torus} consists of the four vertices $\mathbf{v}_{1,1}$ $\mathbf{v}_{2,1}$, $\mathbf{v}_{2,2}$ and $\mathbf{v}_{1,2}$ and the four edges

$$\mathbf{v}_{1,1}\,\mathbf{v}_{2,1}, \quad \mathbf{v}_{2,1}\,\mathbf{v}_{2,2}, \quad \mathbf{v}_{2,2}\,\mathbf{v}_{1,2} \quad \text{and} \quad \mathbf{v}_{1,2}\,\mathbf{v}_{1,1}.$$

Then $H_0(L\cap M;\mathbb{Z}) \cong \mathbb{Z}$ and $H_1(L\cap M;\mathbb{Z}) \cong \mathbb{Z}$, and moreover $H_0(L\cap M;\mathbb{Z})$ is generated by $[\langle \mathbf{v}_{1,1}\rangle]_{L\cap M}$ and $H_1(L\cap M;\mathbb{Z})$ is generated by $[z_3]_{L\cap M}$, where

$$z_3 = \langle \mathbf{v}_{1,1}\,\mathbf{v}_{1,2}\rangle + \langle \mathbf{v}_{1,2}\,\mathbf{v}_{2,2}\rangle + \langle \mathbf{v}_{2,2}\,\mathbf{v}_{2,1}\rangle + \langle \mathbf{v}_{2,1}\,\mathbf{v}_{1,1}\rangle.$$

We now have the necessary information to compute the homology groups of K_{Torus} using the Mayer-Vietoris exact sequence associated with the decomposition of K_{Torus} as the union of subcomplexes L and M as described above. The homomorphisms

$$i_*\colon H_0(L\cap M;\mathbb{Z}) \to H_0(L;\mathbb{Z}) \quad \text{and} \quad j_*\colon H_0(L\cap M;\mathbb{Z}) \to H_0(M;\mathbb{Z})$$

induced by the inclusions $i\colon L\cap M \hookrightarrow L$ and $j\colon L\cap M \hookrightarrow M$ are isomorphisms of Abelian groups that satisfy

$$i_*([\langle \mathbf{v}_{1,1}\rangle]_{L\cap M}) = [\langle \mathbf{v}_{1,1}\rangle]_L = [\langle \mathbf{v}_{0,0}\rangle]_L \quad \text{and} \quad j_*(\langle[\mathbf{v}_{1,1}\rangle]_{L\cap M}) = [\langle \mathbf{v}_{1,1}\rangle]_M.$$

Next we note that the homology group $H_1(L \cap M; \mathbb{Z})$ is generated by $[z_3]_{L\cap M}$, the homology group $H_1(L; \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbf{Z}$ and is freely generated by $[z_1]_L$ and $[z_2]_L$, where

$$\begin{aligned} z_1 &= \langle \mathbf{v}_{0,0}\, \mathbf{v}_{1,0} \rangle + \langle \mathbf{v}_{1,0}\, \mathbf{v}_{2,0} \rangle + \langle \mathbf{v}_{2,0}\, \mathbf{v}_{0,0} \rangle \\ z_2 &= \langle \mathbf{v}_{0,0}\, \mathbf{v}_{0,1} \rangle + \langle \mathbf{v}_{0,1}\, \mathbf{v}_{0,2} \rangle + \langle \mathbf{v}_{0,2}\, \mathbf{v}_{0,0} \rangle, \end{aligned}$$

and moreover the homomorphism $i_*\colon H_1(L \cap M; \mathbb{Z})$ is the zero homomorphism. Also

$$H_2(L; \mathbb{Z}) = 0, \quad H_2(M; \mathbb{Z}) = 0 \quad \text{and} \quad H_1(M; \mathbb{Z}) = 0.$$

It follows from the exactness of the Mayer-Vietoris sequence that the following sequence of Abelian groups and homomorphisms is exact:—

$$\begin{aligned} 0 \longrightarrow H_2(K_{\text{Torus}}; \mathbb{Z}) \xrightarrow{\alpha_2} H_1(L \cap M; \mathbb{Z}) \xrightarrow{i_*} H_1(L; \mathbb{Z}) \xrightarrow{u_*} H_1(K_{\text{Torus}}; \mathbb{Z}) \\ \xrightarrow{\alpha_1} H_0(L \cap M; \mathbb{Z}) \xrightarrow{k_*} H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z}), \end{aligned}$$

where $u_*\colon H_1(L; \mathbb{Z}) \to H_1(K_{\text{Torus}}; \mathbb{Z})$ is induced by the inclusion map $u\colon L \hookrightarrow K_{\text{Torus}}$, the homomorphims α_2 and α_1 are defined as described in Proposition 10.1, and

$$\begin{aligned} k_*([\langle \mathbf{v}_{1,1} \rangle]_{L\cap M}) &= (i_*([\langle \mathbf{v}_{1,1} \rangle]_{L\cap M}), -j_*([\langle \mathbf{v}_{1,1} \rangle]_{L\cap M})) \\ &= ([\langle \mathbf{v}_{0,0} \rangle]_L, -[\langle \mathbf{v}_{1,1} \rangle]_M). \end{aligned}$$

Now $[\langle \mathbf{v}_{1,1} \rangle]_{L\cap M}$ generates $H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z})$, and $k_*([\langle \mathbf{v}_{1,1} \rangle]_{L\cap M}) \neq 0$. It follows that

$$k_*\colon H_0(L \cap M; \mathbb{Z}) \to H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z})$$

is injective. The exactness of the Mayer-Vietoris sequence at $H_0(L \cap M; \mathbb{Z})$ then ensures that the homomorphism $\alpha_1 H_1(K_{\text{Torus}} \to H_0(L \cap M; \mathbb{Z})$ occuring in the Mayer-Vietoris sequence is the zero homomorphism. It then follows from the exactness of the Mayer-Vietoris sequence $H_1(K_{\text{Torus}}$ that the homomorphism

$$u_*\colon H_1(L; \mathbb{Z}) \to H_1(K_{\text{Torus}}; \mathbb{Z})$$

is surjective. Thus the sequence

$$0 \longrightarrow H_2(K_{\text{Torus}}; \mathbb{Z}) \xrightarrow{\alpha_2} H_1(L \cap M; \mathbb{Z}) \xrightarrow{i_*} H_1(L; \mathbb{Z}) \xrightarrow{u_*} H_1(K_{\text{Torus}}; \mathbb{Z}) \longrightarrow 0$$

derived from the Mayer-Vietoris sequence is exact. However $i_*\colon H_1(L \cap M; \mathbb{Z}) to H_1(L; \mathbb{Z})$ is the zero homomorphism. It follows from exactness that $\alpha_2\colon H_2(K_{\text{Torus}}; \mathbb{Z}) \to H_1(L \cap M; \mathbb{Z})$ and $u_*\colon H_1(L; \mathbb{Z}) \to H_1(K_{\text{Torus}}; \mathbb{Z})$ are isomorphisms. We deduce that

$$H_2(K_{\text{Torus}}; \mathbb{Z}) \cong H_1(L \cap M; \mathbb{Z}) \cong \mathbb{Z}$$

and

$$H_1(K_{\text{Torus}}; \mathbb{Z}) \cong H_1(L; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Now the polyhedron of K_{Torus} is connected. It follows from Theorem 8.6 that $H_0(K_{\text{Torus}}; \mathbb{Z}) \cong \mathbb{Z}$. This result can also be deduced from the exactness of the the portion

$$H_0(L \cap M; \mathbb{Z}) \xrightarrow{k_*} H_0(L; \mathbb{Z}) \oplus H_0(M; \mathbb{Z}), \xrightarrow{w_*} H_0(K_{\text{Torus}}; \mathbb{Z}) \longrightarrow 0$$

of the Mayer-Vietoris sequence.

To summarize, the homology groups of the simplicial complex K_{Torus} triangulating the torus are as follows:

$$H_2(K_{\text{Torus}}; \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K_{\text{Torus}}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_0(K_{\text{Torus}}; \mathbb{Z}) \cong \mathbb{Z}.$$

10.3 The Homology Groups of a Klein Bottle

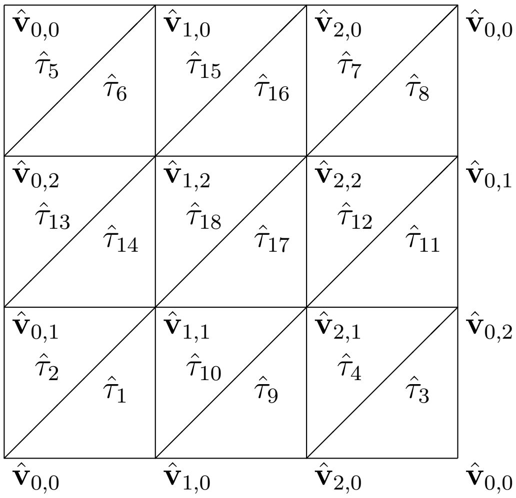
Let K_{Sq} be the simplicial complex triangulating the square $[0,3] \times [0,3]$ defined as in the above discussion of the homology groups of the torus.

There exists a simplicial complex K_{Klein} in $\mathbb{R}^4$ with vertices $\hat{\mathbf{v}}_{i,j}$ for $i = 0, 1, 2$ and $j = 0, 1, 2$ whose polyhedron is homeomorphic to a Klein Bottle, and a simplicial map $\hat{s}\colon K_{\text{Sq}} \to K_{\text{Klein}}$ mapping the simplicial complex K_{Sq} onto the simplicial complex K_{Klein}, where this simplicial map is defined such that

$$\begin{aligned}
\hat{s}_{\text{Vert}}(\mathbf{u}_{i,j}) &= \hat{\mathbf{v}}_{i,j} \quad \text{for } i = 0, 1, 2 \text{ and } j = 0, 1, 2; \\
\hat{s}_{\text{Vert}}(\mathbf{u}_{i,3}) &= \hat{\mathbf{v}}_{i,0} \quad \text{for } i = 0, 1, 2; \\
\hat{s}_{\text{Vert}}(\mathbf{u}_{3,0}) &= \hat{\mathbf{v}}_{0,0}; \\
\hat{s}_{\text{Vert}}(\mathbf{u}_{3,1}) &= \hat{\mathbf{v}}_{0,2}; \\
\hat{s}_{\text{Vert}}(\mathbf{u}_{3,2}) &= \hat{\mathbf{v}}_{0,1}; \\
\hat{s}_{\text{Vert}}(\mathbf{u}_{3,3}) &= \hat{\mathbf{v}}_{0,0}.
\end{aligned}$$

Each triangle of K_{Klein} is then the image under this simplicial map of exactly one triangle of K_{Sq}. We do not discuss here the details of how the simplicial complex representing the Klein Bottle is embedded in $\mathbb{R}^4$.

The following diagram represents the simplicial complex K_{Klein}. The 18 triangles in this diagram represent the 18 triangles of K_{Klein} and are labelled $\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{18}$. Moreover the vertices of each triangle in the diagram are labelled by the vertices of the corresponding triangle of the simplicial complex K_{Klein}.



These 18 triangles $\tau_1, \tau_2, \ldots, \tau_{18}$ are determined by their vertices as follows:

$$\hat{\tau}_1 = \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{1,1}, \quad \hat{\tau}_2 = \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{0,1}, \quad \hat{\tau}_3 = \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,2},$$

$$\hat{\tau}_4 = \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{2,1}, \quad \hat{\tau}_5 = \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{0,0}, \quad \hat{\tau}_6 = \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{1,0},$$

$$\hat{\tau}_7 = \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{2,0}, \quad \hat{\tau}_8 = \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{0,0}, \quad \hat{\tau}_9 = \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{2,1},$$

$$\hat{\tau}_{10} = \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{1,1}, \quad \hat{\tau}_{11} = \hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,1}, \quad \hat{\tau}_{12} = \hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{2,2},$$

$$\hat{\tau}_{13} = \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{0,2}, \quad \hat{\tau}_{14} = \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{1,2}, \quad \hat{\tau}_{15} = \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{1,0},$$

$$\hat{\tau}_{16} = \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{2,0}, \quad \hat{\tau}_{17} = \hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{2,2}, \quad \hat{\tau}_{18} = \hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{1,2}.$$

Let $\hat{L}_0$ be the subcomplex of K_{Klein} consisting of the five vertices

$$\hat{\mathbf{v}}_{0,0}, \ \hat{\mathbf{v}}_{1,0}, \ \hat{\mathbf{v}}_{2,0}, \ \hat{\mathbf{v}}_{0,1} \text{ and } \hat{\mathbf{v}}_{0,2}$$

and the six edges

$$\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,0}, \ \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,0}, \ \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,0}, \ \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,1}, \ \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{0,2} \text{ and } \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,0},$$

and let $\hat{L}$ be the subcomplex of K_{Klein} consisting of the vertices and edges of $\hat{L}_0$ together with the 16 triangles $\hat{\tau}_i$ for $0 \leq i \leq 16$ and all the vertices and edges of those triangles. This subcomplex $\hat{L}$ is the subcomplex of K_{Klein} obtained from removing from K_{Klein} the two triangles $\hat{\tau}_{17}$ and $\hat{\tau}_{18}$ together with the edge $\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,2}$ of K_{Klein} that is common to $\hat{\tau}_{17}$ and $\hat{\tau}_{18}$.

Now the inclusion map $i_0\colon \hat{L}_0 \hookrightarrow L$ induces isomorphisms

$$i_{0*}\colon H_q(\hat{L}_0;\mathbb{Z}) \to H_q(\hat{L};\mathbb{Z})$$

of homology groups for all non-negative integers q. The justification for this corresponds to the justification of the corresponding result in the preceding discussion of the homology of the torus. The subcomplex $\hat{L}$ is obtained $\hat{L}_0$ by the successive addition of 16 triangles together with their vertices and edges. At each stage the intersection of the triangle to be added with the polygon of the subcomplex built up prior to the addition of the triangle under consideration is either a single edge of the added triangle or else is the union of two edges of the added triangle. It then follows from applications of Lemma 7.4 and Lemma 7.5 that the addition of new triangles in the specified sequence does not change homology groups, and therefore the inclusion of $\hat{L}_0$ in $\hat{L}$ induces isomorphisms of homology groups.

Now $H_1(\hat{L}_0;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed let z_1 and z_2 be the 1-cycles of $\hat{L}_0$ with integer coefficients defined such that

$$\begin{aligned} z_1 &= \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,0}\rangle + \langle \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,0}\rangle + \langle \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,0}\rangle \\ z_2 &= \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,1}\rangle + \langle \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{0,2}\rangle + \langle \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,0}\rangle. \end{aligned}$$

A simple calculation shows that $Z_2(\hat{L}_0;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and moreover, given any 1-cycle z of $\hat{L}_0$, there exist uniquely-determined integers r_1 and r_2 such that $z = r_1 z_1 + r_2 z_2$. It follows that, given any 1-cycle z of $\hat{L}$, there exist uniquely-determined integers r_1 and r_2 such that $[z]_{\hat{L}} = r_1[z_1]_{\hat{L}} + r_2[z_2]_{\hat{L}}$, where $[z]_{\hat{L}}$, $[z_1]_{\hat{L}}$ and $[z_2]_{\hat{L}}$ denote the homology classes of the 1-cycles z, z_1 and z_2 in $H_1(\hat{L};\mathbb{Z})$. In consequence, given any 1-cycle z of $\hat{L}$, there exist uniquely-determined integers r_1 and r_2 such that $z - r_1 z_1 - r_2 z_2 \in B_1(\hat{L};\mathbb{Z})$.

Let

$$z_3 = \langle \hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{1,2}\rangle + \langle \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{2,2}\rangle + \langle \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{2,1}\rangle + \langle \hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{1,1}\rangle.$$

Then $[z_3]_L = -2[z_2]_L$. Indeed each triangle $\hat{\tau}_i$ determines a corresponding generator $\hat{\gamma}_i$ of $C_2(\hat{L};\mathbb{Z})$ for $i = 1, 2, \ldots, 16$ that is determined by an anti-clockwise ordering of the vertices of $\hat{\tau}_i$, so that

$$\hat{\gamma}_1 = \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{1,1}\rangle, \quad \hat{\gamma}_2 = \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{0,1}\rangle, \quad \hat{\gamma}_3 = \langle \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,2}\rangle \text{ etc.},$$

and direct computation shows that if $c \in C_2(\hat{L};\mathbb{Z})$ is the 2-chain of $\hat{L}$ defined such that

$$c = \hat{\gamma}_1 + \hat{\gamma}_2 + \cdots + \hat{\gamma}_{16},$$

then $\partial_2 c = -2z_2 - z_3$. Indeed terms corresponding to the edges

$$\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,1}, \quad \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{1,1}, \quad \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,1}, \quad \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{2,1}, \quad \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,2}, \quad \hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{0,2},$$

$$\hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{0,1},\quad \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{0,1},\quad \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{0,0},\quad \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{2,0},\quad \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{2,0},\quad \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{2,0},$$
$$\hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{1,0},\quad \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{1,0},\quad \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{1,2},\quad \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{1,2}\quad \text{and}\quad \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{1,1}$$

cancel off in pairs, with the result that

$$\begin{aligned}
\partial_2 c &= \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,0}\rangle + \langle \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,0}\rangle + \langle \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,0}\rangle \\
&\qquad + \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,2}\rangle + \langle \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,1}\rangle + \langle \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{0,0}\rangle \\
&\qquad + \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{2,0}\rangle + \langle \hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{1,0}\rangle + \langle \hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{0,0}\rangle \\
&\qquad + \langle \hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,2}\rangle + \langle \hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,1}\rangle + \langle \hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{0,0}\rangle \\
&\qquad - \langle \hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{1,2}\rangle - \langle \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{2,2}\rangle - \langle \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{1,2}\rangle - \langle \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{1,1}\rangle \\
&= z_1 - z_2 - z_1 - z_2 - z_3 \\
&= = -2z_2 - z_3
\end{aligned}$$

(The contributing edges may be identified by working round the outer boundary of the large square in the diagram above depicting the structure of the simplicial complex K_{Klein} in an anticlockwise direction, starting at the bottom left hand corner of the large square, and then subtracting off terms corresponding to the edges of the small inner square.)

It follows from this computation that $[z_3]_{\hat{L}} = -2[z_2]_{\hat{L}}$ in $H_1(\hat{L};\mathbb{Z})$.

The subcomplex $\hat{L}_0$ is connected, and therefore $H_0(\hat{L}_0,\mathbb{Z}) \cong \mathbb{Z}$. Indeed $H_0(\hat{L}_0,\mathbb{Z})$ is generated by $[\langle \hat{\mathbf{v}}_{0,0}\rangle]_{\hat{L}_0}$. It follows that $H_0(\hat{L};\mathbb{Z}) \cong \mathbb{Z}$, and indeed the homology class $[\langle \hat{\mathbf{v}}_{i,j}\rangle]$ of any vertex of K_{Klein} in $H_0(\hat{L};\mathbb{Z})$ generates $H_0(\hat{L};\mathbb{Z})$.

Let $\hat{M}$ be the subcomplex of K_{Klein} consisting of the union of the two triangles τ_{17} and τ_{18}, together with the vertices and edges of those triangles. Then $\hat{M}$ has 4 vertices, 5 edges and 2 triangles. The vertices of $\hat{M}$ are $\hat{\mathbf{v}}_{1,1}$ $\hat{\mathbf{v}}_{2,1}$, $\hat{\mathbf{v}}_{2,2}$ and $\hat{\mathbf{v}}_{1,2}$, the edges of $\hat{M}$ are

$$\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,1},\quad \hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{2,2},\quad \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{1,2},\quad \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{1,1}\quad \text{and}\quad \hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,2},$$

and the triangles of $\hat{M}$ are

$$\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{2,2}\quad \text{and}\quad \hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{1,2}.$$

Then $H_0(\hat{M},\mathbb{Z}) \cong \mathbb{Z}$, and $H_q(\hat{M},\mathbb{Z}) = 0$ for all integers q satisfying $q > 0$.

The intersection $\hat{L} \cap \hat{M}$ of the subcomplexes $\hat{L}$ and $\hat{M}$ of K_{Klein} consists of the four vertices $\hat{\mathbf{v}}_{1,1}$ $\hat{\mathbf{v}}_{2,1}$, $\hat{\mathbf{v}}_{2,2}$ and $\hat{\mathbf{v}}_{1,2}$ and the four edges

$$\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{2,1},\quad \hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{2,2},\quad \hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{1,2}\quad \text{and}\quad \hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{1,1}.$$

Then $H_0(\hat{L}\cap\hat{M};\mathbb{Z})\cong\mathbb{Z}$ and $H_1(\hat{L}\cap\hat{M};\mathbb{Z})\cong\mathbb{Z}$, and moreover $H_0(\hat{L}\cap\hat{M};\mathbb{Z})$ is generated by $[\langle\mathbf{v}_{1,1}\rangle]_{\hat{L}\cap\hat{M}}$ and $H_1(\hat{L}\cap\hat{M};\mathbb{Z})$ is generated by $[z_3]_{\hat{L}\cap\hat{M}}$, where

$$z_3=\langle\hat{\mathbf{v}}_{1,1}\,\hat{\mathbf{v}}_{1,2}\rangle+\langle\hat{\mathbf{v}}_{1,2}\,\hat{\mathbf{v}}_{2,2}\rangle+\langle\hat{\mathbf{v}}_{2,2}\,\hat{\mathbf{v}}_{2,1}\rangle+\langle\hat{\mathbf{v}}_{2,1}\,\hat{\mathbf{v}}_{1,1}\rangle.$$

We now have the necessary information to compute the homology groups of K_{Klein} using the Mayer-Vietoris exact sequence associated with the decomposition of K_{Klein} as the union of subcomplexes $\hat{L}$ and $\hat{M}$ as described above. The homomorphisms

$$i_*\colon H_0(\hat{L}\cap\hat{M};\mathbb{Z})\to H_0(\hat{L};\mathbb{Z})\quad\text{and}\quad j_*\colon H_0(\hat{L}\cap\hat{M};\mathbb{Z})\to H_0(\hat{M};\mathbb{Z})$$

induced by the inclusions $i\colon\hat{L}\cap\hat{M}\hookrightarrow\hat{L}$ and $j\colon\hat{L}\cap\hat{M}\hookrightarrow\hat{M}$ are isomorphisms of Abelian groups that satisfy

$$i_*([\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{L}\cap\hat{M}})=[\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{L}}=[\langle\hat{\mathbf{v}}_{0,0}\rangle]_{\hat{L}}\quad\text{and}\quad j_*(\langle[\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{L}\cap\hat{M}})=[\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{M}}.$$

Next we note that the homology group $H_1(\hat{L}\cap\hat{M};\mathbb{Z})$ is generated by $[z_3]_{\hat{L}\cap\hat{M}}$, the homology group $H_1(\hat{L};\mathbb{Z})$ is isomorphic to $\mathbb{Z}\oplus\mathbf{Z}$ and is freely generated by $[z_1]_{\hat{L}}$ and $[z_2]_{\hat{L}}$, where

$$\begin{aligned}z_1&=\langle\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{1,0}\rangle+\langle\hat{\mathbf{v}}_{1,0}\,\hat{\mathbf{v}}_{2,0}\rangle+\langle\hat{\mathbf{v}}_{2,0}\,\hat{\mathbf{v}}_{0,0}\rangle\\z_2&=\langle\hat{\mathbf{v}}_{0,0}\,\hat{\mathbf{v}}_{0,1}\rangle+\langle\hat{\mathbf{v}}_{0,1}\,\hat{\mathbf{v}}_{0,2}\rangle+\langle\hat{\mathbf{v}}_{0,2}\,\hat{\mathbf{v}}_{0,0}\rangle,\end{aligned}$$

and moreover the homomorphism $i_*\colon H_1(\hat{L}\cap\hat{M};\mathbb{Z})$ satisfies

$$i_*([z_3]_{\hat{L}\cap\hat{M}})=[z_3]_{\hat{L}}=-2[z_2]_{\hat{L}}.$$

Also

$$H_2(\hat{L};\mathbb{Z})=0,\quad H_2(\hat{M};\mathbb{Z})=0\quad\text{and}\quad H_1(\hat{M};\mathbb{Z})=0.$$

It follows from the exactness of the Mayer-Vietoris sequence that the following sequence of Abelian groups and homomorphisms is exact:—

$$\begin{aligned}0\longrightarrow H_2(K_{\mathrm{Klein}};\mathbb{Z})&\xrightarrow{\alpha_2}H_1(\hat{L}\cap\hat{M};\mathbb{Z})\xrightarrow{i_*}H_1(\hat{L};\mathbb{Z})\xrightarrow{u_*}H_1(K_{\mathrm{Klein}};\mathbb{Z})\\&\xrightarrow{\alpha_1}H_0(\hat{L}\cap\hat{M};\mathbb{Z})\xrightarrow{k_*}H_0(\hat{L};\mathbb{Z})\oplus H_0(\hat{M};\mathbb{Z}),\end{aligned}$$

where $u_*\colon H_1(\hat{L};\mathbb{Z})\to H_1(K_{\mathrm{Klein}};\mathbb{Z})$ is induced by the inclusion map $u\colon\hat{L}\hookrightarrow K_{\mathrm{Klein}}$, the homomorphims α_2 and α_1 are defined as described in Proposition 10.1, and

$$\begin{aligned}k_*([\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{L}\cap\hat{M}})&=(i_*([\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{L}\cap\hat{M}}),-j_*([\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{L}\cap\hat{M}}))\\&=([\langle\hat{\mathbf{v}}_{0,0}\rangle]_{\hat{L}},-[\langle\hat{\mathbf{v}}_{1,1}\rangle]_{\hat{M}}).\end{aligned}$$

Now $[\langle \hat{\mathbf{v}}_{1,1} \rangle]_{\hat{L}\cap\hat{M}}$ generates $H_0(\hat{L};\mathbb{Z}) \oplus H_0(\hat{M};\mathbb{Z})$, and $k_*([\langle \hat{\mathbf{v}}_{1,1} \rangle]_{\hat{L}\cap\hat{M}}) \neq 0$. It follows that

$$k_*\colon H_0(\hat{L}\cap\hat{M};\mathbb{Z}) \to H_0(\hat{L};\mathbb{Z}) \oplus H_0(\hat{M};\mathbb{Z})$$

is injective. The exactness of the Mayer-Vietoris sequence at $H_0(\hat{L}\cap\hat{M};\mathbb{Z})$ then ensures that the homomorphism $\alpha_1 H_1(K_{\text{Klein}} \to H_0(\hat{L}\cap\hat{M};\mathbb{Z})$ occuring in the Mayer-Vietoris sequence is the zero homomorphism. It then follows from the exactness of the Mayer-Vietoris sequence $H_1(K_{\text{Klein}}$ that the homomorphism

$$u_*\colon H_1(\hat{L};\mathbb{Z}) \to H_1(K_{\text{Klein}};\mathbb{Z})$$

is surjective. Thus the sequence

$$0 \longrightarrow H_2(K_{\text{Klein}};\mathbb{Z}) \xrightarrow{\alpha_2} H_1(\hat{L}\cap\hat{M};\mathbb{Z}) \xrightarrow{i_*} H_1(\hat{L};\mathbb{Z}) \xrightarrow{u_*} H_1(K_{\text{Klein}};\mathbb{Z}) \longrightarrow 0$$

derived from the Mayer-Vietoris sequence is exact. It follows from exactness that

$$H_2(K_{\text{Klein}};\mathbb{Z}) \cong \ker(i_*\colon H_1(\hat{L}\cap\hat{M};\mathbb{Z}) \to H_1(\hat{L};\mathbb{Z}))$$

and

$$H_1(K_{\text{Klein}};\mathbb{Z}) \cong H_1(K_{\text{Klein}};\mathbb{Z})/i_*(H_1(\hat{L}\cap\hat{M};\mathbb{Z})).$$

Let $\varphi\colon H_1(\hat{L};\mathbb{Z}) \to \mathbb{Z}\oplus\mathbb{Z}$ be the isomorphism of Abelian groups defined such that $\varphi(r_1[z_1]_{\hat{L}} + r_2[z_2]_{\hat{L}}) = (r_1, r_2)$ for all $r_1, r_2 \in \mathbb{Z}$. Then

$$\varphi(i_*[z_3]_{\hat{L}\cap\hat{M}}) = \varphi(-2[z_2]_{\hat{L}}) = (0, -2).$$

It follows that $\varphi(i_*(H_1(\hat{L}\cap\hat{M};\mathbb{Z}))) = K$, where K is the subgroup of $\mathbb{Z}\oplus\mathbb{Z}$ such that $K = \{(0, 2r) : r \in \mathbb{Z}\}$. Then

$$H_1(K_{\text{Klein}}) \cong \mathbb{Z}\oplus\mathbb{Z}/K \cong \mathbb{Z}\oplus\mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Also $i_*\colon H_1(\hat{L}\cap\hat{M};\mathbb{Z}) \to H_1(\hat{L};\mathbb{Z}))$ is injective, and therefore $H_2(K_{\text{Klein}};\mathbb{Z}) = 0$.

Now the polyhedron of K_{Klein} is connected. It follows from Theorem 8.6 that $H_0(K_{\text{Klein}};\mathbb{Z}) \cong \mathbb{Z}$. This result can also be deduced from the exactness of the the portion

$$H_0(\hat{L}\cap\hat{M};\mathbb{Z}) \xrightarrow{k_*} H_0(\hat{L};\mathbb{Z}) \oplus H_0(\hat{M};\mathbb{Z}), \xrightarrow{w_*} H_0(K_{\text{Klein}};\mathbb{Z}) \longrightarrow 0$$

of the Mayer-Vietoris sequence.

To summarize, the homology groups of the simplicial complex K_{Klein} triangulating the Klein Bottle are as follows:

$$H_2(K_{\text{Klein}};\mathbb{Z}) = 0, \quad H_1(K_{\text{Klein}};\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}_2, \quad H_0(K_{\text{Klein}};\mathbb{Z}) \cong \mathbb{Z}.$$

10.4 The Homology Groups of a Real Projective Plane

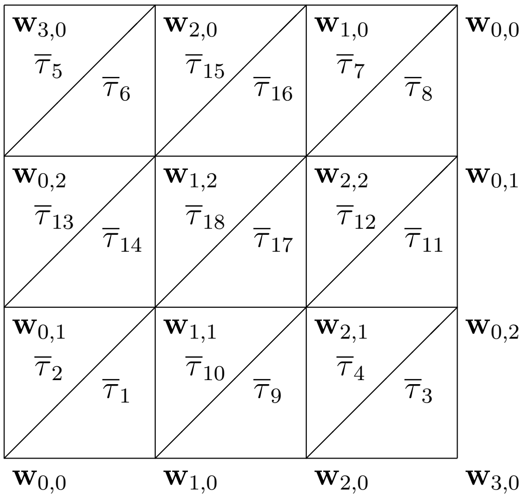
Let K_{Sq} be the simplicial complex triangulating the square $[0,3] \times [0,3]$ defined as in the above discussions of the homology groups of the torus and the Klein Bottle.

There exists a simplicial complex $K_{\mathbb{R}P^2}$ in $\mathbb{R}^4$ with vertices $\mathbf{w}_{i,0}$ for $i = 0,1,2,3$ and $\mathbf{w}_{i,j}$ for $i = 1,2$ and $j = 0,1,2$ whose polyhedron is homeomorphic to a real projective plane, and a simplicial map $\overline{s}\colon K_{\mathrm{Sq}} \to K_{\mathbb{R}P^2}$ mapping the simplicial complex K_{Sq} onto the simplicial complex $K_{\mathbb{R}P^2}$, where this simplicial map is defined such that

$$\begin{array}{rcll} \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{i,j}) & = & \mathbf{w}_{i,j} & \text{for } i = 0,1,2 \text{ and } j = 0,1,2; \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{3,0}) & = & \mathbf{w}_{3,0}; & \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{3,1}) & = & \mathbf{w}_{0,2}; & \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{3,2}) & = & \mathbf{w}_{0,1}; & \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{0,3}) & = & \mathbf{w}_{3,0}; & \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{1,3}) & = & \mathbf{w}_{0,2}; & \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{2,3}) & = & \mathbf{w}_{0,1}; & \\ \overline{s}_{\mathrm{Vert}}(\mathbf{u}_{3,3}) & = & \mathbf{w}_{0,0}. & \end{array}$$

Each triangle of $K_{\mathbb{R}P^2}$ is then the image under this simplicial map of exactly one triangle of K_{Sq}. We do not discuss here the details of how the simplicial complex representing the real projective plane is embedded in $\mathbb{R}^4$.

The following diagram represents the simplicial complex $K_{\mathbb{R}P^2}$. The 18 triangles in this diagram represent the 18 triangles of $K_{\mathbb{R}P^2}$ and are labelled $\overline{\tau}_1, \overline{\tau}_2, \ldots, \overline{\tau}_{18}$. Moreover the vertices of each triangle in the diagram are labelled by the vertices of the corresponding triangle of the simplicial complex $K_{\mathbb{R}P^2}$.



These 18 triangles $\tau_1, \tau_2, \ldots, \tau_{18}$ are determined by their vertices as follows:

$$\overline{\tau}_1 = \mathbf{w}_{0,0}\,\mathbf{w}_{1,0}\,\mathbf{w}_{1,1}, \quad \overline{\tau}_2 = \mathbf{w}_{0,0}\,\mathbf{w}_{1,1}\,\mathbf{w}_{0,1}, \quad \overline{\tau}_3 = \mathbf{w}_{2,0}\,\mathbf{w}_{3,0}\,\mathbf{w}_{0,2},$$

$$\overline{\tau}_4 = \mathbf{w}_{2,0}\,\mathbf{w}_{0,2}\,\mathbf{w}_{2,1}, \quad \overline{\tau}_5 = \mathbf{w}_{0,2}\,\mathbf{w}_{2,0}\,\mathbf{w}_{3,0}, \quad \overline{\tau}_6 = \mathbf{w}_{0,2}\,\mathbf{w}_{1,2}\,\mathbf{w}_{2,0},$$

$$\overline{\tau}_7 = \mathbf{w}_{2,2}\,\mathbf{w}_{0,0}\,\mathbf{w}_{1,0}, \quad \overline{\tau}_8 = \mathbf{w}_{2,2}\,\mathbf{w}_{0,1}\,\mathbf{w}_{0,0}, \quad \overline{\tau}_9 = \mathbf{w}_{1,0}\,\mathbf{w}_{2,0}\,\mathbf{w}_{2,1},$$

$$\overline{\tau}_{10} = \mathbf{w}_{1,0}\,\mathbf{w}_{2,1}\,\mathbf{w}_{1,1}, \quad \overline{\tau}_{11} = \mathbf{w}_{2,1}\,\mathbf{w}_{0,2}\,\mathbf{w}_{0,1}, \quad \overline{\tau}_{12} = \mathbf{w}_{2,1}\,\mathbf{w}_{0,1}\,\mathbf{w}_{2,2},$$

$$\overline{\tau}_{13} = \mathbf{w}_{0,1}\,\mathbf{w}_{1,2}\,\mathbf{w}_{0,2}, \quad \overline{\tau}_{14} = \mathbf{w}_{0,1}\,\mathbf{w}_{1,1}\,\mathbf{w}_{1,2}, \quad \overline{\tau}_{15} = \mathbf{w}_{1,2}\,\mathbf{w}_{1,0}\,\mathbf{w}_{2,0},$$

$$\overline{\tau}_{16} = \mathbf{w}_{1,2}\,\mathbf{w}_{2,2}\,\mathbf{w}_{1,0}, \quad \overline{\tau}_{17} = \mathbf{w}_{1,1}\,\mathbf{w}_{2,1}\,\mathbf{w}_{2,2}, \quad \overline{\tau}_{18} = \mathbf{w}_{1,1}\,\mathbf{w}_{2,2}\,\mathbf{w}_{1,2}.$$

Let $\overline{L}_0$ be the subcomplex of $K_{\mathbb{R}P^2}$ consisting of the six vertices

$$\mathbf{w}_{0,0}, \ \mathbf{w}_{1,0}, \ \mathbf{w}_{2,0}, \ \mathbf{w}_{3,0}, \ \mathbf{w}_{0,2} \text{ and } \mathbf{w}_{0,1}$$

and the six edges

$$\mathbf{w}_{0,0}\,\mathbf{w}_{1,0}, \ \mathbf{w}_{1,0}\,\mathbf{w}_{2,0}, \ \mathbf{w}_{2,0}\,\mathbf{w}_{3,0}, \ \mathbf{w}_{3,0}\,\mathbf{w}_{0,2}, \ \mathbf{w}_{0,2}\,\mathbf{w}_{0,1} \text{ and } \mathbf{w}_{0,1}\,\mathbf{w}_{0,0},$$

and let $\overline{L}$ be the subcomplex of $K_{\mathbb{R}P^2}$ consisting of the vertices and edges of $\overline{L}_0$ together with the 16 triangles $\overline{\tau}_i$ for $0 \leq i \leq 16$ and all the vertices and edges of those triangles. This subcomplex $\overline{L}$ is the subcomplex of $K_{\mathbb{R}P^2}$ obtained from removing from $K_{\mathbb{R}P^2}$ the two triangles $\overline{\tau}_{17}$ and $\overline{\tau}_{18}$ together with the edge $\mathbf{w}_{1,1}\,\mathbf{w}_{2,2}$ of $K_{\mathbb{R}P^2}$ that is common to $\overline{\tau}_{17}$ and $\overline{\tau}_{18}$.

Now the inclusion map $i_0\colon \overline{L}_0 \hookrightarrow L$ induces isomorphisms

$$i_{0*}\colon H_q(\overline{L}_0;\mathbb{Z}) \to H_q(\overline{L};\mathbb{Z})$$

of homology groups for all non-negative integers q. The justification for this corresponds to the justification of the corresponding results in the preceding discussions of the homology of the torus and the Klein Bottle. The subcomplex $\overline{L}$ is obtained $\overline{L}_0$ by the successive addition of 16 triangles together with their vertices and edges. At each stage the intersection of the triangle to be added with the polygon of the subcomplex built up prior to the addition of the triangle under consideration is either a single edge of the added triangle or else is the union of two edges of the added triangle. It then follows from applications of Lemma 7.4 and Lemma 7.5 that the addition of new triangles in the specified sequence does not change homology groups, and therefore the inclusion of $\overline{L}_0$ in $\overline{L}$ induces isomorphisms of homology groups.

Let z_0 be the 1-cycle of $\overline{L}_0$ with integer coefficients defined such that

$$\begin{aligned} z_0 \;&=\; \langle \mathbf{w}_{0,0}\,\mathbf{w}_{1,0}\rangle + \langle \mathbf{w}_{1,0}\,\mathbf{w}_{2,0}\rangle + \langle \mathbf{w}_{2,0}\,\mathbf{w}_{3,0}\rangle \\ &\qquad + \langle \mathbf{w}_{3,0}\,\mathbf{w}_{0,2}\rangle + \langle \mathbf{w}_{0,2}\,\mathbf{w}_{0,1}\rangle + \langle \mathbf{w}_{0,1}\,\mathbf{w}_{0,0}\rangle. \end{aligned}$$

A simple calculation shows that $Z_2(\overline{L}_0;\mathbb{Z}) \cong \mathbb{Z}$, and moreover, given any 1-cycle z of $\overline{L}_0$, there exist a uniquely-determined integer r such that $z = rz_0$. It follows that, given any 1-cycle z of $\overline{L}$, there exist a uniquely-determined integer r such that $[z]_{\overline{L}} = r[z_0]_{\overline{L}}$, where $[z]_{\overline{L}}$ and $[z_0]_{\overline{L}}$ denote the homology classes of the 1-cycles z and z_0 in $H_1(\overline{L};\mathbb{Z})$. In consequence, given any 1-cycle z of $\overline{L}$, there exist a uniquely-determined integer r such that $z - rz_0 \in B_1(\overline{L};\mathbb{Z})$.

Let

$$z_3 = \langle \mathbf{w}_{1,1}\,\mathbf{w}_{1,2}\rangle + \langle \mathbf{w}_{1,2}\,\mathbf{w}_{2,2}\rangle + \langle \mathbf{w}_{2,2}\,\mathbf{w}_{2,1}\rangle + \langle \mathbf{w}_{2,1}\,\mathbf{w}_{1,1}\rangle.$$

Then $[z_3]_L = 2[z_0]_L$. Indeed each triangle $\overline{\tau}_i$ determines a corresponding generator $\overline{\gamma}_i$ of $C_2(\overline{L};\mathbb{Z})$ for $i = 1, 2, \ldots, 16$ that is determined by an anti-clockwise ordering of the vertices of $\overline{\tau}_i$, so that

$$\overline{\gamma}_1 = \langle \mathbf{w}_{0,0}\,\mathbf{w}_{1,0}\,\mathbf{w}_{1,1}\rangle, \quad \overline{\gamma}_2 = \langle \mathbf{w}_{0,0}\,\mathbf{w}_{1,1}\,\mathbf{w}_{0,1}\rangle, \quad \overline{\gamma}_3 = \langle \mathbf{w}_{2,0}\,\mathbf{w}_{3,0}\,\mathbf{w}_{0,2}\rangle \text{ etc.},$$

and direct computation shows that if $c \in C_2(\overline{L};\mathbb{Z})$ is the 2-chain of $\overline{L}$ defined such that

$$c = \overline{\gamma}_1 + \overline{\gamma}_2 + \cdots + \overline{\gamma}_{16},$$

then $\partial_2 c = 2z_0 - z_3$. Indeed terms corresponding to the edges

$$\mathbf{w}_{0,0}\,\mathbf{w}_{1,1}, \quad \mathbf{w}_{1,0}\,\mathbf{w}_{1,1}, \quad \mathbf{w}_{1,0}\,\mathbf{w}_{2,1}, \quad \mathbf{w}_{2,0}\,\mathbf{w}_{2,1}, \quad \mathbf{w}_{0,2}\,\mathbf{w}_{0,2}, \quad \mathbf{w}_{2,1}\,\mathbf{w}_{0,2},$$

$$\mathbf{w}_{2,1}\,\mathbf{w}_{0,1},\quad \mathbf{w}_{2,2}\,\mathbf{w}_{0,1},\quad \mathbf{w}_{2,2}\,\mathbf{w}_{0,0},\quad \mathbf{w}_{2,2}\,\mathbf{w}_{2,0},\quad \mathbf{w}_{2,2}\,\mathbf{w}_{2,0},\quad \mathbf{w}_{1,2}\,\mathbf{w}_{2,0},$$
$$\mathbf{w}_{1,2}\,\mathbf{w}_{1,0},\quad \mathbf{w}_{0,2}\,\mathbf{w}_{1,0},\quad \mathbf{w}_{0,2}\,\mathbf{w}_{1,2},\quad \mathbf{w}_{0,1}\,\mathbf{w}_{1,2}\quad \text{and}\quad \mathbf{w}_{0,1}\,\mathbf{w}_{1,1}$$

cancel off in pairs, with the result that

$$\begin{aligned}\partial_2 c &= \langle \mathbf{w}_{0,0}\,\mathbf{w}_{1,0}\rangle + \langle \mathbf{w}_{1,0}\,\mathbf{w}_{2,0}\rangle + \langle \mathbf{w}_{2,0}\,\mathbf{w}_{3,0}\rangle \\ &\qquad + \langle \mathbf{w}_{3,0}\,\mathbf{w}_{0,2}\rangle + \langle \mathbf{w}_{0,2}\,\mathbf{w}_{0,1}\rangle + \langle \mathbf{w}_{0,1}\,\mathbf{w}_{0,0}\rangle \\ &\qquad + \langle \mathbf{w}_{0,0}\,\mathbf{w}_{2,0}\rangle + \langle \mathbf{w}_{2,0}\,\mathbf{w}_{1,0}\rangle + \langle \mathbf{w}_{1,0}\,\mathbf{w}_{3,0}\rangle \\ &\qquad + \langle \mathbf{w}_{3,0}\,\mathbf{w}_{0,2}\rangle + \langle \mathbf{w}_{0,2}\,\mathbf{w}_{0,1}\rangle + \langle \mathbf{w}_{0,1}\,\mathbf{w}_{0,0}\rangle \\ &\qquad - \langle \mathbf{w}_{1,1}\,\mathbf{w}_{1,2}\rangle - \langle \mathbf{w}_{1,2}\,\mathbf{w}_{2,2}\rangle - \langle \mathbf{w}_{2,2}\,\mathbf{w}_{1,2}\rangle - \langle \mathbf{w}_{1,2}\,\mathbf{w}_{1,1}\rangle \\ &= 2z_0 - z_3 \\ &= \; = 2z_0 - z_3\end{aligned}$$

(The contributing edges may be identified by working round the outer boundary of the large square in the diagram above depicting the structure of the simplicial complex $K_{\mathbb{R}P^2}$ in an anticlockwise direction, starting at the bottom left hand corner of the large square, and then subtracting off terms corresponding to the edges of the small inner square.)

It follows from this computation that $[z_3]_{\overline{L}} = 2[z_0]_{\overline{L}}$ in $H_1(\overline{L};\mathbb{Z})$.

The subcomplex $\overline{L}_0$ is connected, and therefore $H_0(\overline{L}_0,\mathbb{Z}) \cong \mathbb{Z}$. Indeed $H_0(\overline{L}_0,\mathbb{Z})$ is generated by $[\langle \mathbf{w}_{0,0}\rangle]_{\overline{L}_0}$. It follows that $H_0(\overline{L};\mathbb{Z}) \cong \mathbb{Z}$, and indeed the homology class $[\langle \mathbf{w}_{i,j}\rangle]$ of any vertex of $K_{\mathbb{R}P^2}$ in $H_0(\overline{L};\mathbb{Z})$ generates $H_0(\overline{L};\mathbb{Z})$.

Let $\overline{M}$ be the subcomplex of K_{Torus} consisting of the union of the two triangles τ_{17} and τ_{18}, together with the vertices and edges of those triangles. Then $\overline{M}$ has 4 vertices, 5 edges and 2 triangles. The vertices of $\overline{M}$ are $\mathbf{w}_{1,1}$ $\mathbf{w}_{2,1}$, $\mathbf{w}_{2,2}$ and $\mathbf{w}_{1,2}$, the edges of $\overline{M}$ are

$$\mathbf{w}_{1,1}\,\mathbf{w}_{2,1},\quad \mathbf{w}_{2,1}\,\mathbf{w}_{2,2},\quad \mathbf{w}_{2,2}\,\mathbf{w}_{1,2},\quad \mathbf{w}_{1,2}\,\mathbf{w}_{1,1}\quad \text{and}\quad \mathbf{w}_{1,1}\,\mathbf{w}_{2,2},$$

and the triangles of $\overline{M}$ are

$$\mathbf{w}_{1,1}\,\mathbf{w}_{2,1}\,\mathbf{w}_{2,2}\quad \text{and}\quad \mathbf{w}_{1,1}\,\mathbf{w}_{2,2}\,\mathbf{w}_{1,2}.$$

Then $H_0(\overline{M},\mathbb{Z}) \cong \mathbb{Z}$, and $H_q(\overline{M},\mathbb{Z}) = 0$ for all integers q satisfying $q > 0$.

The intersection $\overline{L} \cap \overline{M}$ of the subcomplexes $\overline{L}$ and $\overline{M}$ of $K_{\mathbb{R}P^2}$ consists of the four vertices $\mathbf{w}_{1,1}$ $\mathbf{w}_{2,1}$, $\mathbf{w}_{2,2}$ and $\mathbf{w}_{1,2}$ and the four edges

$$\mathbf{w}_{1,1}\,\mathbf{w}_{2,1},\quad \mathbf{w}_{2,1}\,\mathbf{w}_{2,2},\quad \mathbf{w}_{2,2}\,\mathbf{w}_{1,2}\quad \text{and}\quad \mathbf{w}_{1,2}\,\mathbf{w}_{1,1}.$$

Then $H_0(\overline{L}\cap\overline{M};\mathbb{Z}) \cong \mathbb{Z}$ and $H_1(\overline{L}\cap\overline{M};\mathbb{Z}) \cong \mathbb{Z}$, and moreover $H_0(\overline{L}\cap\overline{M};\mathbb{Z})$ is generated by $[\langle \mathbf{v}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}$ and $H_1(\overline{L}\cap\overline{M};\mathbb{Z})$ is generated by $[z_3]_{\overline{L}\cap\overline{M}}$, where

$$z_3 = \langle \mathbf{w}_{1,1}\,\mathbf{w}_{1,2}\rangle + \langle \mathbf{w}_{1,2}\,\mathbf{w}_{2,2}\rangle + \langle \mathbf{w}_{2,2}\,\mathbf{w}_{2,1}\rangle + \langle \mathbf{w}_{2,1}\,\mathbf{w}_{1,1}\rangle.$$

We now have the necessary information to compute the homology groups of $K_{\mathbb{R}P^2}$ using the Mayer-Vietoris exact sequence associated with the decomposition of $K_{\mathbb{R}P^2}$ as the union of subcomplexes $\overline{L}$ and $\overline{M}$ as described above. The homomorphisms

$$i_*\colon H_0(\overline{L}\cap\overline{M};\mathbb{Z}) \to H_0(\overline{L};\mathbb{Z}) \quad\text{and}\quad j_*\colon H_0(\overline{L}\cap\overline{M};\mathbb{Z}) \to H_0(\overline{M};\mathbb{Z})$$

induced by the inclusions $i\colon \overline{L}\cap\overline{M} \hookrightarrow \overline{L}$ and $j\colon \overline{L}\cap\overline{M} \hookrightarrow \overline{M}$ are isomorphisms of Abelian groups that satisfy

$$i_*([\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}) = [\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}} = [\langle \mathbf{w}_{0,0}\rangle]_{\overline{L}} \quad\text{and}\quad j_*(\langle[\mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}) = [\langle \mathbf{w}_{1,1}\rangle]_{\overline{M}}.$$

Next we note that the homology group $H_1(\overline{L}\cap\overline{M};\mathbb{Z})$ is generated by $[z_3]_{\overline{L}\cap\overline{M}}$, the homology group $H_1(\overline{L};\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and is freely generated by $[z_0]_{\overline{L}}$, where

$$\begin{aligned} z_0 &= \langle \mathbf{w}_{0,0}\,\mathbf{w}_{1,0}\rangle + \langle \mathbf{w}_{1,0}\,\mathbf{w}_{2,0}\rangle + \langle \mathbf{w}_{2,0}\,\mathbf{w}_{3,0}\rangle \\ &\qquad + \langle \mathbf{w}_{3,0}\,\mathbf{w}_{0,1}\rangle + \langle \mathbf{w}_{0,1}\,\mathbf{w}_{0,2}\rangle + \langle \mathbf{w}_{0,2}\,\mathbf{w}_{0,0}\rangle, \end{aligned}$$

and moreover the homomorphism $i_*\colon H_1(\overline{L}\cap\overline{M};\mathbb{Z})$ satisfies

$$i_*([z_3]_{\overline{L}\cap\overline{M}}) = [z_3]_{\overline{L}} = 2[z_0]_{\overline{L}}.$$

Also

$$H_2(\overline{L};\mathbb{Z}) = 0, \quad H_2(\overline{M};\mathbb{Z}) = 0 \quad\text{and}\quad H_1(\overline{M};\mathbb{Z}) = 0.$$

It follows from the exactness of the Mayer-Vietoris sequence that the following sequence of Abelian groups and homomorphisms is exact:—

$$\begin{aligned} 0 \longrightarrow H_2(K_{\mathbb{R}P^2};\mathbb{Z}) &\xrightarrow{\alpha_2} H_1(\overline{L}\cap\overline{M};\mathbb{Z}) \xrightarrow{i_*} H_1(\overline{L};\mathbb{Z}) \xrightarrow{u_*} H_1(K_{\mathbb{R}P^2};\mathbb{Z}) \\ &\xrightarrow{\alpha_1} H_0(\overline{L}\cap\overline{M};\mathbb{Z}) \xrightarrow{k_*} H_0(\overline{L};\mathbb{Z}) \oplus H_0(\overline{M};\mathbb{Z}), \end{aligned}$$

where $u_*\colon H_1(\overline{L};\mathbb{Z}) \to H_1(K_{\mathbb{R}P^2};\mathbb{Z})$ is induced by the inclusion map $u\colon \overline{L} \hookrightarrow K_{\mathbb{R}P^2}$, the homomorphims α_2 and α_1 are defined as described in Proposition 10.1, and

$$\begin{aligned} k_*([\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}) &= (i_*([\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}), -j_*([\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}})) \\ &= ([\langle \mathbf{w}_{0,0}\rangle]_{\overline{L}}, -[\langle \mathbf{w}_{1,1}\rangle]_{\overline{M}}). \end{aligned}$$

Now $[\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}$ generates $H_0(\overline{L};\mathbb{Z}) \oplus H_0(\overline{M};\mathbb{Z})$, and $k_*([\langle \mathbf{w}_{1,1}\rangle]_{\overline{L}\cap\overline{M}}) \neq 0$. It follows that

$$k_*\colon H_0(\overline{L}\cap\overline{M};\mathbb{Z}) \to H_0(\overline{L};\mathbb{Z}) \oplus H_0(\overline{M};\mathbb{Z})$$

is injective. The exactness of the Mayer-Vietoris sequence at $H_0(\overline{L} \cap \overline{M}; \mathbb{Z})$ then ensures that the homomorphism $\alpha_1 H_1(K_{\mathbb{R}P^2} \to H_0(\overline{L} \cap \overline{M}; \mathbb{Z})$ occuring in the Mayer-Vietoris sequence is the zero homomorphism. It then follows from the exactness of the Mayer-Vietoris sequence $H_1(K_{\mathbb{R}P^2}$ that the homomorphism

$$u_*\colon H_1(\overline{L}; \mathbb{Z}) \to H_1(K_{\mathbb{R}P^2}; \mathbb{Z})$$

is surjective. Thus the sequence

$$0 \longrightarrow H_2(K_{\mathbb{R}P^2}; \mathbb{Z}) \xrightarrow{\alpha_2} H_1(\overline{L} \cap \overline{M}; \mathbb{Z}) \xrightarrow{i_*} H_1(\overline{L}; \mathbb{Z}) \xrightarrow{u_*} H_1(K_{\mathbb{R}P^2}; \mathbb{Z}) \longrightarrow 0$$

derived from the Mayer-Vietoris sequence is exact. It follows from exactness that

$$H_2(K_{\mathbb{R}P^2}; \mathbb{Z}) \cong \ker(i_*\colon H_1(\overline{L} \cap \overline{M}; \mathbb{Z}) \to H_1(\overline{L}; \mathbb{Z}))$$

and

$$H_1(K_{\mathbb{R}P^2}; \mathbb{Z}) \cong H_1(K_{\mathbb{R}P^2}; \mathbb{Z})/i_*(H_1(\overline{L} \cap \overline{M}; \mathbb{Z})).$$

Now $H_1(K_{\mathbb{R}P^2}; \mathbb{Z})$ is generated by $[z_0]_{\overline{L}}$, $H_1(\overline{L} \cap \overline{M}; \mathbb{Z})$ is generated by $[z_3]_{\overline{L} \cap \overline{M}}$ and $i_*([z_3]_{\overline{L} \cap \overline{M}}) = 2[z_0]_{\overline{L}}$. It follows that

$$H_1(K_{\mathbb{R}P^2}) \cong \mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Also $i_*\colon H_1(\overline{L} \cap \overline{M}; \mathbb{Z}) \to H_1(\overline{L}; \mathbb{Z}))$ is injective, and therefore $H_2(K_{\mathbb{R}P^2}; \mathbb{Z}) = 0$.

Now the polyhedron of $K_{\mathbb{R}P^2}$ is connected. It follows from Theorem 8.6 that $H_0(K_{\mathbb{R}P^2}; \mathbb{Z}) \cong \mathbb{Z}$. This result can also be deduced from the exactness of the the portion

$$H_0(\overline{L} \cap \overline{M}; \mathbb{Z}) \xrightarrow{k_*} H_0(\overline{L}; \mathbb{Z}) \oplus H_0(\overline{M}; \mathbb{Z}), \xrightarrow{w_*} H_0(K_{\mathbb{R}P^2}; \mathbb{Z}) \longrightarrow 0$$

of the Mayer-Vietoris sequence.

To summarize, the homology groups of the simplicial complex $K_{\mathbb{R}P^2}$ triangulating the real projective plane are as follows:

$$H_2(K_{\mathbb{R}P^2}; \mathbb{Z}) = 0, \quad H_1(K_{\mathbb{R}P^2}; \mathbb{Z}) \cong \mathbb{Z}_2, \quad H_0(K_{\mathbb{R}P^2}; \mathbb{Z}) \cong \mathbb{Z}.$$