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Contents

1 Rings and Modules **1**

1.1 Rings and Fields 1
1.2 Left Modules 2
1.3 Submodules and Quotient Modules 3
1.4 Homomorphisms of Left Modules 5
1.5 Direct Sums of Left Modules 7

2 Free Modules **9**

2.1 Linear Independence 9
2.2 Free Generators 9
2.3 The Free Module on a Given Set 11
2.4 The Free Module on a Finite Set 16

3 Simplicial Complexes **20**

3.1 Geometrical Independence 20
3.2 Simplices 21
3.3 Barycentric Coordinates 23
3.4 Simplicial Complexes in Euclidean Spaces 25
3.5 Triangulations 26
3.6 Simplicial Maps 27

4 The Chain Groups of a Simplicial Complex **28**
4.1 Basic Properties of Permutations of a Finite Set 28
4.2 The Chain Groups of a Simplicial Complex 28
4.3 Homomorphisms defined on Chain Groups 31
4.4 Free Bases for Chain Groups 33
4.5 Homomorphisms of Chain Groups induced by Simplicial Maps 36

5 The Homology Groups of a Simplicial Complex **37**
5.1 Orientations on Simplices 37
5.2 Boundary Homomorphisms 40
5.3 The Homology Groups of a Simplicial Complex 45

6 Homology Calculations **47**
6.1 The Homology Groups of an Octahedron 47
6.2 Another Homology Example 52

7 The Homology Groups of Filled Polygons **55**
7.1 The Homology of a Simple Polygonal Chain 55
7.2 The Homology of a Simple Polygon 56
7.3 The Two-Dimensional Homology Group of a Simplicial Complex triangulating a Region of the Plane 59
7.4 Attaching Triangles to Two-Dimensional Simplicial Complexes 59
7.5 Homology of a Planar Region bounded by a Simple Polygon . 64

8 General Theorems concerning the Homology of Simplical Complexes **69**
8.1 The Homology of Cone-Shaped Simplicial Complexes 69
8.2 Simplicial Maps and Induced Homomorphisms 71
8.3 Connectedness and $H_0(K;R)$ 71
8.4 The Homology Groups of the Boundary of a Simplex 74
8.5 The Reduced Homology of a Simplicial Complex 76

9 Introduction to Homological Algebra **78**
9.1 Exact Sequences 78
9.2 Chain Complexes 81

10 The Mayer-Vietoris Exact Sequence **86**
10.1 The Mayer Vietoris Sequence of Homology Groups 86
10.2 The Homology Groups of a Torus 87
10.3 The Homology Groups of a Klein Bottle 93
10.4 The Homology Groups of a Real Projective Plane 99

1 Rings and Modules

1.1 Rings and Fields

Definition A *ring* consists of a set R on which are defined operations of *addition* and *multiplication* that satisfy the following properties:

- the ring is an Abelian group with respect to the operation of addition;
- the operation of multiplication on the ring is associative, and thus $x(yz) = (xy)z$ for all elements x, y and z of the ring.
- the operations of addition and multiplication satisfy the *Distributive Law*, and thus $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all elements x, y and z of the ring.

Let R be a ring. Then R is an Abelian group with respect to the operation of addition, and therefore $x + (y + z) = (x + y) + z$ and $x + y = y + x$ for all $x, y \in R$. Also the ring R contains a unique *zero element* 0_R characterized by the property that $x + 0_R = x$ for all $x \in R$. Moreover given any element x of R, there exists a unique element $-x$ of R for which $x + (-x) = 0_R$. This element $-x$ is the *negative* of the element x. An element x of a ring R is said to be *non-zero* if $x \neq 0_R$.

The operation of *subtraction* in a ring R is defined such that $x - y = x + (-y)$ for all $x, y \in R$, where $-y$ is the unique element of R for which $y + (-y) = 0_R$.

Let R be a ring, and let 0_R be the zero element of R. It is a straightforward exercise to verify from the defining properties of rings that $x0_R = 0_R$, $0_R x = 0_R$, $(-x)y = -(xy)$ and $x(-y) = -(xy)$ for all elements x and y of R.

Definition A subset S of a ring R is said to be a *subring* of R if $0_R \in S$, $a + b \in S$, $-a \in S$ and $ab \in S$ for all $a, b \in S$, where 0_R denotes the zero element of the ring R.

Definition A ring R is said to be *commutative* if $xy = yx$ for all $x, y \in R$.

Not every ring is commutative: an example of a non-commutative ring is provided by the ring of $n \times n$ matrices with real or complex coefficients when $n > 1$.

Definition A ring R is said to be *unital* if it possesses a (necessarily unique) non-zero multiplicative identity element 1_R satisfying $1_R x = x = x1_R$ for all $x \in R$.

Definition A *field* consists of a set on which are defined operations of *addition* and *multiplication* that satisfy the following properties:

- the field is an Abelian group with respect to the operation of addition;
- the non-zero elements of the field constitute an Abelian group with respect to the operation of multiplication;
- the operations of addition and multiplication satisfy the *Distributive Law*, and thus $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$ for all elements x, y and z of the field.

An examination of the relevant definitions shows that a unital commutative ring R is a field if and only if, given any non-zero element x of R, there exists an element x^{-1} of R such that $xx^{-1} = 1_R$. Moreover a ring R is a field if and only if the set of non-zero elements of R is an Abelian group with respect to the operation of multiplication.

1.2 Left Modules

Definition Let R be a unital ring. A set M is said to be a *left module over the ring R* (or *left R-module*) if

(i) given any $x, y \in M$ and $r \in R$, there are well-defined elements $x + y$ and rx of M,

(ii) M is an Abelian group with respect to the operation $+$ of addition,

(iii) the identities

$$r(x+y) = rx + ry, \qquad (r+s)x = rx + sx,$$
$$(rs)x = r(sx), \qquad 1_R x = x$$

are satisfied for all $x, y \in M$ and $r, s \in R$, where 1_R denotes the multiplicative identity element of the ring R.

Let M be a left module over a unital ring R. Then M is an Abelian group with respect to the operation of addition, and therefore $x+(y+z) = (x+y)+z$ and $x+y = y+x$ for all $x, y \in M$. Also the left module M contains a unique *zero element* 0_M characterized by the property that $x+0_M = x$ for all $x \in M$. Moreover given any element x of M, there exists a unique element $-x$ of M for which $x + (-x) = 0_M$. This element $-x$ is the *negative* of the element x. An element x of a left module M is said to be *non-zero* if $x \neq 0_M$.

The operation of *subtraction* in a left module M is defined such that $x - y = x + (-y)$ for all $x, y \in M$, where $-y$ is the unique element of M for which $y + (-y) = 0_M$.

Lemma 1.1 *Let M be a left module over a unital ring R, and let and let 0_R and 0_M be the zero elements of R and M respectively. Then $0_R x = 0_M$, $r0_M = 0_M$ and $(-r)x = r(-x) = -(rx)$ for all $r \in R$ and $x \in M$.*

Proof Let $r \in R$ and $x \in M$. Then

$$rx = (r + 0_R)x = rx + 0_R x.$$

On subtracting rx from both sides of this equation, we find that $0_R x = 0_M$. Similarly

$$rx = r(x + 0_M) = rx + r0_M,$$

and therefore $r0_M = 0_M$. Also

$$(-r)x + rx = ((-r) + r)x = 0_R x = 0_M$$

and

$$r(-x) + rx = r((-x) + x) = r0_M = 0_M,$$

and therefore $(-r)x = r(-x) = -(rx)$, as required. ∎

1.3 Submodules and Quotient Modules

Definition Let R be a unital ring, and let M be a left R-module. A non-empty subset L of M is said to be a *submodule* of M if $x + y \in L$ and $rx \in L$ for all $x, y \in L$ and $r \in R$.

Let M be a left module over a unital ring R, and let L be a submodule of M. Then L contains at least one element x, and therefore contains the zero element 0_M of M, because $0_M = 0_R x$. Thus every submodule of a left module contains the zero element of that module. Also $-x \in L$ for all $x \in L$, because $-x = (-1_R)x$, where 1_R denotes the multiplicative identity element of the unital ring R.

Example A subset L of a ring R is said to be a *left ideal* of R if $0_R \in L$, $-x \in L$, $x + y \in L$ and $rx \in L$ for all $x, y \in L$ and $r \in R$. Any unital ring R may be regarded as a left R-module, where multiplication on the left by elements of R is defined in the obvious fashion using the multiplication operation on the ring R itself. A subset of R is then a submodule of R (when R is regarded as a left module over itself) if and only if this subset is a left ideal of R.

Given any submodule L of the left R-module M, we denote by M/L the set of cosets of L in M. These cosets are the subsets of M that are of the form $L + x$ for some $x \in M$, where

$$L + x = \{l + x : l \in L\}.$$

Let x and y be elements of M. If $y \in L + x$ then $y = l_y + x$ for some $l_y \in L$. But then $x = (-l_y) + y$, and therefore $x \in L + y$. Moreover

$$l + y = l + l_y + x \in L + x$$

and

$$l + x = l + (-l_y) + y \in L + y$$

for all $l \in L$. Thus if $y \in L + x$ then $L + y = L + x$. It follows that $L + x = L + y$ if and only if $x - y \in L$.

Let $x, x', y, y' \in M$ and $r \in R$. Suppose that $L + x = L + x'$ and $L + y = L + y'$. Then $x' - x \in L$ and $y' - y \in L$. But then

$$(x + y) - (x' + y') = (x - x') + (y - y') \in L,$$

because the operation of addition on M is both commutative and associative, and

$$rx - rx' = r(x - x') \in L,$$

and therefore $L + (x + y) = L + (x' + y')$ and $L + rx = L + rx'$. It follows that there is a well-defined operation of addition on the set M/L of cosets of L in M, where

$$(L + x) + (L + y) = L + (x + y)$$

for all $x, y \in M$. This addition operation on M/L is associative and commutative. Also $L+(L+x) = (L+0_M)+(L+x) = L+x$ and $(L+(-x))+(L+x) = L + ((-x) + x) = L + 0_M = L$ for all $x \in M$. It follows that the set M/L of cosets of L in M is an Abelian group with respect to the operation of addition of cosets. We define $r(L + x) = L + rx$ for all $r \in R$. Then

$$\begin{aligned}
r((L + x) + (L + y)) &= r(L + (x + y)) = L + r(x + y) \\
&= L + (rx + ry) = (L + rx) + (L + ry) \\
&= r(L + x) + r(L + y), \\
(r + s)(L + x) &= L + (r + s)x = L + (rx + sx) \\
&= (L + rx) + (L + sx) \\
&= r(L + x) + s(L + x), \\
(rs)(L + x) &= L + (rs)x = L + r(sx) = r(L + sx) \\
&= r(s(L + x)),
\end{aligned}$$

and

$$1_R(L + x) = L + 1_R x = L + x$$

for all $r, s \in R$ and $x, y \in M$. It follows that the set M/L of left cosets of L in M is itself a left module over the unital ring R.

Definition Let M be a left module over a unital ring R, and let L be a submodule of M. The corresponding *quotient module* M/L is the left R-module M/L whose elements are the cosets of L in M, with operations of addition of cosets and left multiplication of cosets by elements of the ring R defined such that

$$(L + x) + (L + y) = L + x + y \quad \text{and} \quad r(L + x) = L + rx$$

for all $x, y \in M$ and $r \in R$.

1.4 Homomorphisms of Left Modules

Definition Let M and N be left modules over some unital ring R. A function $\varphi\colon M \to N$ is said to be a *homomorphism of left R-modules* if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$ and $r \in R$. A homomorphism of R-modules is said to be an isomorphism if it is invertible.

Let M and N be left modules over a unital ring R. A homomorphism $\varphi\colon M \to N$ from M to N is said to be a *monomorphism* if it is injective. A homomorphism $\varphi\colon M \to N$ from M to N is said to be a *epimorphism* if it is surjective. A homomorphism $\varphi\colon M \to N$ from M to N is said to be an *isomorphism* if it is bijective. A homomorphism $\varphi\colon M \to M$ from M to itself is referred to as an *endomorphism* of M. An isomorphism $\varphi\colon M \to M$ from M to itself is referred to as an *automorphism* of M.

Let $\varphi\colon M \to N$ be an isomorphism from M to N. Then the function φ has a well-defined inverse $\varphi^{-1}\colon N \to M$. Let $u, v \in N$, and let $x = \varphi^{-1}(u)$ and $y = \varphi^{-1}(v)$. Then $\varphi(x) = u$ and $\varphi(y) = v$, and therefore

$$\varphi(x + y) = \varphi(x) + \varphi(y) = u + v \quad \text{and} \quad \varphi(rx) = r\varphi(x) = ru.$$

It follows that

$$\varphi^{-1}(u + v) = \varphi^{-1}(u) + \varphi^{-1}(v) \quad \text{and} \quad \varphi^{-1}(ru) = r\varphi^{-1}(u).$$

Thus the inverse $\varphi^{-1}\colon N \to M$ of any left R-module isomorphism $\varphi\colon M \to N$ is itself a left R-module isomorphism.

Lemma 1.2 *Let M and N be left modules over a unital ring R, and let $\varphi: M \to N$ be a left R-module homomorphism from M to N. Then $\varphi(0_M) = 0_N$, where 0_M and 0_N denote the zero elements of the left modules M and N respectively. Moreover $\varphi(-x) = -\varphi(x)$ for all $x \in M$.*

Proof Let $x \in M$. Then

$$\varphi(x) = \varphi(x + 0_M) = \varphi(x) + \varphi(0_M).$$

On subtracting $\varphi(x)$ from both sides of this identity, we find that $0_N = \varphi(0_M)$. It follows that

$$\varphi(x) + \varphi(-x) = \varphi(x + (-x)) = \varphi(0_M) = 0_N,$$

and therefore $\varphi(-x) = -\varphi(x)$, as required. ∎

Definition Let M and N be left modules over some unital ring R, and let $\varphi: M \to N$ be a left R-module homomorphism. The *kernel* $\ker \varphi$ of the homomorphism φ is defined so that

$$\ker \varphi = \{x \in M : \varphi(x) = 0_N\},$$

where 0_N denotes the zero element of the module N.

The kernel $\ker \varphi$ of a left R-module homomorphism $\varphi: M \to N$ is itself a left R-module. Indeed let $x, y \in \ker \varphi$ and $r \in R$. Then

$$\varphi(x + y) = \varphi(x) + \varphi(y) = 0_N + 0_N = 0_N$$

and

$$\varphi(rx) = r\varphi(x) = r0_N = 0_N,$$

and therefore $x + y \in \ker \varphi$ and $rx \in \ker \varphi$.

The *image* or *range* $\varphi(M)$ of a left R-module homomorphism $\varphi: M \to N$ is defined such that

$$\varphi(N) = \{\varphi(x) : x \in M\}.$$

The image of any left R-module homomorphism is itself a left R-module.

Proposition 1.3 *Let M and N be left modules over a unital ring R, let $\varphi: M \to N$ be a left R-module homomorphism from M and N, and let L be a submodule of M. Suppose that $L \subset \ker \varphi$. Then $\varphi: M \to N$ induces a homomorphism $\overline{\varphi}: M/L \to N$ defined on the quotient module M/L, where $\overline{\varphi}(L + x) = \varphi(x)$ for all $x \in M$. This induced homomorphism is injective if and only if $L = \ker \varphi$.*

Proof Let $x, x' \in M$. Then $L + x = L + x'$ if and only if $x' - x \in L$. Also $\varphi(x' - x) = \varphi(x') - \varphi(x)$, and therefore $\varphi(x) = \varphi(x')$ if and only if $x' - x \in \ker\varphi$. But $L \subset \ker\varphi$. It follows that if $L+x = L+x'$ then $\varphi(x) = \varphi(x')$, and therefore there exists a well-defined function $\overline{\varphi}: M/L \to N$ characterized by the property that $\overline{\varphi}(L + x) = \varphi(x)$ for all $x \in M$. The function from M/L to N characterized by this property is uniquely determined. Moreover the function $\overline{\varphi}$ is injective if and only if $L + x = L + x'$ whenever $\varphi(x) = \varphi(x')$. It follows that $\overline{\varphi}: M/L \to N$ is injective if and only if $L = \ker\varphi$.

Let $x, y \in M$. Then

$$\begin{aligned} \overline{\varphi}((L + x) + (L + y)) &= \overline{\varphi}(L + x + y) = \varphi(x + y) = \varphi(x) + \varphi(y) \\ &= \overline{\varphi}(L + x) + \overline{\varphi}(L + y). \end{aligned}$$

Also

$$\overline{\varphi}(r(L + x)) = \overline{\varphi}(L + rx) = \varphi(rx) = r\varphi(x)$$

for all $r \in R$. It follows that $\overline{\varphi}: M/L \to N$ is a homomorphism of left R-modules with the required properties. ∎

The following corollary follows immediately on applying Proposition 1.3.

Corollary 1.4 *Let M and N be left modules over a unital ring R, and let $\varphi: M \to N$ be a left R-module homomorphism from M and N. Then $\varphi(M) \cong M/\ker\varphi$.*

1.5 Direct Sums of Left Modules

Definition Let $M_1, M_2, \ldots, M_k$ be left modules over a unital ring R. The *direct sum* $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ of the modules $M_1, M_2, \ldots, M_k$ is defined to be the set of ordered k-tuples $(x_1, x_2, \ldots, x_k)$, where $x_i \in M_i$ for $i = 1, 2, \ldots, k$. This direct sum is itself a left R-module, where

$$\begin{aligned} (x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) &= (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k), \\ r(x_1, x_2, \ldots, x_k) &= (rx_1, rx_2, \ldots, rx_k) \end{aligned}$$

for all $x_i, y_i \in M_i$ and $r \in R$.

Definition Let R be a unital ring, and let n be a positive integer. We define the left R-module R^n to be the direct sum of n copies of the ring R. The elements of this left R-module R^n are thus represented as n-tuples $(r_1, r_2, \ldots, r_n)$ whose components are elements of the ring R.

Definition Let M be a left module over some unital ring R. Given any subset X of M, the submodule of M generated by the set X is defined to be the intersection of all submodules of M that contain the set X. It is therefore the smallest submodule of M that contains the set X. A left R-module M is said to be *finitely-generated* if it is generated by some finite subset of itself.

Lemma 1.5 *Let M be a left module over some unital ring R. Then the submodule of M generated by some finite subset $\{x_1, x_2, \ldots, x_k\}$ of M consists of all elements of M that are of the form*

$$r_1x_1 + r_2x_2 + \cdots + r_kx_k$$

for some $r_1, r_2, \ldots, r_k \in R$.

Proof The subset of M consisting of all elements of M of this form is clearly a submodule of M. Moreover it is contained in every submodule of M that contains the set $\{x_1, x_2, \ldots, x_k\}$. The result follows. ∎

2 Free Modules

2.1 Linear Independence

Let M be a left module over a unital ring R, and let $b_1, b_2, \ldots, b_k$ be elements of M. A *linear combination* of the elements $b_1, b_2, \ldots, b_k$ with *coefficients* $r_1, r_2, \ldots, r_k$ is an element of M that is represented by means of an expression of the form

$$r_1 b_1 + r_2 b_2 + \cdots + r_k b_k,$$

where $r_1, r_2, \ldots, r_k$ are elements of the ring R.

Definition Let M be a left module over a unital ring R. The elements of a subset X of M are said to be *linearly dependent* if there exist distinct elements $b_1, b_2, \ldots, b_k$ of X (where $b_i \neq b_j$ for $i \neq j$) and elements $r_1, r_2, \ldots, r_k$ of the ring R, not all zero, such that

$$r_1 b_1 + r_2 b_2 + \cdots + r_k b_k = 0_M,$$

where 0_M denotes the zero element of the module M.

The elements of a subset X of M are said to be *linearly independent* over the ring R if they are not linearly dependent over R. Thus the elements of X are linearly independent over R, if and only if, given distinct elements $b_1, b_2, \ldots, b_k$ of X, and given elements $r_1, r_2, \ldots, r_k$ of R satisfying

$$r_1 b_1 + r_2 b_2 + \cdots + r_k b_k = 0,$$

it must necessarily follow that $r_j = 0$ for $j = 1, 2, \ldots, k$.

2.2 Free Generators

Let M be a left module over a unital ring R, and let X be a (finite or infinite) subset of M. The set X generates M as a left R-module if and only if, given any non-zero element m of M, there exist $b_1, b_2, \ldots, b_k \in X$ and $r_1, r_2, \ldots, r_k \in R$ such that

$$m = r_1 b_1 + r_2 b_2 + \cdots + r_k b_k$$

(see Lemma 1.5). In particular, a left module M over a unital ring R is generated by a finite set $\{b_1, b_2, \ldots, b_k\}$ if and only if any element of M can be represented as a linear combination of $b_1, b_2, \ldots, b_k$ with coefficients in the ring R.

A left module over a unital ring is freely generated by the empty set if and only if it is the zero module.

Definition Let M be a left module over a unital ring R, and let X be a subset of M. The left module M is said to be *freely generated* by the set X if the following conditions are satisfied:

(i) the elements of X are linearly independent over the ring R;

(ii) the module M is generated by the subset X.

Definition Let M be a left module over a unital ring R. Elements

$$b_1, b_2, \ldots, b_k$$

of M are said to constitute a *free basis* of M if these elements are distinct, and if the left R-module M is freely generated by the set $\{b_1, b_2, \ldots, b_k\}$.

Example Let K be a field, let V be a finite-dimensional vector space over K, and let $b_1, b_2, \ldots, b_m$ be a basis of V over the field K. Then V is a left K-module, and moreover V is freely generated by the set B, where $B = \{b_1, b_2, \ldots, b_m\}$.

Example The additive group $\mathbb{Z}^3$ whose elements are ordered triples of integers is a left module over the ring $\mathbb{Z}$ of integers. The triples $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ constitute a free basis of $\mathbb{Z}^3$ over the coefficient ring $\mathbb{Z}$.

Definition A module M over a unital ring R is said to be *free* if there exists a free basis for M over R.

Lemma 2.1 *Let M be a left module over an unital ring R. Elements*

$$b_1, b_2, \ldots, b_k$$

of M constitute a free basis of that left module if and only if, given any element m of M, there exist uniquely determined elements $r_1, r_2, \ldots, r_k$ of the ring R such that

$$m = r_1 b_1 + r_2 b_2 + \cdots + r_k b_k.$$

Proof First suppose that $b_1, b_2, \ldots, b_k$ is a list of elements of M with the property that, given any element m of M, there exist uniquely determined elements $r_1, r_2, \ldots, r_k$ of R such that

$$m = r_1 b_1 + r_2 b_2 + \cdots + r_k b_k.$$

Then the elements $b_1, b_2, \ldots, b_k$ generate M. Also the uniqueness of the coefficients $r_1, r_2, \ldots, r_k$ ensures that the zero element 0_M of M cannot be expressed as a linear combination of $b_1, b_2, \ldots, b_k$ unless the coeffients involved are all zero. Therefore these elements are linearly independent and thus constitute a free basis of the left module M.

Conversely suppose that $b_1, b_2, \ldots, b_k$ is a free basis of M. Then any element of M can be expressed as a linear combination of the free basis vectors. We must prove that the coefficients involved are uniquely determined. Let $r_1, r_2, \ldots, r_k$ and $s_1, s_2, \ldots, s_k$ be elements of the coefficient ring R satisfying

$$r_1b_1 + r_2b_2 + \cdots + r_kb_k = s_1b_1 + s_2b_2 + \cdots + s_kb_k.$$

Then

$$(r_1 - s_1)b_1 + (r_2 - s_2)b_2 + \cdots + (r_k - s_k)b_k = 0_M.$$

But then $r_j - s_j = 0$ and thus $r_j = s_j$ for $j = 1, 2, \ldots, n$, since the elements of any free basis are required to be linearly independent. This proves that any element of M can be represented in a unique fashion as a linear combination of the elements of a free basis of M, as required. ∎

Lemma 2.2 *Let M be a left module over a unital ring that is freely generated by elements $b_1, b_2, \ldots, b_n$ of M. Then there is an isomorphism from R^n to M that sends each element $(r_1, r_2, \ldots, r_n)$ of R^n to*

$$r_1b_1 + r_2b_2 + \cdots + r_nb_n.$$

Proof Let the homomorphism $\varphi: R^n \to M$ be defined such that

$$\varphi(r_1, r_2, \ldots, r_n) = r_1b_1 + r_2b_2 + \cdots + r_nb_n$$

for all $r_1, r_2, \ldots, r_n \in R$. Lemma 2.1 then ensures that $\varphi: R^n \to M$ is both surjective and injective. This homomorphism is thus an isomorphism from $R^n \to M$, as required. ∎

2.3 The Free Module on a Given Set

Definition Let X be a set, let R be a unital ring with zero element 0_R and multiplicative identity element 1_R. We say that a function $\sigma: X \to R$ from X to R is *finitely-supported* if

$$\{x \in X : \sigma(x) \neq 0_R\}$$

is a finite subset of X.

Let X be a set, let R be a unital ring with zero element 0_R and multiplicative identity element 1_R, and let $R^{(X)}$ denote the set of finitely-supported functions from X to the ring R. For each $\sigma \in R^{(X)}$, let

$$X_\sigma = \{x \in X : \sigma(x) \neq 0_R\}.$$

If σ and τ are finitely-supported functions from X to R, then so is $\sigma+\tau$, where $(\sigma + \tau)(x) = \sigma(x) + \tau(x)$ for all $x \in X$. Indeed $X_{\sigma+\tau} \subset X_\sigma \cup X_\tau$, and thus if both X_σ and X_τ are finite subsets of X then so is $X_{\sigma+\tau}$. Also $X_{r\sigma} \subset X_\sigma$ for all $r \in R$, and therefore $r\sigma \colon X \to R$ is a finitely-supported function for all $r \in R$. Thus there are well-defined operations of addition and scalar multiplication defined on $R^{(X)}$ defined such that $(\sigma+\tau)(x) = \sigma(x)+\tau(x)$ and $(r\sigma)(x) = r\sigma(x)$ for all $\sigma, \tau \in R^{(X)}$, $r \in R$ and $x \in X$. These operations give $R^{(X)}$ the structure of a left module over the unital ring R. Each element x of the set X determines a corresponding finitely-supported function $\delta_x \colon X \to R$, where

$$\delta_x(x') = \begin{cases} 1_R & \text{if } x' = x; \\ 0_R & \text{if } x' \neq x. \end{cases}$$

Proposition 2.3 *Let X be a set, let R be a unital ring with zero element 0_R and multiplicative identity element 1_R, and let $R^{(X)}$ be the left R-module whose elements are finitely-supported functions from X to R, with operations of addition and scalar multiplication defined such that*

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x) \quad \textit{and} \quad (r\sigma)(x) = r\sigma(x)$$

for all $\sigma, \tau \in R^{(X)}$ $r \in R$ and $x \in X$. Then the left R-module $R^{(X)}$ is freely generated by $(\delta_x : x \in X)$, where $\delta_x \colon X \to R$ is defined for each $x \in X$ so that $\delta_x(x) = 1_R$ and $\delta_x(x') = 0_R$ for all $x' \in X$ satisfing $x' \neq x$.

Proof First we note that each of the functions $\delta_x \colon X \to R$ is a finitely-supported function from X to R and is thus an element of $R^{(X)}$. Let $\sigma \in R^{(X)}$, and let $X_\sigma = \{x \in X : \sigma(x) \neq 0_R\}$. Then X_σ is a finite subset of X. Let $x_1, x_2, \ldots, x_k$ be a list of distinct elements of X that includes all elements of X_σ, and let $r_j = \sigma(x_j)$ for $j = 1, 2, \ldots, k$. Then

$$\sigma = \sum_{j=1}^{k} r_j \delta_{x_j}.$$

Thus the elements $(\delta_x : x \in X)$ generate $R^{(X)}$.

We must show that the generators $(\delta_x : x \in X)$ are linearly independent over the coefficient ring R. Suppose that

$$\sum_{j=1}^{k} r_j \delta_{x_j} = 0_{R^{(X)}},$$

where $x_1, x_2, \ldots, x_k$ are distinct elements of X and $r_1, r_2, \ldots, r_k$ are elements of the coefficient ring R. Then

$$0_R = \left(\sum_{j=1}^{k} r_j \delta_{x_j} \right) (x_i) = \sum_{j=1}^{k} r_j \delta_{x_j}(x_i) = r_i$$

for $i = 1, 2, \ldots, k$, because $\delta_{x_i}(x_i) = 1_R$ and $\delta_{x_j}(x_i) = 0_R$ when $j \neq i$. Thus $\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k}$ are linearly independent whenever $x_1, x_2, \ldots, x_k$ are distinct. It follows that $R^{(X)}$ is freely generated by $(\delta_x : x \in X)$, as required. ∎

Definition Let X be a set, and let R be a unital ring with zero element 0_R and multiplicative identity element 1_R. The *free left R-module on the set X* is defined to be the module $R^{(X)}$ whose elements are represented as finitely-supported functions from X to R, with operations of addition and scalar multiplication defined such that

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x) \quad \text{and} \quad (r\sigma)(x) = r\sigma(x)$$

for all $\sigma, \tau \in R^{(X)}$, $r \in R$ and $x \in X$. The *natural embedding* $\iota_X : X \to R^{(X)}$ of the set X in the R-module $R^{(X)}$ is the injective function that sends each element x of X to the corresponding finitely-supported function $\delta_x : X \to R$ defined so that $\delta_x(x) = 1_R$ and $\delta_x(x') = 0_R$ for all $x' \in X$ satisfying $x' \neq x$.

Proposition 2.4 *Let X be a set, let R be a unital ring, let $R^{(X)}$ denote the free left R-module on set X, and let $\iota_X : X \to R^{(X)}$ denote the natural embedding that maps the set X into the free left R-module $R^{(X)}$. Let N be a left R-module, and let $f : X \to N$ be a function from X to N. Then there exists a uniquely-determined R-module homomorphism $\varphi : R^{(X)} \to N$ such that $f = \varphi \circ \iota_X$.*

Proof Let 0_R and 1_R denote the zero element and multiplicative identity element respectively of the unital ring R. We represent the elements of $R^{(X)}$ as finitely-supported functions from X to R, as in the statement and proof of Proposition 2.3. Then $\iota_X(x) = \delta_x$ for all $x \in X$, where $\delta_x(x) = 1_R$ and $\delta_x(x') = 0_R$ for all $x' \in X$ satisfying $x' \neq x$.

For each element σ of $R^{(X)}$ let $X_\sigma = \{x \in X : \sigma(x) \neq 0_R\}$. We define a function $\varphi\colon R^{(X)} \to N$ so that φ maps the zero element of $R^{(X)}$ to the zero element of N and

$$\varphi(\sigma) = \sum_{x \in X_\sigma} \sigma(x) f(x)$$

for all non-zero elements σ of $R^{(X)}$. Moreover

$$\varphi(\sigma) = \sum_{x \in Z} \sigma(x) f(x)$$

for all supersets Z of X_σ.

Let σ and τ be elements of $R^{(X)}$, and let r be an element of the coefficient ring R. Then $X_{r\sigma}$ is a subset of X_σ, and X_σ, X_τ and $X_{\sigma+\tau}$ are all subsets of $X_\sigma \cup X_\tau$. It follows that

$$\begin{aligned} \varphi(\sigma + \tau) &= \sum_{x \in X_\sigma \cup X_\tau} (\sigma(x) + (\tau(x)) f(x) \\ &= \sum_{x \in X_\sigma \cup X_\tau} \sigma(x) f(x) + \sum_{x \in X_\sigma \cup X_\tau} \tau(x) f(x) \\ &= \varphi(\sigma) + \varphi(\tau) \end{aligned}$$

and

$$\varphi(r\sigma) = \sum_{x \in X_\sigma} r\sigma(x) = r\varphi(\sigma).$$

It follows that $\varphi\colon R^{(X)} \to N$ is a homomorphism of R-modules.

Now

$$\varphi(\iota_X(x)) = \varphi(\delta_x) = f(x)\delta_x(x) = f(x)$$

for all $x \in X$. It follows that $\varphi \circ \iota_X = f$. Moreover if $\psi\colon R^{(X)} \to N$ is a R-module homomorphism that satisfies $\psi \circ \iota_X = f$, then $\psi(\delta_x) = f(x)$ for all $x \in X$. Let σ be an element of $R^{(X)}$. Then there exist distinct elements $x_1, x_2, \ldots, x_k$ of X and elements $r_1, r_2, \ldots, r_k$ of R such that $\sigma = \sum_{j=1}^{k} r_j \delta_{x_j}$ (see Proposition 2.3). Moreover $r_j = \sigma(x_j)$ for $j = 1, 2, \ldots, k$. But then

$$\psi(\sigma) = \psi\left(\sum_{j=1}^{k} r_j \delta_{x_j}\right) = \sum_{j=1}^{k} r_j \psi(\delta_{x_j}) = \sum_{j=1}^{k} r_j f(x_j) = \varphi(\sigma)$$

for all $\sigma\colon R^{(X)}$, and therefore $\psi = \varphi$. Thus $\varphi\colon R^{(X)} \to N$ is the unique R-module homomorphism satisfying $\varphi \circ \iota_X = f$, as required. ∎

Corollary 2.5 *Let R be a unital ring, and let M be a left R-module that is freely generated by X, where $X \subset M$. Let $R^{(X)}$ be the free left R-module on the set X, and let $\iota_X\colon X \to R^{(X)}$ be the natural embedding that maps the set X into the free left R-module $R^{(X)}$. Then there exists a uniquely-determined R-module isomorphism $\nu\colon R^{(X)} \to M$ such that $\nu(\iota_X(x)) = x$ for all $x \in X$.*

Proof Let $e\colon X \to M$ be the inclusion function from X to M defined such that $e(x) = x$ for all $x \in X$. It follows from Proposition 2.4 that there exists a uniquely-determined R-module homomorphism $\nu\colon R^{(X)} \to M$ such that $e = \nu \circ \iota_X$. Now $x \in \nu(R^{(X)})$ for all $x \in X$, because $x = e(x) = \nu(\iota_X(x))$. Moreover the module M is generated by the subset X of M. It follows that the homomorphism $\nu\colon R^{(X)} \to M$ is surjective.

Let $\sigma \in \ker \nu$. Then there is some finite list $x_1, x_2, \ldots, x_k$ of distinct elements of X that includes all elements of X at which the finitely-supported function σ has a non-zero value. Then

$$\begin{aligned} 0_M &= \nu(\sigma) = \nu\left(\sum_{x \in X_\sigma} \sigma(x)\delta_x\right) = \sum_{x \in X_\sigma} \sigma(x)\nu(\delta_x) = \sum_{x \in X_\sigma} \sigma(x)x \\ &= \sum_{j=1}^{k} \sigma(x_i)x_i. \end{aligned}$$

But the elements $x_1, x_2, \ldots, x_k$ are linearly independent over R, because M is freely generated by X. It follows that $\sigma(x_j) = 0_R$ for $j = 1, 2, \ldots, k$, and therefore σ is the zero element of the R-module $R^{(X)}$. Thus proves that $\nu\colon R^{(X)} \to M$ is injective.

We have now shown that the homomorphism $\nu\colon R^{(X)} \to M$ is both surjective and injective. It follows that this homomorphism is an isomorphism, as required. ∎

Proposition 2.4 establishes the universal property satisfied by the free module $R^{(X)}$ on a given set X: given any left R-module N, and given any function $f\colon X \to N$, there exists a unique homomorphism $\varphi\colon R^{(X)} \to N$ of left R-modules that satisfies $\varphi \circ \iota_X = f$, where $\iota_X\colon X \to R^{(X)}$ denotes the natural embedding mapping the set X into the free module $R^{(X)}$.

Corollary 2.6 *Let M be a free left module over a unital ring R, and let X be a subset of M that freely generates M. Then, given any left R-module N, and given any function $f\colon X \to N$ from X to N, there exists a unique left R-module homomorphism $\varphi\colon M \to N$ such that $\varphi|X = f$.*

Proof Let $R^{(X)}$ be the free module on the set X, and let $\iota_X: X \to R^{(X)}$ be the natural embedding from X to $R^{(X)}$. The inclusion function $i: X \to M$ then induces an isomorphism $\nu: R^{(X)} \to M$ with the property that $\nu \circ \iota_X = i$. Also the function $f: X \to N$ induces a homomorphism $\psi: R^{(X)} \to N$ with the property that $\psi \circ \iota_X = f$ (Proposition 2.4). Let $\varphi: M \to N$ be defined such that $\varphi = \psi \circ \nu^{-1}$. Then $\psi = \varphi \circ \nu$, and therefore $\varphi \circ i = \varphi \circ \nu \circ \iota_X = \psi \circ \iota_X = f$.

Now let $\varphi': M \to N$ be a homomorphism that satisfies $\varphi' \circ i = f$. Then

$$\varphi' \circ \nu \circ \iota_X = \varphi' \circ i = f = \psi \circ \iota_X.$$

But $\psi: R^{(X)} \to N$ is the unique homomorphism from $R^{(X)}$ to N satisfying $\psi \circ \iota_X = f$ (Proposition 2.4). It follows that $\varphi' \circ \nu = \psi$, and therefore $\varphi' = \varphi$. Thus the homomorphism $\varphi: M \to N$ is uniquely determined by the requirement that $\varphi|X = f$. ∎

Let R be a unital ring with zero element 0_R and multiplicative identity element 1_R, let X be a set, let $R^{(X)}$ be the free left module over R on the set X, and let $\iota_X: X \to R^{(X)}$ be the natural embedding mapping the set X into the left R-module $R^{(X)}$. Let us denote $\iota_X(x) by (x)$ for all $x \in X$. Thus if the elements of the free module $R^{(X)}$ are represented as finitely-supported functions from X to R, then the element (x) of $R^{(X)}$ corresponding to an element x of X is represented by the function $\delta_x: X \to R$ that takes the value 1_R at x and takes the value 0_R throughout $X \setminus \{x\}$. Then $R^{(X)}$ is freely generated by $((x) : x \in X)$. It follows that, given any element σ of $R^{(X)}$, there exist elements $x_1, x_2, \ldots, x_k$ of X and $r_1, r_2, \ldots, r_k$ of R such that

$$\sigma = r_1(x_1) + r_2(x_2) + \cdots + r_k(x_k).$$

Moreover if

$$r_1(x_1) + r_2(x_2) + \cdots + r_k(x_k) = 0_{R^(X)},$$

and if $x_1, x_2, \ldots, x_k$ are distinct, then $r_j = 0_R$ for $j = 1, 2, \ldots, k$ (see Lemma 2.1).

2.4 The Free Module on a Finite Set

Let X be a finite set with n elements, let R be a unital ring with zero element 0_R and multiplicative identity element 1_R, let $R^{(X)}$ be the free R-module over the ring R on the set X, and let $\iota_X: X \to R^{(X)}$ be the natural embedding mapping the set X into the R-module $R^{(X)}$. Let $x_1, x_2, \ldots, x_n$ be a listing of the elements of X, where every element of X occurs exactly once in the list, and let $(x_j) = \iota_X(x_j)$ for $j = 1, 2, \ldots, n$. Then $R^{(X)}$ is freely generated by

$(x_1), (x_2), \dots, (x_n)$. It follows that, given any element σ of $R^{(X)}$, there exist uniquely determined elements $r_1, r_2, \dots, r_n$ of R such that

$$\sigma = r_1(x_1) + r_2(x_2) + \cdots + r_n(x_n).$$

(see Lemma 2.1). It follows that the free left R-module $R^{(X)}$ on the set X is isomorphic to the direct sum R^n of n copies of the coefficient ring R.

Example Let K be a field, let V be a finite-dimensional vector space over K, and let $b_1, b_2, \dots, b_m$ be a basis of V. Then V is a free left K-module that is freely generated by the basis. Then, given any vector space W over K, and given any function $f\colon B \to W$, there is a unique linear transformation $\varphi\colon V \to W$ from V to B that extends f. Moreover

$$\varphi\left(\sum_{j=1}^{m} v_j b_j\right) = \sum_{j=1}^{m} v_j f(b_j)$$

for all $v_1, v_2, \dots, v_m \in K$. This linear transformation is a homomorphism of left modules over the field K.

If the vector space W is finite-dimensional, and if $c_1, c_2, \dots, c_n$ is a basis for W over K, then there exist elements $T_{i,j}$ of K for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ such that

$$f(b_j) = T_{1,j}c_1 + T_{2,j}c_2 + \cdots + T_{n,j}c_n$$

for $j = 1, 2, \dots, m$. Then

$$\varphi\left(\sum_{j=1}^{m} v_j b_j\right) = \sum_{i=1}^{n} w_i c_i,$$

where

$$w_i = \sum_{j=1}^{m} T_{i,j} v_j$$

for $i = 1, 2, \dots, n$. The elements $T_{i,j}$ of the coefficient field K are thus the elements of the $n \times m$ matrix over the field K that represents the linear transformation φ with respect to the basis $b_1, b_2, \dots, b_m$ of V and the basis $c_1, c_2, \dots, c_n$ of W.

Let R be an integral domain, and let M be a free left module over M that is freely generated by some finite subset of M. Then it can be shown that the number of elements in any free basis of M is finite and is independent of the choice of free basis. The *rank* of the free R-module M is defined to be the number of elements in a free basis of M.

Example Abelian groups are left modules over the ring $\mathbb{Z}$ of integers. Let M be a free Abelian group (i.e., a free left Z-module) that is freely generated by a subset of M with exactly n elements. Then $M \cong \mathbb{Z}^n$. Let p be a positive integer, and let $pM = \{pm : m \in M\}$. Then pM is a submodule of M, and the quotient module M/pM is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. This quotient module is a finite Abelian group with p^n elements. Now the number of elements in the quotient group M/pM does not depend in any way on a choice of a free basis for M. It follows that every free basis of M has n elements. This shows that any finitely-generated free Abelian group is isomorphic to $\mathbb{Z}^n$ for exactly one value of n. This non-negative integer n is the rank of the free Abelian group M.

Lemma 2.7 *Let R be an integral domain, let M be a free left R-module of rank m, let N be a free left R-module of rank n, and let $\varphi: M \to N$ be an R-module homomorphism from M to N. Let $b_1, b_2, \ldots, b_m$ be a free basis of M, and let $c_1, c_2, \ldots, c_n$ be a free basis of N. Then there exists an $n \times m$ matrix*

$$\begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n,1} & T_{n,2} & \cdots & T_{n,m} \end{pmatrix}$$

with coefficients $T_{i,j}$ in the coefficient ring R, so that

$$\varphi\left(\sum_{j=1}^{m} r_j b_j\right) = \sum_{i=1}^{n} s_i c_i,$$

for all $r_1, r_2, \ldots, r_m \in R$, where

$$\varphi(b_j) = \sum_{i=1}^{n} T_{i,j} c_i$$

for $j = 1, 2, \ldots, m$, and

$$s_i = \sum_{j=1}^{m} T_{i,j} r_j$$

for $i = 1, 2, \ldots, n$.

Proof The module N is generated by the free basis $c_1, c_2, \ldots, c_n$. Therefore there exists elements $T_{i,j}$ of R for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ such that

$$\varphi(b_j) = \sum_{i=1}^{n} T_{i,j} c_i.$$

Then

$$\varphi\left(\sum_{j=1}^{m} r_j b_j\right) = \sum_{j=1}^{m} r_j \varphi(b_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} T_{i,j} r_j c_i.$$

The result follows. ▮

3 Simplicial Complexes

3.1 Geometrical Independence

Definition Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum_{j=0}^{q} s_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^{q} s_j & = & 0 \end{array} \right.$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

Lemma 3.1 *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension k. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.*

Proof Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$, where $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$s_0 = s_1 = \cdots = s_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $s_0, s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$. Then $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} s_j \mathbf{v}_j = s_0 \mathbf{v}_0 + \sum_{j=1}^{q} s_j \mathbf{v}_j = \sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$s_1 = s_2 = \cdots = s_q = 0.$$

But then $s_0 = 0$ also, because $s_0 = -\sum_{j=1}^{q} s_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required. ∎

It follows from Lemma 3.1 that any set of affinely independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of affinely independent points in $\mathbb{R}^k$, then so does every subset of that set.

3.2 Simplices

Definition A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex.

Example A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

Example A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let λ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned} \lambda &= \{ s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1 \} \\ &= \{ (1-t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1 \}, \end{aligned}$$

and thus λ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

Example A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let τ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{ r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1 \}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1-r}$. Moreover $0 \leq r < 1$ and $0 \leq p \leq 1$. Moreover the above formula determines a point of the 2-simplex τ for each pair of real numbers r and p satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$. Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as p increases from 0 to 1. It follows that τ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Example A 3-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least three is a tetrahedron on that space. Indeed let $\mathbf{x}$ be a point of a 3-simplex σ in $\mathbb{R}^3$ with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then there exist non-negative real numbers s, t, u and v such that

$$\mathbf{x} = s\,\mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c} + v\,\mathbf{d},$$

and $s+t+u+v = 1$. These real numbers s, t, u and v all have values between 0 and 1, and moreover $0 \leq t \leq 1-s$, $0 \leq u \leq 1-s$ and $0 \leq v \leq 1-s$. Suppose that $\mathbf{x} \neq \mathbf{a}$. Then $0 \leq s < 1$ and $\mathbf{x} = s\,\mathbf{a} + (1-s)\mathbf{y}$, where

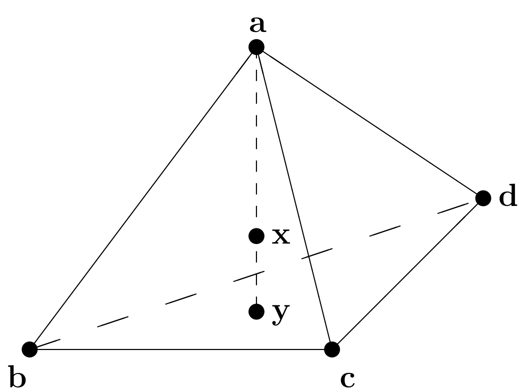
$$\mathbf{y} = \frac{t}{1-s}\,\mathbf{b} + \frac{u}{1-s}\,\mathbf{c} + \frac{v}{1-s}\,\mathbf{d}.$$

Moreover $\mathbf{y}$ is a point of the triangle $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$, because

$$0 \leq \frac{t}{1-s} \leq 1, \quad 0 \leq \frac{u}{1-s} \leq 1, \quad 0 \leq \frac{v}{1-s} \leq 1$$

and

$$\frac{t}{1-s} + \frac{u}{1-s} + \frac{v}{1-s} = 1.$$



It follows that the point $\mathbf{x}$ lies on a line segment with one endpoint at the vertex $\mathbf{a}$ of the 3-simplex and the other at some point $\mathbf{y}$ of the triangle $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$. Thus the 3-simplex σ has the form of a tetrahedron (i.e., it has the form of a pyramid on a triangular base $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ with apex $\mathbf{a}$).

A simplex of dimension q in $\mathbb{R}^k$ determines a subset of $\mathbb{R}^k$ that is a translate of a q-dimensional vector subspace of $\mathbb{R}^k$. Indeed let the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-dimensional simplex σ in $\mathbb{R}^k$. Then these points are affinely independent. It follows from Lemma 3.1 that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent. These vectors therefore span a k-dimensional vector subspace V of $\mathbb{R}^k$. Now, given any point $\mathbf{x}$ of σ, there exist real numbers $t_0, t_1, \ldots, t_q$ such that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$ and $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$. Then

$$\mathbf{x} = \left(\sum_{j=0}^{q} t_j \right) \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows that

$$\sigma = \left\{ \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) : 0 \leq t_j \leq 1 \text{ for } j = 1, 2, \ldots, q \text{ and } \sum_{j=1}^{q} t_j \leq 1 \right\},$$

and therefore $\sigma \subset \mathbf{v_0} + V$. Moreover the q-dimensional vector subspace V of $\mathbb{R}^k$ is the unique q-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between each pair of points belonging to the simplex σ.

3.3 Barycentric Coordinates

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^{q} s_j \mathbf{v}_j = \sum_{j=0}^{q} t_j \mathbf{v}_j$ and $\sum_{j=0}^{q} s_j = \sum_{j=0}^{q} t_j = 1$, then $\sum_{j=0}^{q} (t_j - s_j) \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, and therefore $t_j - s_j = 0$ for $j = 0, 1, \ldots, q$, because the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent.

Lemma 3.2 *Let q be a non-negative integer, let σ be a q-simplex in $\mathbb{R}^m$, and let τ be a q-simplex in $\mathbb{R}^n$, where $m \geq q$ and $n \geq q$. Then σ and τ are homeomorphic.*

Proof Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of τ. The required homeomorphism $h\colon \sigma \to \tau$ is given by

$$h\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \mathbf{w}_j$$

for all $t_0, t_1, \ldots, t_q$ satisfying $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. ∎

A homeomorphism between two q-simplices defined as in the above proof is referred to as a *simplicial homeomorphism*.

It follows from Lemma 3.2 that every q-simplex is homeomorphic to the *standard q-simplex* in $\mathbb{R}^{q+1}$ whose vertices are the points

$$(1, 0, 0, \ldots, 0),\ \ (0, 1, 0, \ldots, 0), \ldots,\ \ (0, 0, 0, \ldots, 1).$$

This standard q-simplex is the subset of $\mathbb{R}^{q+1}$ consisting of those points $(t_0, t_1, \ldots, t_q)$ of $\mathbb{R}^{q+1}$ which satisfy $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$.

Example Consider the triangle σ in $\mathbb{R}^2$ with vertices at $(1, 2)$, $(3, 3)$ and $(4, 5)$. Let t_0, t_1 and t_2 be the barycentric coordinates of a point (x, y) of this triangle. Then t_0, t_1, t_2 are non-negative real numbers, and $t_0 + t_1 + t_2 = 1$. Moreover

$$(x, y) = (1 - t_1 - t_2)(1, 2) + t_1(3, 3) + t_2(4, 5),$$

and thus

$$x = 1 + 2t_1 + 3t_2 \quad \text{and} \quad y = 2 + t_1 + 3t_2.$$

It follows that

$$t_1 = x - y + 1 \quad \text{and} \quad t_2 = \tfrac{1}{3}(x - 1 - 2t_1) = \tfrac{2}{3}y - \tfrac{1}{3}x - 1,$$

and therefore

$$t_0 = 1 - t_1 - t_2 = \tfrac{1}{3}y - \tfrac{2}{3}x + 1.$$

In order to verify these formulae it suffices to note that $(t_0, t_1, t_2) = (1, 0, 0)$ when $(x, y) = (1, 2)$, $(t_0, t_1, t_2) = (0, 1, 0)$ when $(x, y) = (3, 3)$ and $(t_0, t_1, t_2) = (0, 0, 1)$ when $(x, y) = (4, 5)$.

Let the map $h: \sigma \to \mathbb{R}^3$ from σ to $\mathbb{R}^3$ be defined such that

$$h(x, y) = \left(\tfrac{1}{3}y - \tfrac{2}{3}x + 1,\ \ x - y + 1,\ \ \tfrac{2}{3}y - \tfrac{1}{3}x - 1\right).$$

Then the components of this map h are the barycentric coordinate functions on the triangle σ. It follows that h maps this triangle homeomorphically onto the triangle in $\mathbb{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

3.4 Simplicial Complexes in Euclidean Spaces

Definition Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

Definition The *interior* of a simplex σ is defined to be the set consisting of all points of σ that do not belong to any proper face of σ.

Definition A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the *polyhedron* of K. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

Example Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 3.3 *Let K be a simplicial complex, and let X be a topological space. A function $f: |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.*

Proof If a topological space can be expressed as a finite union of closed subsets, then a function is continuous on the whole space if and only if its restriction to each of the closed subsets is continuous on that closed set. The required result is a direct application of this general principle. ∎

We shall denote by $\operatorname{Vert} K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

3.5 Triangulations

Definition A *triangulation* (K, h) of a topological space X consists of a simplicial complex K in some Euclidean space, together with a homeomorphism $h: |K| \to X$ mapping the polyhedron $|K|$ of K onto X.

The polyhedron of a simplicial complex is a compact Hausdorff space. Thus if a topological space admits a triangulation then it must itself be a compact Hausdorff space.

Lemma 3.4 *Let X be a Hausdorff topological space, let K be a simplicial complex, and let $h: |K| \to X$ be a bijection mapping $|K|$ onto X. Suppose that the restriction of h to each simplex of K is continuous on that simplex. Then the map $h: |K| \to X$ is a homeomorphism, and thus (K, h) is a triangulation of X.*

Proof Each simplex of K is a closed subset of $|K|$, and the number of simplices of K is finite. It follows from Lemma 3.3 that $h: |K| \to X$ is continuous. Also the polyhedron $|K|$ of K is a compact topological space. But every continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism. Thus (K, h) is a triangulation of X. ∎

3.6 Simplicial Maps

Definition A *simplicial map* $\varphi: K \to L$ between simplicial complexes K and L is a function $\varphi: \operatorname{Vert} K \to \operatorname{Vert} L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Note that a simplicial map $\varphi: K \to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

A simplicial map $\varphi: K \to L$ also induces in a natural fashion a continuous map $\varphi: |K| \to |L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. The continuity of this map follows immediately from a straightforward application of Lemma 3.3. Note that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

4 The Chain Groups of a Simplicial Complex

4.1 Basic Properties of Permutations of a Finite Set

A *permutation* of a set T is a bijection mapping T onto itself. The set of all permutations of some set T is a group with respect to the operation of composition of permutations. A *transposition* is a permutation of a set T which interchanges two elements of T, leaving the remaining elements of the set fixed. If T is finite, and has more than one element, then any permutation of T can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and j for some j.

Associated to any permutation π of a finite set T is a number ϵ_π, known as the *parity* or *signature* of the permutation, which can take on the values ± 1. If π can be expressed as the product of an even number of transpositions, then $\epsilon_\pi = +1$; if π can be expressed as the product of an odd number of transpositions then $\epsilon_\pi = -1$. The function $\pi \mapsto \epsilon_\pi$ is a homomorphism from the group of permutations of a finite set T to the multiplicative group $\{+1, -1\}$ (i.e., $\epsilon_{\pi\rho} = \epsilon_\pi \epsilon_\rho$ for all permutations π and ρ of the set T). Note in particular that the parity of any transposition is -1.

4.2 The Chain Groups of a Simplicial Complex

Let K be a simplicial complex. For each non-negative integer q, let $W_{q,K}$ denote the set of all ordered $(q+1)$-tuples of vertices of K that span simplices of K. An element of $W_{q,K}$ is thus an ordered $(q+1)$-tuple $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. The vertices in the list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not required to be distinct.

Let R be a unital ring. We refer to this ring in the following discussion as the *coefficient ring*. We denote by $\Delta_q(K; R)$ the free left R-module on the set $W_{q,K}$, and we denote by $\iota_q \colon W_{q,K} \to \Delta_q(K; R)$ the natural embedding that maps the set $W_{q,K}$ bijectively onto a free basis of $\Delta_q(K; R)$. Then, given any element θ of $\Delta_q(K; R)$, there exist uniquely-determined elements r_w of the coefficient ring R for all $w \in W_{q,K}$ such that

$$\theta = \sum_{w \in W_{q,K}} r_w \delta_w,$$

where $\delta_w = \iota_q(w)$ for all $w \in W_{q,K}$.

We now give a formal definition of the qth *chain group* of a simplicial complex K with coefficients in a unital ring R.

Definition Let K be a simplicial complex, let q be a non-negative integer, and let R be a unital ring. Let $W_{q,K}$ denote the set of ordered $(q+1)$-tuples of vertices of K that span simplices of K, let $\Delta_q(K;R)$ denote the free left R-module on the set $W_{q,K}$, let $\iota_q\colon W_{q,K} \to \Delta_q(K;R)$ denote the natural embedding that maps the set $W_{q,K}$ bijectively onto the corresponding free basis of $\Delta_q(K;R)$, and let $\delta_w = \iota_q(w)$ for all $w \in W_{q,K}$. Let $P_{q,K}$ be the subset of $\Delta_q(K;R)$ consisting of those basis elements $\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$ for which the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct, let $Q_{q,K}$ be the subset of $\Delta_q(K;R)$ consisting of elements of the form

$$\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are vertices of K that span some simplex of K and π is a permutation of $\{0, 1, \ldots, q\}$ with parity ϵ_π, and let $\Delta_q^0(K;R)$ denote the submodule of $\Delta_q(K;R)$ generated by $P_{q,K} \cup Q_{q,K}$. The qth *chain group* $C_q(K;R)$ with coefficients in the unital ring R is then defined to be the quotient module $\Delta_q(K;R)/\Delta_q^0(K;R)$.

We now discuss in more detail the essential features of this definition of the chain groups of a simplicial complex. We have defined the chain group $C_q(K;R)$ to be the quotient module $\Delta_q(K;R)/\Delta_q^0(K;R)$. It follows that each element of $C_q(K;R)$ can be represented in the form $\Delta_q^0(K;R) + \theta$ for some $\theta \in \Delta_q(K;R)$. Moreover elements θ and θ' satisfy $\Delta_q^0(K;R) + \theta = \Delta_q^0(K;R) + \theta'$ if and only if $\theta - \theta' \in \Delta_q^0(K;R)$. Now the algebraic operations on $\Delta_q(K;R)/\Delta_q^0(K;R)$ are defined so that

$$(\Delta_q^0(K;R) + \theta) + (\Delta_q^0(K;R) + \theta') = \Delta_q^0(K;R) + \theta + \theta'$$

and

$$r(\Delta_q^0(K;R) + \theta) = \Delta_q^0(K;R) + r\theta$$

for all $\theta, \theta' \in \Delta^q(K;R)$ and $r \in R$. It follows that there is a well-defined quotient homomorphism $\rho_q\colon \Delta_q(K;R) \to C_q(K;R)$, where

$$\rho_q(\theta) = \Delta_q^0(K;R) + \theta$$

for all $\theta \in \Delta_q(K;R)$. This quotient homomorphism is surjective, and $\ker \rho_q = \Delta_q^0(K;R)$.

We now establish some notation for representing elements of the qth chain group.

Given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span some simplex of K, we denote by

$$\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle,$$

the element of $C_q(K;R)$ defined so that

$$\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = \rho_q \left(\delta_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)} \right)$$

where $\rho_q \colon \Delta_q(K;R) \to C_q(K;R)$ is the quotient homomorphism discussed above.

A list consisting of $(q+1)$ vertices of K that span some simplex of K determines an element w of the set $W_{q,K}$, which in turn determines a corresponding generator δ_w of $\Delta_q(K;R)$. We denote by $\langle w \rangle$ the image of δ_w under the quotient homomorphism $\rho_q \colon \Delta_q(K;R) \to C_q(K;R)$, so that $\langle w \rangle = \rho_q(\delta_w)$ for all $w \in W_{q,K}$. If

$$w = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q),$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are vertices of K that span some simplex of K, then

$$\langle w \rangle = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle.$$

Let c be an element of $C_q(K;R)$. Then $c = \rho_q(\theta)$ for some element θ of $\Delta_q(K;R)$. This element θ may be represented (uniquely) as a linear combination of elements of the free basis $(\delta_w : w \in W_{q,K})$. Therefore there exist elements r_w of the coefficient ring R for all $w \in W_{q,K}$ such that

$$\theta = \sum_{w \in W_{q,K}} r_w \delta_w.$$

But then

$$c = \rho_q(\theta) = \sum_{w \in W_{q,K}} r_w \langle w \rangle.$$

Thus any element of $C_q(K;R)$ can be represented as a linear combination of generator elements $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$, where each of these generator elements corresponds to some ordered list consisting of $q+1$ vertices of K that span some simplex of K. However these generator elements are not linearly independent. The following lemma establishes the basic identities used in performing calculations with linear combinations of these generator elements.

Lemma 4.1 *Let K be a simplicial complex, let R be a unital ring, let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices of K that span a simplex of K. Then the following identities are satisfied within the R-module $C_q(K;R)$:—*

(i) $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = 0$ *if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct;*

(ii) $\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle = \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ *for any permutation π of the set $\{0, 1, \ldots, q\}$.*

Proof If the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct then $\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$ belongs to the kernel $\Delta^0(K;R)$ of the quotient homomorphism $\rho_q\colon \Delta_q(K;R) \to \Delta_q(K;R)$, and therefore

$$\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = \rho_q(\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}) = 0.$$

This proves (i).

Now suppose that the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K span a simplex of K but are not necessarily distinct. Let π be a permutation of the set $\{0, 1, \ldots, q\}$. Then

$$\begin{aligned}
&\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle - \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \\
&\qquad = \rho_q(\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})}) - \epsilon_\pi \rho_q(\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle)}) \\
&\qquad = \rho_q\left(\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle)}\right) \\
&\qquad = 0,
\end{aligned}$$

because the element

$$\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$$

of $\Delta_q(K;R)$ is one of the generators of the kernel $\Delta_q^0(K;R)$ of the quotient homomorphism $\rho_q\colon \Delta_q(K;R) \to C_q(K;R)$ specified in the definition of $\Delta_q^0(K;R)$. This proves (ii). ∎

4.3 Homomorphisms defined on Chain Groups

Lemma 4.2 *Let K be a simplicial complex, let R be a unital ring, and let N be a left module over R with zero element 0_N. Let $W_{q,K}$ denote the set consisting of all $(q+1)$-tuples of vertices of K that span simplices of K, and let $f\colon W_{q,K} \to N$ be a function from $W_{q,K}$ to N. Suppose that this function f has the following properties:—*

- *$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0_N$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct;*
- *$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign on interchanging any two adjacent vertices $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$.*

Then there exists a unique R-module homomorphism $\varphi\colon C_q(K;R) \to N$ characterized by the property that

$$\varphi(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Proof Let $\iota_q\colon W_{q,K} \to \Delta_q(K;R)$ denote the natural embedding of the set $W_{q,K}$ into the free left R-module $\Delta_q(K;R)$, and let $\delta_w = \iota_q(w)$ for all $w \in W_{q,K}$. Then $(\delta_w : w \in W_{q,k})$ is a free basis of $\Delta_q(K)$. The function $f\colon W_{q,K} \to N$ then induces a homomorphism $\psi\colon \Delta_q(K;R) \to N$ for which $f = \psi \circ \iota$ (see Proposition 2.4). Then $\psi(\delta_w) = f(w)$ for all $w \in W_{q,K}$.

Let $\rho_q\colon \Delta_q(K;R) \to C_q(K;R)$ denote the quotient homomorphism from $\Delta_q(K;R)$ to $C_q(K;R)$ whose kernel is the submodule $\Delta_q^0(K;R)$ of $\Delta_q(K;R)$ generated by the set $P_{q,K} \cup Q_{q,K}$, where $P_{q,K}$ is the subset of $\Delta_q(K;R)$ consisting of those basis elements $\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$ for which the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct, and $Q_{q,K}$ is the subset of $\Delta_q(K;R)$ consisting of elements of the form

$$\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are vertices of K that span some simplex of K and π is a permutation of $\{0, 1, \ldots, q\}$ with parity ϵ_π. The requirement that $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0_N$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct ensures that

$$\psi(\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}) = 0_N$$

unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. It follows that $P_{q,K} \subset \ker \psi$. Also the requirement that $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign on interchanging any two adjacent vertices $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ ensures that

$$\begin{aligned}\psi(\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})}) &= f(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) = \epsilon_\pi f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \\ &= \epsilon_\pi \psi(\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)})\end{aligned}$$

for all $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in W_{q,K}$ and for all permutations π of $\{0, 1, \ldots, q\}$. It follows from that

$$\delta_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)} \in \ker \psi$$

for all $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in W_{q,K}$ and for all permutations π of $\{0, 1, \ldots, q\}$, and thus $Q_{q,K} \subset \ker \psi$.

We have now shown that $P_{q,K} \subset \psi$ and $Q_{q,K} \subset \psi$. Now the kernel $\Delta_q^0(K;R)$ of the quotient homomorphism $\rho_q\colon \Delta_q(K;R) \to C_q(K;R)$ is generated by $P_{q,K} \cup Q_{q,K}$. It follows that $\Delta_q^0(K;R) \subset \ker \psi$.

Now $C_q(K;R) = \Delta_q(K;R)/\Delta_q^0(K;R)$. Therefore the R-module homomorphism $\psi\colon \Delta_q(K;R) \to N$ induces a well-defined R-module homomorphism $\varphi\colon C_q(K;R) \to N$ characterized by the property that $\varphi(\rho_q(\theta)) = \psi(\theta)$ for all $\theta \in C_q(K;R)$. Then

$$\begin{aligned}\varphi(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) &= \varphi(\rho_q(\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)})) = \psi(\delta_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}) \\ &= f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K, as required. ∎

4.4 Free Bases for Chain Groups

The chain groups $C_q(K; R)$ have been defined as quotients of free left R-modules. We shall show that they are themselves free left R-modules, and that each chain group $C_q(K; R)$ has a free basis whose elements are in one-to-one correspondence with the q-simplices of the simplicial complex K. However, in order to construct such a free basis, it is necessary to choose an ordering of the vertices of each q-simplex of K.

Lemma 4.3 *Let K be a simplicial complex, let q be a non-negative integer, let R be a unital ring, let $W_{q,K}$ denote the set consisting of all $(q+1)$-tuples of vertices of K that span simplices of K, let $\Delta_q(K; R)$ be the free left R-module on the set $W_{q,K}$, and let $\iota_q\colon W_{q,K} \to \Delta_q(K; R)$ denote the natural embedding of $W_{q,K}$ into $\Delta_q(K; R)$. For each q-simplex σ of K let $\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma$ be a listing of the vertices of σ in some chosen order. Then there is a well-defined homomorphism*

$$\lambda_q\colon C_q(K; R) \to \Delta_q(K; R)$$

characterized by the property that

$$\lambda_q(\langle \mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma \rangle) = \iota_q(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)$$

for all q-simplices σ of K.

Proof Let $W_{q,K}^{\mathrm{dist}}$ denote the subset of $W_{q,K}$ consisting of those $(q+1)$-tuples $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)$ of vertices of K for which $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q$ are distinct and span some simplex of K, and let $W_{q,K}^{\mathrm{rep}}$ denote the complement of $W_{q,K}^{\mathrm{dist}}$ in $W_{q,K}$. An element $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)$ of $W_{q,K}$ belongs of $W_{q,K}^{\mathrm{rep}}$ if and only if some vertex of K occurs more than once in the list $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q$.

Let $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q) \in W_{q,K}^{\mathrm{dist}}$. Then the vertices $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q$ span some q-simplex σ of K. All vertices of σ occur exactly once in the list $\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma$ that is determined by the chosen ordering of the vertices of σ. It follows that there exists some permutation τ of the set $\{0, 1, \ldots, q\}$ such that $\mathbf{u}_j = \mathbf{v}_{\tau(j)}^\sigma$ for $j = 0, 1, \ldots, q$. The simplex σ and the permutation τ are uniquely determined by the $(q+1)$-tuple $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)$. It follows that there is a well-defined function $f\colon W_{q,K} \to \Delta_q(K; R)$ characterized by the following two properties:

- $f(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)$ is the zero element of $\Delta_q(K; R)$ for all $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q) \in W_{q,K}^{\mathrm{rep}}$;
- $f(\mathbf{v}_{\tau(0)}^\sigma, \mathbf{v}_{\tau(1)}^\sigma, \ldots, \mathbf{v}_{\tau(q)}^\sigma) = \epsilon_\tau \iota_q(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)$ for all q-simplices σ of K and for all permutations τ of the set $\{0, 1, \ldots, q\}$.

Now $f(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)$ is the zero element of $\Delta_q(K; R)$ unless the vertices $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q$ are distinct. Suppose that $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q$ are distinct, and therefore span a q-simplex σ of K. Then there exists some permutaion τ of the set $\{0, 1, \ldots, q\}$ such that $\mathbf{u}_j = \mathbf{v}^\sigma_{\tau(j)}$ for $j = 0, 1, \ldots, q$. Let π be a permutation of $\{0, 1, \ldots, q\}$. Then

$$\begin{aligned} f(\mathbf{u}_{\pi(0)}, \mathbf{u}_{\pi(1)}, \ldots, \mathbf{u}_{\pi(q)}) &= f(\mathbf{v}^\sigma_{\tau(\pi(0))}, \mathbf{v}^\sigma_{\tau(\pi(1))}, \ldots, \mathbf{v}^\sigma_{\tau(\pi(q))}) \\ &= \epsilon_{\tau\circ\pi} \iota_q(\mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q) \\ &= \epsilon_\pi \epsilon_\tau \iota_q(\mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q) \\ &= \epsilon_\pi f(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q) \end{aligned}$$

It now follows from Lemma 4.2 that the function $f\colon W_{q,K} \to \Delta_q(K; R)$ induces a well-defined homomorphism $\lambda_q\colon C_q(K; R) \to \Delta_q(K; R)$ with the property that

$$\lambda_q\left(\langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q\rangle\right) = f(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)$$

for all $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q) \in W_{q,K}$. Then

$$\lambda_q(\langle \mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q\rangle) = f(\mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q) = \iota_q(\mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q)$$

for all q-simplices σ of K, as required. ∎

Proposition 4.4 *Let K be a simplicial complex, let q be a non-negative integer, let R be a unital ring, and let $C_q(K; R)$ be the qth chain group of K with coefficients in R. For each q-simplex σ of K let $\mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q$ be a listing of the vertices of σ in some chosen order, and let $\gamma_\sigma = \langle \mathbf{v}^\sigma_0, \mathbf{v}^\sigma_1, \ldots, \mathbf{v}^\sigma_q\rangle$ for each q-simplex σ of K. Then $C_q(K; R)$ is freely generated by the set*

$$\{\gamma_\sigma : \sigma \in K \text{ and } \dim \sigma = q\},$$

and thus the qth chain group $C_q(K; R)$ of K with coefffficients in the unital ring R is a free left R-module. on the q-simplices of K. Thus, given any element c of $C_q(K; R)$, there exist uniquely-determined elements r_σ of the coefficient ring R such that

$$c = \sum_{\substack{\sigma \in K \\ \dim \sigma = q}} r_\sigma \gamma_\sigma.$$

Proof Let $\mathrm{Simp}_q(K)$ denote the set of q-simplices of K, let $W_{q,K}$ denote the set consisting of all $(q+1)$-tuples of vertices of K that span simplices of K, let $\Delta_q(K; R)$ be the free left R-module on the set $W_{q,K}$, let $\iota_q\colon W_{q,K} \to \Delta_q(K; R)$ denote the natural embedding of $W_{q,K}$ into $\Delta_q(K; R)$, and let $\delta_w = \iota_q(w)$ for all $w \in W_{q,K}$. It then follows from Lemma 4.3 that there is a well-defined homomorphism

$$\lambda_q\colon C_q(K;R) \to \Delta_q(K;R)$$

characterized by the property that

$$\lambda_q(\langle \mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma \rangle) = \iota_q(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma) = \delta_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)}$$

for all q-simplices σ of K. Let $\Gamma_q(K;R)$ be the submodule of $\Delta_q(K;R)$ generated by

$$\{\delta_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)} : \sigma \in \operatorname{Simp}_q K\}.$$

The elements of this generating set are independent, because $(\delta_w : w \in W_{q,K})$ is a free basis of $\Delta_q(K;R)$. It follows that the submodule $\Gamma_q(K;R)$ is a free left R-module, and that

$$\{\delta_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)} : \sigma \in \operatorname{Simp}_q K\}$$

is a free basis of $\Gamma_q(K;R)$.

Now

$$\lambda_q(\langle \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q \rangle) \in \Gamma_q(K;R)$$

for all $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q \in W_{q,K}$, and therefore $\lambda_q(C_q(K;R)) \subset \Gamma_q(K;R)$. But $\delta_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)} \in \lambda_q(C_q(K;R))$ for all q-simplices σ of K, and therefore $\Gamma_q(K;R) \subset \lambda_q(C_q(K;R))$. We conclude therefore that $\lambda_q(C_q(K;R)) = \Gamma_q(K;R)$.

Let $\rho_q\colon \Delta_q(K;R) \to C_q(K;R)$ denote the quotient homomorphism from $\Delta_q(K;R) \to C_q(K;R)$. Then

$$\langle \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q \rangle = \rho_q(\delta_{(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q)})$$

for all $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_q) \in W_{q,K}$. Now

$$\rho_q(\lambda_q(\gamma_\sigma)) = \rho_q(\lambda_q(\langle \mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma \rangle)) = \rho_q(\delta_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma)}) = \langle \mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma \rangle = \gamma_\sigma$$

for all $\sigma \in \operatorname{Simp}_q K$. The properties of $C_q(K;R)$ stated in Lemma 4.1 ensure that every element $C_q(K;R)$ can be expressed as a linear combination of elements of the set $\{\gamma_\sigma : \sigma \in \operatorname{Simp}_q K\}$. It follows that $\rho_q(\lambda_q(c)) = c$ for all $c \in C_q(K;R)$. Therefore the homomorphism $\lambda_q\colon C_q(K;R) \to \Delta_q(K;R)$ is injective. Now we have shown that $\lambda_q(C_q(K;R)) = \Gamma_q(K;R)$. An injective homorphism maps its domain isomorphically into its image. We conclude therefore that the homomorphism λ_q maps the qth chain group $C_q(K;R)$ isomorphically onto the free left R-module $\Gamma_q(K;R)$. Therefore $C_q(K;R)$ must itself be a free left R-module. Moreover $(\lambda_q(\gamma_\sigma) : \sigma \in \operatorname{Simp}_q K)$ is a free basis of $\Gamma(K;R)$. It follows that $(\gamma_\sigma : \sigma \in \operatorname{Simp}_q K)$ is a free basis of $C_q(K;R)$, as required. ∎

4.5 Homomorphisms of Chain Groups induced by Simplicial Maps

Proposition 4.5 *Let K and L be simplical complexes, and let $\varphi\colon K \to L$ be a simplicial map, and let R be a unital ring. Then the simplicial map φ induces well-defined homomorphisms $\varphi_q\colon C_q\colon C_q(K;R) \to C_q(L;R)$ of chain groups, where*

$$\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Proof Let $W_{q,K}$ be the set consisting of all $(q+1)$-tuples of vertices of K that span simplices of K. If $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are vertices of K that span a simplex of K then their images $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ under the simplicial map φ are vertices of L that span a simplex of L. It follows that there is a well-defined function $f\colon W_{q,k} \to C_q(L;R)$, where

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

for all $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in W_{q,K}$. Moreover

$$f(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) = \epsilon_\pi f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

for all $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in W_{q,K}$ and for all permutations π of the set $\{0, 1, 2, \ldots, q\}$. (Here ϵ_π denotes the parity of the permutation π, defined such that $\epsilon_\pi = +1$ when π is an even permutation, and $\epsilon_\pi = -1$ when π is an odd permutation.) Also if the list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ contains repeated vertices then the list $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ also contains repeated vertices, and therefore $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ is the zero element of $C_q(L;R)$. It now follows from Lemma 4.2 that there is a well-defined homomorphism $\varphi_q\colon C_q(K;R) \to C_q(L;R)$ that satisfies

$$\varphi_q\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

$(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in W_{q,K}$, as required. ∎

5 The Homology Groups of a Simplicial Complex

5.1 Orientations on Simplices

Let V be a finite-dimensional real vector space. Then each ordered basis of V determines one of two possible orientations on this vector space. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ and $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_q$ be two ordered bases of a vector space V of dimension q. Then there exists a non-singular $q \times q$ matrix (A^j_k) such that $\mathbf{f}_k = \sum_{j=1}^{q} A^j_k \mathbf{e}_j$ for $k = 1, 2, \ldots, q$. If this matrix (A^j_k) has positive determinant then the two bases determine the same orientation on the vector space V. On the other hand, if the matrix (A^j_k) has negative determinant then the two bases determine the opposite orientation on the vector space V. In particular if any two elements of an ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of the vector space V are interchanged with one another, then this reverses the orientation of the vector space.

Let π be a permutation of the set $\{1, \ldots, q\}$, and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ be an ordered basis of the vector space V, determining a particular orientation of this vector space. If the permutation π is even then the basis $\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(q)}$ of V obtained on reordering the elements of the given basis by means of the permutation π determines the same orientation on the vector space V as the original basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$. On the other hand, if the permutation π is odd then the basis $\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(q)}$ determines the opposite orientation on V to that determined by the original basis.

Let σ be a q-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$, and let V be the unique q-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between any two points of σ.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ an ordered list of the vertices of σ. Then these vertices are affinely independent and determine an ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of the vector space V, where $\mathbf{e}_j = \mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$. This ordered basis then determines an orientation on the vector space V. We see therefore that each ordering of the vertices of the q-simplex σ determines a corresponding orientation on the q-dimensional vector space V determined by the q-simplex σ.

Proposition 5.1 *Let σ be a q-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$, and let V be the unique q-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between any two points of σ (so that V is parallel to the tangent space to σ at each point in the interior of σ). Given any ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the vertices of σ, let the corresponding orientation on the vector space V be the orientation determined by the ordered* *basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of V, where $\mathbf{e}_j = \mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$. Then any even permutation of the order of the vertices in the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ preserves the orientation on the vector space V, whereas any odd permutation of the order of these vertices reverses the orientation on V.*

Proof Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the ordered list of vertices determining the orientation on the vector space V. If the vertex $\mathbf{v}_j$ is transposed with $\mathbf{v}_k$, where $j > 0$ and $k > 0$, then the corresponding basis elements $\mathbf{e}_j$ and $\mathbf{e}_k$ in the ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of V are also transposed, and this reverses the orientation on V determined by that ordered basis.

If the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ are interchanged, then this has the effect of replacing the ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ corresponding to the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ by the ordered basis $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_q$, where

$$\mathbf{f}_1 = \mathbf{v}_0 - \mathbf{v}_1 = -\mathbf{e}_1$$

and

$$\mathbf{f}_j = \mathbf{v}_j - \mathbf{v}_1 = \mathbf{e}_j - \mathbf{e}_1 \quad \text{for } j = 2, 3, \ldots, q.$$

The non-singular $q \times q$ matrix that implements this change of basis is the upper triangular matrix A, where

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The determinant of an upper triangular matrix is the product of the matrix elements along the leading diagonal, and therefore $\det A = -1$. It follows that transposing the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ occurring in the first two positions in the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of vertices of σ reverses the orientation on the vector space V determined by the ordering of the vertices of σ.

It now follows from standard properties of permutations of finite sets that interchanging any two of the vertices in any ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the vertices of the q-simplex σ reverses the orientation on the q-dimensional real vector space V that is determined by the ordering of these vertices. Indeed if the positions in the list are numbered from 0 to q then the vertex in position 0 can be transposed with the vertex in position j, where $j > 1$, by first transposing the vertices in positions 1 and j, then transposing the vertices in positions 0 and 1, and then again transposing the vertices in positions 1 and j. This involves three transpositions of vertices in the list, and each of these transpositions reverses the orientation on the vector space V. It follows that any even permutation of the ordering of the vertices in the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ preserves the corresponding orientation on the vector space V, whereas any odd permutation of the ordering of these vertices reverses the orientation on this vector space, as required. ∎

We can regard the orientation on the vector space V as an orientation of the simplex σ itself. Indeed this orientation may be viewed as an orientation on the q-dimensional tangent space to the simplex σ at any interior point of σ. In this fashion any ordering of the vertices of a simplex σ determines a corresponding orientation on that simplex. If the ordering of the vertices is permuted by means of an even permutation then the orientation of the simplex is preserved. But if the ordering of the vertices is permuted by means of an odd permutation then the orientation of the simplex is reversed.

Example Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the vertices of a triangle in a Euclidean space $\mathbb{R}^k$ of dimension at least two. Then this triangle determines a 2-dimensional vector subspace V of $\mathbb{R}^k$. This 2-dimensional subspace V is spanned by the displacement vectors $\mathbf{v} - \mathbf{u}$ and $\mathbf{w} - \mathbf{u}$, and is parallel to the tangent plane to the triangle at any interior point of the triangle.

Now it follows from Proposition 5.1 that the orientation of the triangle should be preserved under cyclic permutations of its vertices. Now the ordering $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of these vertices determines an ordered basis $\mathbf{b}_1, \mathbf{b}_2$ of the vector space V, where $\mathbf{b}_1 = \mathbf{v} - \mathbf{u}$ and $\mathbf{b}_2 = \mathbf{w} - \mathbf{u}$. The ordering $\mathbf{v}, \mathbf{w}, \mathbf{u}$ of the vertices of the triangle corresponds to the orientation on the vector space V determined by the ordered basis $\mathbf{w} - \mathbf{v}$, $\mathbf{u} - \mathbf{v}$. Now $\mathbf{w} - \mathbf{v} = \mathbf{b}_2 - \mathbf{b}_1$ and $\mathbf{u} - \mathbf{v} = -\mathbf{b}_1$. Moreover the 2×2 matrix implementing the change of basis from the ordered basis $\mathbf{b}_1, \mathbf{b}_2$ to the ordered basis $\mathbf{b}_2 - \mathbf{b}_1, -\mathbf{b}_1$ is the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

and this matrix has determinant 1. Similarly the ordering $\mathbf{w}, \mathbf{u}, \mathbf{v}$ of the vertices of the triangle determines a corresponding ordered basis $\mathbf{u} - \mathbf{w}$, $\mathbf{v} - \mathbf{w}$ of the vector space V. Moreover $\mathbf{u} - \mathbf{w} = -\mathbf{b}_2$ and $\mathbf{v} - \mathbf{w} = \mathbf{b}_1 - \mathbf{b}_2$, and the 2×2 matrix implementing the change of basis from the ordered basis $\mathbf{b}_1, \mathbf{b}_2$ to the ordered basis $-\mathbf{b}_2$, $\mathbf{b}_1 - \mathbf{b}_2$ is the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

and this matrix also has determinant 1. It follows that an even permutation of the ordering of the vertices of the triangle (resulting from a cyclic permutation of those vertices) preserves the orientation on the vector space V determined by the ordering of the vertices.

On the other hand the 2×2 matrices that implement the change of ordered basis of the vector space V resulting from odd permutations of the order of the vertices $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

and these three matrices all have determinant -1. It follows that any odd permutation of the vertices (resulting from a transposition of two of those vertices that fixes the remaining vertex) results in a reversal of the orientation on the vector space V.

Thus even permutations of the ordering of the vertices of the triangle preserve the orientation of the triangle determined by the ordering of its vertices, whereas odd permutations of the ordering reverse the orientation determined by the ordering.

Let K be a simplicial complex, and let σ be a q-simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then σ, with the chosen ordering of its vertices, determines a corresponding element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_q(K; \mathbb{Z})$. This element is in fact determined by the orientation on the simplex σ. If the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the simplex are reordered by means of an even permutation of the vertices in the list then both the orientation on the simplex determined by the ordering of its vertices remains unchanged and the corresponding element of $C_q(K; \mathbb{Z})$ determined by the ordered list of the vertices of the simplex also remains unchanged. On the other hand, if the vertices are reordered through an odd permutation of the vertices in the list then both the orientation of the simplex determined by the ordering of its vertices is reversed, and the corresponding element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of $C_q(K; \mathbb{Z})$ determined by the ordered list of those vertices is replaced by the element $-\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$.

5.2 Boundary Homomorphisms

Let K be a simplicial complex, and let R be an integral domain. We introduce below *boundary homomorphisms* $\partial_q: C_q(K; R) \to C_{q-1}(K; R)$ between the chain groups of K with coefficients in R.

In order to define and investigate the properties of this boundary homomorphism, we introduce a notation that is frequently used to indicate that some particular vertex is to be omitted from a ordered list of vertices of a simplex. Let $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ be the element of the chain group $C_q(K; R)$ determined by some ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of vertices of K that span a simplex of K. We denote by $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ the element

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q \rangle$$

of $C_{q-1}(K; R)$ obtained on omitting the vertex $\mathbf{v}_j$ from the list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of vertices of K. Thus

$$\begin{aligned}
\langle \hat{\mathbf{v}}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle &= \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle, \\
\langle \mathbf{v}_0, \hat{\mathbf{v}}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle &= \langle \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle, \\
\langle \mathbf{v}_0, \mathbf{v}_1, \hat{\mathbf{v}}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle &= \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle, \\
&\vdots \\
\langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \hat{\mathbf{v}}_q \rangle &= \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{q-1} \rangle.
\end{aligned}$$

We may employ analogous notation when omitting two or more vertices from an ordered list of vertices. Thus if j and k are integers between 0 and q, where $j < k$, we denote by

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots \mathbf{v}_q \rangle$$

the element $\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_q \rangle$ of $C_{q-2}(K; R)$ determined by the ordered list of vertices that results on omitting both vertices $\mathbf{v}_j$ and $\mathbf{v}_k$ from the list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.

If the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct then they are the vertices of a q-simplex σ of K, and this simplex is represented by the corresponding generators $\pm\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_q(K; \mathbb{Z})$. Moreover there are exactly two such generators in $C_q(K; \mathbb{Z})$ corresponding to the simplex σ, and these two generators represent the two possible orientations on the simplex. The elements $\pm\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_{q-1}(K; \mathbb{Z})$ obtained by omitting the vertex $\mathbf{v}_j$ from the list of vertices then represent the unique $(q-1)$-dimensional face of the simplex σ that does not contain the vertex $\mathbf{v}_j$.

Proposition 5.2 *Let K be a simplicial complex, and let R be a unital ring. Then there exist well-defined homomorphisms*

$$\partial_q : C_q(K; R) \to C_{q-1}(K; R)$$

for all integers q characterized by the requirement that

$$\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

whenever the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K span a simplex of K.

Proof If $q \leq 0$, or if $q > \dim K$, then at least one of the R-modules $C_q(K;R)$ and $C_{q-1}(K;R)$ is the zero module: in those case we define $\partial_q\colon C_q(K;R) \to C_{q-1}(K;R)$ to be the zero homomorphism.

Suppose then that $0 < q \leq \dim K$. We prove the existence of the required homomorphism ∂_q by means of Lemma 4.2.

Given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ spanning a simplex of K, let

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Let i be an integer between 1 and q. If $0 \leq j < i - 1$ then

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \ldots, \mathbf{v}_q \rangle$$

changes sign (i.e., it is replaced by the negative of itself) when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed. Similarly if $i < j \leq q$ then

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

changes sign when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed. Also

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_{i-1}, \ldots, \mathbf{v}_q \rangle \quad \text{and} \quad \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle$$

are transposed when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed. It follows that the $(q-1)$-chain $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed for some integer i satisfying $1 \leq i \leq q$.

Next suppose that $\mathbf{v}_i = \mathbf{v}_k$ for some i and k satisfying $i < k$. Then

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = (-1)^i \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle + (-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle,$$

since the remaining terms in the expression defining $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ contain both $\mathbf{v}_i$ and $\mathbf{v}_k$ and are therefore equal to the zero element of $C_{q-1}(K;R)$ when $\mathbf{v}_i = \mathbf{v}_k$. Also

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{k-i-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle.$$

Indeed this identity is immediate when $k = i+1$. Suppose that $k > i+1$. Let $\mathbf{w} = \mathbf{v}_i = \mathbf{v}_k$. Then the vertex $\mathbf{w}$ occurs in the ordered list $\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q$ before $\mathbf{v}_{i+1}$ but is omitted after $\mathbf{v}_{k-1}$, whereas the vertex $\mathbf{w}$ occurs in the ordered list $\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q$ after $\mathbf{v}_{k-1}$ but is omitted before $\mathbf{v}_{i+1}$. Thus, in order to convert the first ordered list to the second by successively transposing vertices, it suffices to transpose the vertex $\mathbf{w}$ occurring before $\mathbf{v}_{i+1}$ in the first list successively with the vertices $\mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_{k-1}$, shuffling it along the list until it occurs after $\mathbf{v}_{k-1}$. This process requires $k-i-1$ successive transpositions and is thus results in a permutation of the vertices in the list which is of parity $(-1)^{k-i-1}$. It follows that

$$(-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{i-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle$$

and thus

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$$

whenever $\mathbf{v}_i = \mathbf{v}_k$, where $0 \leq i < k \leq q$. We conclude therefore that $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ unless the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct.

It now follows directly from Lemma 4.2 that there is a well-defined homomorphism $\partial_q \colon C_q(K;R) \to C_{q-1}(K;R)$, characterized by the property that

$$\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. ∎

Let K be a simplicial complex, and let R be an integral domain. The R-module homomorphism $\partial_q \colon C_q(K;R) \to C_{q-1}(K;R)$ between the chain groups of K in dimensions q and $q-1$ is referred to as the *boundary homomorphism* between these chain groups.

Example Let K be a simplicial complex consisting of a triangle with vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, together with all the vertices and edges of this triangle, and let R be an integral domain. Then

$$C_2(K;R) = \{ r \, \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle : r \in R \}.$$

Now

$$\partial_2 \Big(r \, \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle \Big) = r \, \partial_2 (\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle),$$

because $\partial_2 \colon C_3(K;R) \to C_2(K;R)$ is a homomorphism of R-modules. It follows that this boundary homomorphism is determined by the value of $\partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle)$. Moreover

$$\partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle) = \langle \mathbf{b}, \mathbf{c} \rangle - \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle,$$

and

$$\begin{aligned} \partial_1(\langle \mathbf{b}, \mathbf{c} \rangle) &= \langle \mathbf{c} \rangle - \langle \mathbf{b} \rangle, \\ \partial_1(\langle \mathbf{a}, \mathbf{c} \rangle) &= \langle \mathbf{c} \rangle - \langle \mathbf{a} \rangle, \\ \partial_1(\langle \mathbf{a}, \mathbf{b} \rangle) &= \langle \mathbf{b} \rangle - \langle \mathbf{a} \rangle. \end{aligned}$$

Therefore

$$\partial_1\Big(\partial_2\Big(\langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle\Big)\Big) = \langle \mathbf{c}\rangle - \langle \mathbf{b}\rangle - \langle \mathbf{c}\rangle + \langle \mathbf{a}\rangle + \langle \mathbf{b}\rangle - \langle \mathbf{a}\rangle = 0.$$

It follows that $\partial_1(\partial_2(x)) = 0$ for all $x \in C_2(K;R)$.

Example Let K be a simplicial complex consisting of a tetrahedron with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, together with all the vertices, edges and triangular faces of this tetrahedron, and let R be an integral domain. Then

$$C_3(K;R) = \{r\,\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\rangle : r \in R\}.$$

Now

$$\partial_3\Big(r\,\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\rangle\Big) = r\,\partial_3(\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\rangle),$$

because $\partial_3\colon C_3(K;R) \to C_2(K;R)$ is a homomorphism of R-modules. It follows that this boundary homomorphism is determined by the value of $\partial_3(\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\rangle)$. Moreover

$$\partial_3\Big(\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\rangle\Big) = \langle \mathbf{b},\mathbf{c},\mathbf{d}\rangle - \langle \mathbf{a},\mathbf{c},\mathbf{d}\rangle + \langle \mathbf{a},\mathbf{b},\mathbf{d}\rangle - \langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle,$$

and

$$\begin{array}{rcl}
\partial_2(\langle \mathbf{b},\mathbf{c},\mathbf{d}\rangle) &=& \langle \mathbf{c},\mathbf{d}\rangle - \langle \mathbf{b},\mathbf{d}\rangle + \langle \mathbf{b},\mathbf{c}\rangle,\\
\partial_2(\langle \mathbf{a},\mathbf{c},\mathbf{d}\rangle) &=& \langle \mathbf{c},\mathbf{d}\rangle - \langle \mathbf{a},\mathbf{d}\rangle + \langle \mathbf{a},\mathbf{c}\rangle,\\
\partial_2(\langle \mathbf{a},\mathbf{b},\mathbf{d}\rangle) &=& \langle \mathbf{b},\mathbf{d}\rangle - \langle \mathbf{a},\mathbf{d}\rangle + \langle \mathbf{a},\mathbf{b}\rangle,\\
\partial_2(\langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle) &=& \langle \mathbf{b},\mathbf{c}\rangle - \langle \mathbf{a},\mathbf{c}\rangle + \langle \mathbf{a},\mathbf{b}\rangle.
\end{array}$$

Therefore

$$\begin{array}{rcl}
\partial_2\Big(\partial_3\Big(\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\rangle\Big)\Big) &=& \partial_2(\langle \mathbf{b},\mathbf{c},\mathbf{d}\rangle) - \partial_2(\langle \mathbf{a},\mathbf{c},\mathbf{d}\rangle) + \partial_2(\langle \mathbf{a},\mathbf{b},\mathbf{d}\rangle)\\
&& \quad - \partial_2(\langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle)\\
&=& \langle \mathbf{c},\mathbf{d}\rangle - \langle \mathbf{b},\mathbf{d}\rangle + \langle \mathbf{b},\mathbf{c}\rangle\\
&& \quad - \langle \mathbf{c},\mathbf{d}\rangle + \langle \mathbf{a},\mathbf{d}\rangle - \langle \mathbf{a},\mathbf{c}\rangle\\
&& \quad + \langle \mathbf{b},\mathbf{d}\rangle - \langle \mathbf{a},\mathbf{d}\rangle + \langle \mathbf{a},\mathbf{b}\rangle\\
&& \quad - \langle \mathbf{b},\mathbf{c}\rangle + \langle \mathbf{a},\mathbf{c}\rangle - \langle \mathbf{a},\mathbf{b}\rangle\\
&=& 0.
\end{array}$$

It follows that $\partial_2(\partial_3(x)) = 0$ for all $x \in C_3(K;R)$. Also the boundary homomorphism $\partial_2\colon C_2(K;R) \to C_1(K;R)$ is determined by the values of

$$\partial_2(\langle \mathbf{b},\mathbf{c},\mathbf{d}\rangle),\quad \partial_2(\langle \mathbf{a},\mathbf{c},\mathbf{d}\rangle),\quad \partial_2(\langle \mathbf{a},\mathbf{b},\mathbf{d}\rangle)\quad\text{and}\quad \partial_2(\langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle).$$

It follows from the calculation in the preceding example that $\partial_1(\partial_2(x)) = 0$ for all $x \in C_2(K;R)$.

Lemma 5.3 *Let K be a simplicial complex, let R be an integral domain, and, for each integer q, let $\partial_q\colon C_q(K;R) \to C_{q-1}(K;R)$ be the boundary homomorphism between the chain groups $C_q(K;R)$ and $C_{q-1}(K;R)$. Then $\partial_{q-1}\circ\partial_q = 0$ for all integers q.*

Proof The result is trivial if $q < 2$, since in this case $\partial_{q-1} = 0$. Suppose that $q \geq 2$. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned}
\partial_{q-1}\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1}\left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle\right) \\
&= \sum_{j=1}^{q}\sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\
&\quad + \sum_{j=0}^{q-1}\sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q\rangle \\
&= 0
\end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1}\circ\partial_q$ is determined by its values on the elements of any free basis of $C_q(K;R)$. ∎

5.3 The Homology Groups of a Simplicial Complex

Let K be a simplicial complex, and let R be an integral domain, and, for each non-negative integer q, let $C_q(K;R)$ denote the R-module whose elements are q-chains of K with coefficients in the coefficient ring R. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1}c'$ for some $(q+1)$-chain c'. The R-module consisting of the q-cycles of K with coefficients in the integral domain R is denoted by $Z_q(K;R)$, and the R-module consisisting of the q-boundaries of K with coefficients in R is denoted by $B_q(K;R)$. Thus $Z_q(K;R)$ is the kernel of the boundary homomorphism $\partial_q\colon C_q(K;R) \to C_{q-1}(K;R)$, and $B_q(K;R)$ is the image of the boundary homomorphism $\partial_{q+1}\colon C_{q+1}(K;R) \to C_q(K;R)$. However $\partial_q\circ\partial_{q+1} = 0$ (see Lemma 5.3). It follows that $B_q(K;R) \subset Z_q(K;R)$. But these R-modules are submodules of the R-module $C_q(K;R)$. We can therefore form the quotient module $H_q(K;R)$, where $H_q(K;R) = Z_q(K;R)/B_q(K;R)$. The R-module $H_q(K;R)$ is referred to as the *qth homology group* of the simplicial complex K with coefficients in the integral domain R. Note that $H_q(K;R) = 0$ if $q < 0$ or $q > \dim K$ (since $Z_q(K;R) = 0$ and $B_q(K;R) = 0$ in these cases).

The element $[z] \in H_q(K; R)$ of the homology group $H_q(K; R)$ determined by an element z of $Z_q(K; R)$ is referred to as the *homology class* of the q-cycle z. Note that $[z_1 + z_2] = [z_1] + [z_2]$ for all $z_1, z_2 \in Z_q(K; R)$, and $[z_1] = [z_2]$ if and only if $z_1 - z_2 = \partial_{q+1} c$ for some $(q+1)$-chain c with coefficients in the coefficient ring R.

An important special case of the above definitions is that in which the coefficient ring R is the ring $\mathbb{Z}$ of integers. The resultant Abelian groups $C_q(K; \mathbb{Z})$, $Z_q(K; \mathbb{Z})$, $B_q(K; \mathbb{Z})$ and $H_q(K; \mathbb{Z})$ defined as described above are often denoted simply by $C_q(K)$, $Z_q(K)$, $B_q(K)$ and $H_q(K)$ respectively. Thus if a group of q-dimensional chains, cycles, boundaries or homology classes is specified, but the ring of coefficients is not specified, then the coefficient ring is by default taken to be the ring of integers.

Remark It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex. This fact is far from obvious, and a lot of basic theory must be developed in order to establish the tools to prove this result.

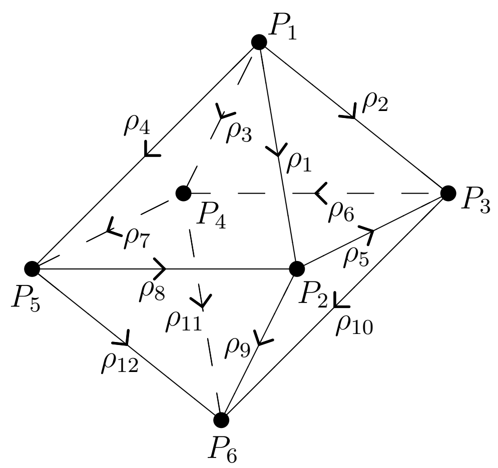
6 Homology Calculations

6.1 The Homology Groups of an Octahedron

Let K be the simplicial complex consisting of the triangular faces, edges and vertices of an octahedron in $\mathbb{R}^3$ with vertices P_1, P_2, P_3, P_4, P_5 and P_6, where

$$P_1 = (0, 0, 1), \quad P_2 = (1, 0, 0), \quad P_3 = (0, 1, 0),$$

$$P_4 = (-1, 0, 0), \quad P_5 = (0, -1, 0), \quad P_6 = (0, 0, -1)$$



This octahedron consists of the four triangular faces $P_1P_2P_3$, $P_1P_3P_4$, $P_1P_4P_5$ and $P_1P_5P_2$ of the pyramid whose base is the square $P_2P_3P_4P_5$ and whose apex is P_1, together with the four triangular faces $P_6P_2P_3$, $P_6P_3P_4$, $P_6P_4P_5$ and $P_6P_5P_2$ of the pyramid whose base is $P_2P_3P_4P_5$ and whose apex is P_6.

A typical 2-chain c_2 of K is a linear combination, with integer coefficients, of eight oriented 2-simplices that represent the triangular faces of the octahedron. Thus we can write

$$c_2 = \sum_{i=1}^{8} n_i \sigma_i,$$

where $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, 8$ and

$$\sigma_1 = \langle P_1, P_2, P_3 \rangle, \quad \sigma_2 = \langle P_1, P_3, P_4 \rangle, \quad \sigma_3 = \langle P_1, P_4, P_5 \rangle,$$

$$\sigma_4 = \langle P_1, P_5, P_2 \rangle, \quad \sigma_5 = \langle P_6, P_3, P_2 \rangle, \quad \sigma_6 = \langle P_6, P_4, P_3 \rangle,$$

$$\sigma_7 = \langle P_6, P_5, P_4 \rangle, \quad \sigma_8 = \langle P_6, P_2, P_5 \rangle.$$

(The orientation on each of these triangles has been chosen such that the vertices of the triangle are listed in anticlockwise order when viewed from a point close to the centre of triangle that lies outside the octahedron.)

Similarly a typical 1-chain c_1 of K is a linear combination, with integer coefficients, of twelve 1-simplices that represent the edges of the octahedron. Thus we can write

$$c_1 = \sum_{j=1}^{12} m_j \rho_j,$$

where $m_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, 12$ and

$$\rho_1 = \langle P_1, P_2 \rangle, \quad \rho_2 = \langle P_1, P_3 \rangle, \quad \rho_3 = \langle P_1, P_4 \rangle, \quad \rho_4 = \langle P_1, P_5 \rangle,$$

$$\rho_5 = \langle P_2, P_3 \rangle, \quad \rho_6 = \langle P_3, P_4 \rangle, \quad \rho_7 = \langle P_4, P_5 \rangle, \quad \rho_8 = \langle P_5, P_2 \rangle,$$

$$\rho_9 = \langle P_2, P_6 \rangle, \quad \rho_{10} = \langle P_3, P_6 \rangle, \quad \rho_{11} = \langle P_4, P_6 \rangle, \quad \rho_{12} = \langle P_5, P_6 \rangle,$$

A typical 0-chain c_0 takes the form

$$c_0 = \sum_{k=1}^{6} r_k \langle P_k \rangle,$$

where $r_k \in \mathbb{Z}$ for $k = 1, 2, \ldots, 6$.

We now calculate the boundary of a 2-chain. It follows from the definition of the boundary homomorphism ∂_2 that

$$\partial_2 \sigma_1 = \partial_2 \langle P_1, P_2, P_3 \rangle = \langle P_2 P_3 \rangle - \langle P_1 P_3 \rangle + \langle P_1 P_2 \rangle = \rho_5 - \rho_2 + \rho_1.$$

Similarly

$$\begin{aligned}
\partial_2 \sigma_2 &= \partial_2 \langle P_1, P_3, P_4 \rangle = \rho_6 - \rho_3 + \rho_2, \\
\partial_2 \sigma_3 &= \partial_2 \langle P_1, P_4, P_5 \rangle = \rho_7 - \rho_4 + \rho_3, \\
\partial_2 \sigma_4 &= \partial_2 \langle P_1, P_5, P_2 \rangle = \rho_8 - \rho_1 + \rho_4, \\
\partial_2 \sigma_5 &= \partial_2 \langle P_6, P_3, P_2 \rangle = -\rho_5 + \rho_9 - \rho_{10}, \\
\partial_2 \sigma_6 &= \partial_2 \langle P_6, P_4, P_3 \rangle = -\rho_6 + \rho_{10} - \rho_{11}, \\
\partial_2 \sigma_7 &= \partial_2 \langle P_6, P_5, P_4 \rangle = -\rho_7 + \rho_{11} - \rho_{12}, \\
\partial_2 \sigma_8 &= \partial_2 \langle P_6, P_2, P_5 \rangle = -\rho_8 + \rho_{12} - \rho_9.
\end{aligned}$$

Thus

$$\partial_2 c_2 = \partial_2 \left(n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3 + n_4 \sigma_4 + n_5 \sigma_5 + n_6 \sigma_6 + n_7 \sigma_7 + n_8 \sigma_8 \right)$$

$$
\begin{aligned}
&= n_1\partial_2\sigma_1 + n_2\partial_2\sigma_2 + n_3\partial_2\sigma_3 + n_4\partial_2\sigma_4 \\
&\qquad + n_5\partial_2\sigma_5 + n_6\partial_2\sigma_6 + n_7\partial_2\sigma_7 + n_8\partial_2\sigma_8 \\
&= (n_1 - n_4)\rho_1 + (n_2 - n_1)\rho_2 + (n_3 - n_2)\rho_3 + (n_4 - n_3)\rho_4 \\
&\qquad + (n_1 - n_5)\rho_5 + (n_2 - n_6)\rho_6 + (n_3 - n_7)\rho_7 + (n_4 - n_8)\rho_8 \\
&\qquad + (n_5 - n_8)\rho_9 + (n_6 - n_5)\rho_{10} + (n_7 - n_6)\rho_{11} + (n_8 - n_7)\rho_{12}
\end{aligned}
$$

It follows that $\partial_2 c_2 = 0$ if and only if

$$n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8.$$

Therefore

$$Z_2(K;\mathbb{Z}) = \ker\partial_2 = \{n\mu : n \in \mathbb{Z}\}, \text{ where } \mu = \sum_{i=1}^{8}\sigma_i.$$

Now $C_3(K;\mathbb{Z}) = 0$, and thus $B_2(K;\mathbb{Z}) = 0$ (where 0 here denotes the zero group), since the complex K has no 3-simplices. Therefore

$$H_2(K;\mathbb{Z}) \cong Z_2(K;\mathbb{Z}) \cong \mathbb{Z}.$$

Next we calculate the boundary of a 1-chain. It follows from the definition of the boundary homomorphism ∂_1 that

$$
\begin{aligned}
\partial_1 c_1 &= \partial_1\left(\sum_{j=1}^{12} m_j\rho_j\right) \\
&= m_1(\langle P_2\rangle - \langle P_1\rangle) + m_2(\langle P_3\rangle - \langle P_1\rangle) \\
&\qquad + m_3(\langle P_4\rangle - \langle P_1\rangle) + m_4(\langle P_5\rangle - \langle P_1\rangle) \\
&\qquad + m_5(\langle P_3\rangle - \langle P_2\rangle) + m_6(\langle P_4\rangle - \langle P_3\rangle) \\
&\qquad + m_7(\langle P_5\rangle - \langle P_4\rangle) + m_8(\langle P_2\rangle - \langle P_5\rangle) \\
&\qquad + m_9(\langle P_6\rangle - \langle P_2\rangle) + m_{10}(\langle P_6\rangle - \langle P_3\rangle) \\
&\qquad + m_{11}(\langle P_6\rangle - \langle P_4\rangle) + m_{12}(\langle P_6\rangle - \langle P_5\rangle) \\
&= -(m_1 + m_2 + m_3 + m_4)\langle P_1\rangle + (m_1 - m_5 + m_8 - m_9)\langle P_2\rangle \\
&\qquad + (m_2 + m_5 - m_6 - m_{10})\langle P_3\rangle + (m_3 + m_6 - m_7 - m_{11})\langle P_4\rangle \\
&\qquad + (m_4 + m_7 - m_8 - m_{12})\langle P_5\rangle + (m_9 + m_{10} + m_{11} + m_{12})\langle P_6\rangle
\end{aligned}
$$

It follows that the 1-chain c_1 is a 1-cycle if and only if

$$m_1 + m_2 + m_3 + m_4 = 0, \quad m_1 - m_5 + m_8 - m_9 = 0,$$

$$m_2 + m_5 - m_6 - m_{10} = 0, \quad m_3 + m_6 - m_7 - m_{11} = 0,$$

$$m_4 + m_7 - m_8 - m_{12} = 0 \quad \text{and} \quad m_9 + m_{10} + m_{11} + m_{12} = 0.$$

On examining the structure of these equations, we see that, when c_1 is a 1-cycle, it is possible to eliminate five of the integer quantities m_j, expressing them in terms of the remaining quantities. For example, we can eliminate m_4, m_6, m_7, m_8 and m_{12}, expressing these quantities in terms of m_1, m_2, m_3, m_5, m_9 m_{10} and m_{11} by means of the equations

$$\begin{aligned}
m_4 &= -m_1 - m_2 - m_3, \\
m_6 &= m_2 - m_{10} + m_5, \\
m_7 &= m_2 + m_3 - m_{10} - m_{11} + m_5, \\
m_8 &= -m_1 + m_9 + m_5, \\
m_{12} &= -m_9 - m_{10} - m_{11}
\end{aligned}$$

It follows that

$$Z_2(K;\mathbb{Z}) = \{m_1 z_1 + m_2 z_2 + m_3 z_3 + m_5 z_5 + m_9 z_9 + m_{10} z_{10} + m_{11} z_{11}\},$$

where

$$\begin{aligned}
z_1 &= \rho_1 - \rho_4 - \rho_8 = -\partial_2 \sigma_4, \\
z_2 &= \rho_2 - \rho_4 + \rho_6 + \rho_7 = \partial_2(\sigma_2 + \sigma_3), \\
z_3 &= \rho_3 - \rho_4 + \rho_7 = \partial_2 \sigma_3, \\
z_5 &= \rho_5 + \rho_6 + \rho_7 + \rho_8 = \partial_2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4), \\
z_9 &= \rho_8 + \rho_9 - \rho_{12} = -\partial_2 \sigma_8, \\
z_{10} &= -\rho_6 - \rho_7 + \rho_{10} - \rho_{12} = \partial_2(\sigma_6 + \sigma_7), \\
z_{11} &= \rho_{11} - \rho_7 - \rho_{12} = \partial_2 \sigma_7.
\end{aligned}$$

From these equations, we see that the generators z_1, z_2, z_3, z_5, z_9, z_{10} and z_{11} of the group $Z_1(K;\mathbb{Z})$ of 1-cycles all belong to the group $B_1(K;\mathbb{Z})$ of 1-boundaries. It follows that $Z_1(K;\mathbb{Z}) = B_1(K;\mathbb{Z})$, and therefore $H_1(K;\mathbb{Z}) = 0$.

In order to determine $H_0(K;\mathbb{Z})$ it suffices to note that the 0-chains

$$\langle P_2 \rangle - \langle P_1 \rangle, \quad \langle P_3 \rangle - \langle P_1 \rangle, \quad \langle P_4 \rangle - \langle P_1 \rangle, \quad \langle P_5 \rangle - \langle P_1 \rangle \quad \text{and} \quad \langle P_6 \rangle - \langle P_1 \rangle$$

are 0-boundaries. Indeed

$$\langle P_2 \rangle - \langle P_1 \rangle = \partial_1 \rho_1, \quad \langle P_3 \rangle - \langle P_1 \rangle = \partial_1 \rho_2, \quad \langle P_4 \rangle - \langle P_1 \rangle = \partial_1 \rho_3,$$

$$\langle P_5 \rangle - \langle P_1 \rangle = \partial_1 \rho_4 \quad \text{and} \quad \langle P_6 \rangle - \langle P_1 \rangle = \partial_1(\rho_1 + \rho_9).$$

Therefore

$$\sum_{k=1}^{6} r_k \langle P_k\rangle - \left(\sum_{k=1}^{6} r_k\right) \langle P_1\rangle \in B_0(K;\mathbb{Z})$$

for all integers r_1, r_2, r_3, r_4, r_5 and r_6. It follows that $B_0(K;\mathbb{Z}) = \ker\varepsilon$, where $\varepsilon: C_0(K;\mathbb{Z}) \to \mathbb{Z}$ is the homomorphism defined such that

$$\varepsilon\left(\sum_{k=1}^{6} r_k\langle P_k\rangle\right) = \sum_{k=1}^{6} r_k$$

for all integers r_k ($k = 1, 2, \ldots, 6$). Now $Z_0(K;\mathbb{Z}) = C_0(K;\mathbb{Z})$ since the homomorphism $\partial_0: C_0(K;\mathbb{Z}) \to C_{-1}(K;\mathbb{Z})$ is the zero homomorphism mapping $C_0(K;\mathbb{Z})$ to the zero group. It follows that

$$H_0(K;\mathbb{Z}) = C_0(K;\mathbb{Z})/B_0(K;\mathbb{Z}) = C_0(K;\mathbb{Z})/\ker\varepsilon \cong \mathbb{Z}.$$

(Here we are using the result that the image of a homomorphism is isomorphic to the quotient of the domain of the homomorphism by the kernel of the homomorphism.)

We have thus shown that

$$H_2(K;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K;\mathbb{Z}) = 0, \quad H_0(K;\mathbb{Z}) \cong \mathbb{Z}.$$

One can show that $Z_1(K;\mathbb{Z}) = B_1(K;\mathbb{Z})$ by employing an alternative approach to that used above. An element z of $Z_1(K;\mathbb{Z})$ is of the form $z = \sum\limits_{j=1}^{12} m_j\rho_j$, where

$$m_1 + m_2 + m_3 + m_4 = 0, \quad m_1 - m_5 + m_8 - m_9 = 0,$$

$$m_2 + m_5 - m_6 - m_{10} = 0, \quad m_3 + m_6 - m_7 - m_{11} = 0,$$

$$m_4 + m_7 - m_8 - m_{12} = 0 \quad \text{and} \quad m_9 + m_{10} + m_{11} + m_{12} = 0.$$

The 1-cycle z belongs to the group $B_1(K;\mathbb{Z})$ if and only if there exists some 2-chain c_2 such that $z = \partial_2 c_2$. It follows that $z \in B_1(K;\mathbb{Z})$ if and only if there exist integers $n_1, n_2, \ldots, n_8$ such that

$$m_1 = n_1 - n_4, \quad m_2 = n_2 - n_1, \quad m_3 = n_3 - n_2, \quad m_4 = n_4 - n_3,$$

$$m_5 = n_1 - n_5, \quad m_6 = n_2 - n_6, \quad m_7 = n_3 - n_7, \quad m_8 = n_4 - n_8,$$

$$m_9 = n_5 - n_8, \quad m_{10} = n_6 - n_5, \quad m_{11} = n_7 - n_6, \quad m_{12} = n_8 - n_7.$$

The integers $n_1, n_2, \ldots, n_8$ solving the above equations are not uniquely determined, since, given one collection of integers $n_1, n_2, \ldots, n_8$ satisfying these equations, another solution can be obtained by adding some fixed integer to each of $n_1, n_2, \ldots, n_8$. It follows from this that if there exists some collection $n_1, n_2, \ldots, n_8$ of integers that solves the above equations, then there exists a solution which satisfies the extra condition $n_1 = 0$. We then find that

$$n_1 = 0, \quad n_2 = m_2, \quad n_3 = m_2 + m_3, \quad n_4 = -m_1,$$

$$n_5 = -m_5, \quad n_6 = m_2 - m_6, \quad n_7 = m_2 + m_3 - m_7, \quad n_8 = -m_1 - m_8.$$

On substituting $n_1, n_2, \ldots, n_8$ into the relevant equations, and making use of the constraints on the values of $m_1, m_2, \ldots, m_{12}$, we find that we do indeed have a solution to the equations that express the integers m_j in terms of the integers n_i. It follows that every 1-cycle of K is a 1-boundary. Thus $Z_1(K;\mathbb{Z}) = B_1(K;\mathbb{Z})$, and therefore $H_1(K;\mathbb{Z}) = 0$.

Note that the results of many of the calculations of boundaries of chains can be verified by consulting the diagram representing the vertices and edges of the octahedron with their labels and orientations. For example, direct calculation using the definition of the boundary homomorphism $\delta_2 \colon C_2(K;\mathbb{Z}) \to C_1(K;\mathbb{Z})$ shows that

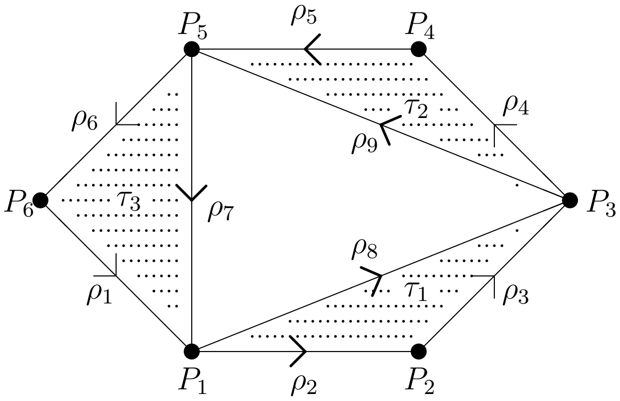
$$\partial_2\sigma_1 = \partial_2\langle P_1, P_2, P_3\rangle = \langle P_2 P_3\rangle - \langle P_1 P_3\rangle + \langle P_1 P_2\rangle = \rho_5 - \rho_2 + \rho_1.$$

Now if we follow round the edges of the triangle $P_1\,P_2\,P_3$ represented by σ, starting at P_1 and proceeding to P_2, then P_3 then back to P_1 we traverse the edge ρ_1 in the direction of the arrow, then the edge ρ_5 in the direction of the arrow, and finally the edge ρ_2 in the reverse direction to the arrow. In consequence, both ρ_1 and ρ_5 occur in the 1-boundary $\partial_2\sigma_1$ with coefficient $+1$, whereas ρ_2 occurs in this 1-boundary with coefficient -1.

Consider also the coefficient corresponding to the vertex P_2 in the 0-boundary $\partial_1 c_1$, where $c_1 = \sum_{j=1}^{12} m_j \rho_j$. The vertex P_2 is an endpoint of four edges. The arrows indicating the orientation on the edges ρ_1 and ρ_8 are directed towards the vertex P_2, whereas the arrows indicating the orientation on the edges ρ_5 and ρ_9 are directed away from the vertex P_2. In consequence, the coefficient of $\langle P_2\rangle$ in $\partial_1 c_1$ is $m_1 - m_5 + m_8 - m_9$.

6.2 Another Homology Example

Let P_1, P_2, P_3, P_4, P_5 and P_6 be the vertices of a hexagon in the plane, listed in cyclic order, and let K be simplicial complex consisting of the triangles $P_1P_2P_3$, $P_3P_4P_5$ and $P_5P_6P_1$, together with all the edges and vertices of these triangles.



Then

$$C_2(K;\mathbb{Z}) = \{n_1\tau_1 + n_2\tau_2 + n_3\tau_3 : n_1, n_2, n_3 \in \mathbb{Z}\},$$

where

$$\tau_1 = \langle P_1P_2P_3\rangle, \quad \tau_2 = \langle P_3P_4P_5\rangle \quad \text{and} \quad \tau_3 = \langle P_5P_6P_1\rangle.$$

(Note τ_1, τ_2 and τ_3 represent the three triangles of the simplicial complex with the orientations that results from an anticyclic ordering of the vertices in the diagram above.) Also

$$C_1(K;\mathbb{Z}) = \left\{\sum_{j=1}^{9} m_j\rho_j : m_j \in \mathbb{Z} \text{ for } j = 1,2,\ldots,9\right\},$$

where

$$\rho_1 = \langle P_6P_1\rangle, \quad \rho_2 = \langle P_1P_2\rangle, \quad \rho_3 = \langle P_2P_3\rangle, \quad \rho_4 = \langle P_3P_4\rangle, \quad \rho_5 = \langle P_4P_5\rangle,$$

$$\rho_6 = \langle P_5P_6\rangle, \quad \rho_7 = \langle P_5P_1\rangle, \quad \rho_8 = \langle P_1P_3\rangle \quad \text{and} \quad \rho_9 = \langle P_3P_5\rangle,$$

and

$$C_0(K;\mathbb{Z}) = \left\{\sum_{k=1}^{6} r_k\langle P_k\rangle : r_k \in \mathbb{Z} \text{ for } k = 1,2,\ldots,6\right\}.$$

(Note that the 1-chains $\rho_1, \rho_2, \ldots, \rho_9$ represent the 9 edges of the simplicial complex with the orientations indicated by the arrows on the above diagram.)

We now calculate the images of the 2-chains τ_1, τ_2 and τ_3 under the boundary homomorphism $\partial_2\colon C_2(K;\mathbb{Z}) \to C_1(K;\mathbb{Z})$. We find that

$$\partial_2\tau_1 = \rho_3 - \rho_8 + \rho_2, \quad \partial_2\tau_2 = \rho_5 - \rho_9 + \rho_4, \quad \partial_2\tau_3 = \rho_1 - \rho_7 + \rho_6,$$

Now

$$
\begin{aligned}
&\partial_2(n_1\tau_1 + n_2\tau_2 + n_3\tau_3) \\
&\quad = \quad n_3\rho_1 + n_1\rho_2 + n_1\rho_3 + n_2\rho_4 + n_2\rho_5 + n_3\rho_6 - n_3\rho_7 - n_1\rho_8 - n_2\rho_9.
\end{aligned}
$$

The simplicial complex K has no non-zero 2-cycles, and therefore $Z_2(K;\mathbb{Z}) = 0$. It follows that $H_2(K;\mathbb{Z}) = 0$.

Let

$$c_1 = \sum_{j=1}^{9} m_j\rho_j.$$

Then

$$
\begin{aligned}
\partial_1 c_1 \quad = \quad & (m_1 - m_2 + m_7 - m_8)\langle P_1\rangle + (m_2 - m_3)\langle P_2\rangle \\
& + (m_3 - m_4 + m_8 - m_9)\langle P_3\rangle + (m_4 - m_5)\langle P_4\rangle \\
& + (m_5 - m_6 + m_9 - m_7)\langle P_5\rangle + (m_6 - m_1)\langle P_6\rangle
\end{aligned}
$$

It follows that c_1 is a 1-cycle of K if and only if

$$m_2 = m_3, \quad m_4 = m_5, \quad m_6 = m_1$$

and

$$m_1 + m_7 = m_3 + m_8 = m_5 + m_9.$$

Moreover c_1 is a 1-boundary of K if and only if

$$m_2 = m_3 = -m_8, \quad m_4 = m_5 = -m_9, \quad m_6 = m_1 = -m_7.$$

We see from this that not every 1-cycle of K is a 1-boundary of K. Indeed

$$Z_1(K;\mathbb{Z}) = \{n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz : n_1, n_2, n_3, n \in \mathbb{Z}\},$$

where $z = \rho_7 + \rho_8 + \rho_9$. Let $\theta\colon Z_1(K;\mathbb{Z}) \to \mathbb{Z}$ be the homomorphism defined such that

$$\theta\left(n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz\right) = n$$

for all $n_1, n_2, n_3, n \in \mathbb{Z}$. Now

$$n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz \in B_1(K;\mathbb{Z}) \quad \text{if and only if} \quad n = 0.$$

It follows that $B_1(K;\mathbb{Z}) = \ker\theta$. Therefore the homomorphism θ induces an isomorphism from $H_1(K;\mathbb{Z})$ to $\mathbb{Z}$, where $H_1(K;\mathbb{Z}) = Z_1(K;\mathbb{Z})/B_1(K;\mathbb{Z})$. Indeed $H_1(K;\mathbb{Z}) = \{n[z] : n \in \mathbb{Z}\}$, where $z = \rho_7 + \rho_8 + \rho_9$ and $[z]$ denotes the homology class of the 1-cycle z.

It is a straightforward exercise to verify that

$$B_0(K;\mathbb{Z}) = \left\{\sum_{k=1}^{6} r_k\langle P_k\rangle : r_k \in \mathbb{Z} \text{ for } k = 1, 2, \ldots, 6 \text{ and } \sum_{k=1}^{6} r_k = 0\right\}.$$

It follows from this that $H_0(K;\mathbb{Z}) \cong \mathbb{Z}$.

7 The Homology Groups of Filled Polygons

7.1 The Homology of a Simple Polygonal Chain

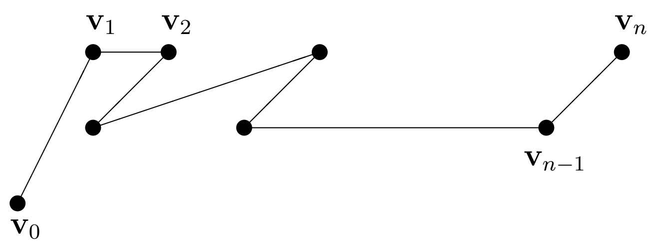
Definition We define a *simple polygonal chain* $\mathbf{v}_0\,\mathbf{v}_1,\ldots,\mathbf{v}_n$ of length n to be a collection consisting of $n+1$ vertices and n edges, where the vertices may be ordered in a finite sequence $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_n$ satisfying the following conditions:—

(i) the vertices $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_n$ of the polygonal chain are distinct;

(ii) the edges of the polygonal chain are

$$\mathbf{v}_0\,\mathbf{v}_1,\;\; \mathbf{v}_1\,\mathbf{v}_2,\ldots,\;\; \mathbf{v}_{n-1}\,\mathbf{v}_n;$$

(iii) two distinct edges of the polygon intersect if and only if they have an endpoint in common, in which case their intersection consists only of that common endpoint.



Lemma 7.1 *Let $\mathbf{v}_0\,\mathbf{v}_1,\ldots,\mathbf{v}_n$ be a simple polygonal chain of length n, let K be the one-dimensional simplicial complex consisting of the vertices $\mathbf{v}_i$ for $i=0,1,2,\ldots,n$ and the edges $\mathbf{v}_{i-1}\,\mathbf{v}_i$ for $i=1,2,\ldots,n$, and let R be a unital ring. Then $H_0(K;R)\cong R$, and $H_q(K;R)=0$ when $q>0$.*

Proof The definitions of the groups $Z_q(K;R)$ and $B_q(K;R)$ of q-cycles and q-boundaries ensure that $Z_q(K;R)=0$ and $B_q(K;R)=0$ when $q>1$. It follows that $H_q(K;R)=0$ when $q>1$.

Let c be a 1-chain of the simplicial complex K with coefficients in the ring R. Then there exist uniquely-determined elements $r_1,r_2,\ldots,r_n$ of the coefficient ring R such that $c=\sum_{i=1}^{n} r_i\langle\mathbf{v}_{i-1}\,\mathbf{v}_i\rangle$. Then

$$\begin{aligned}\partial_1 c &= \sum_{i=1}^{n} r_i\partial_1(\langle\mathbf{v}_{i-1}\,\mathbf{v}_i\rangle) = \sum_{i=1}^{n} r_i(\langle\mathbf{v}_i\rangle - \langle\mathbf{v}_{i-1}\rangle)\\ &= -r_1\langle\mathbf{v}_0\rangle + \sum_{i=1}^{n-1}(r_i - r_{i+1})\langle\mathbf{v}_i\rangle + r_n\langle\mathbf{v}_n\rangle.\end{aligned}$$

Thus if $\partial_1 c = 0$ then $r_1 = r_n = 0_R$, where 0_R denotes the zero element of the coefficient ring R, and $r_{i-1} = r_i$ for $i = 1, 2, \ldots, n$. It follows that if $\partial_1 c = 0$ then $r_i = 0_R$ for $i = 1, 2, \ldots, n$, and therefore $c = 0$. Thus $Z_1(K; R) = 0$. It follows that $H_1(K; R) = 0$.

Let z be a 0-chain of the simplicial complex K with coefficients in the ring R. Then there exist elements $s_0, s_1, \ldots, s_n$ of the coefficient ring R such that $z = \sum\limits_{i=0}^{n} s_i \langle \mathbf{v}_i \rangle$. Let $c = \sum\limits_{i=1}^{n} r_i \langle \mathbf{v}_{i-1}\,\mathbf{v}_i \rangle$, where $r_1, r_2, \ldots, r_n \in R$. The calculation in the previous paragraph ensures that $z = \partial_1 c$ if and only if $s_0 = -r_1$, $s_i = r_i - r_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $s_n = r_n$. It then follows that if $z = \partial_1 c$ then $\sum\limits_{i=0}^{n} s_i = 0_R$. Conversely if $\sum\limits_{i=0}^{n} s_i = 0_R$, then $r_1, r_2, \ldots, r_n$ can be determined such that $r_i = -\sum\limits_{j=0}^{i-1} s_j$ for $i = 1, 2, \ldots, n$. Then $-r_1 = s_0$, $r_i - r_{i+1} = s_i$ for $1, 2, \ldots, n-1$ and $r_n = s_n$, and therefore $z = \partial_1 c$. It follows that $z \in B_0(K; R)$ if and only if $\sum\limits_{i=0}^{n} s_i = 0_R$.

Now $Z_0(K; R) = C_0(K; R)$, and therefore $H_0(K; R) \cong C_0(K; R)/B_0(K; R)$. Let $\varepsilon\colon C_0(K; R) \to R$ be the R-module homomorphism defined such that

$$\varepsilon\left(\sum_{i=0}^{n} s_i \langle \mathbf{v}_i \rangle\right) = \sum_{i=0}^{n} s_i.$$

Then $\ker \varepsilon = B_0(K; R)$. It follows that

$$H_0(K; R) = C_0(K; R)/B_0(K; R) = C_0(K; R)/\ker \varepsilon \cong R,$$

as required. ∎

7.2 The Homology of a Simple Polygon

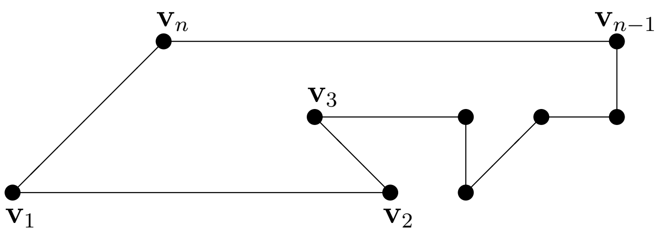
Definition We define a *simple polygon* with n sides of length n to be a collection consisting of n vertices $\mathbf{v}_1\,\mathbf{v}_2, \ldots, \mathbf{v}_n$ and n edges, where $n \leq 3$ and where the vertices may be ordered in a finite sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ satisfying the following conditions:—

(i) the vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of the polygon are distinct;

(ii) the edges of the polygon are

$$\mathbf{v}_1\,\mathbf{v}_2,\ \ \mathbf{v}_2\,\mathbf{v}_3, \ldots,\ \ \mathbf{v}_{n-1}\,\mathbf{v}_n \ \text{ and } \ \mathbf{v}_n\,\mathbf{v}_1;$$

(iii) two distinct edges of the polygon intersect if and only if they have an endpoint in common, in which case their intersection consists only of that common endpoint.



Lemma 7.2 *Let K be the one-dimensional simplicial complex consisting of the vertices and edges of a simple polygon with n sides, where $n \geq 3$, and let R be a unital ring. Then $H_0(K; R) \cong R$, $H_1(K; R) \cong R$ and $H_q(K; R) = 0$ when $q > 0$.*

Proof We order the vertices of the simple polygon in the sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ so that the edges of the polygon, in order round the polygon, are

$$\mathbf{v}_1\,\mathbf{v}_2,\ \ \mathbf{v}_2\,\mathbf{v}_3, \ldots,\ \ \mathbf{v}_{n-1}\,\mathbf{v}_n \ \text{ and } \ \mathbf{v}_n\,\mathbf{v}_1.$$

The definitions of the groups $Z_q(K; R)$ and $B_q(K; R)$ of q-cycles and q-boundaries ensure that $Z_q(K; R) = 0$ and $B_q(K; R) = 0$ when $q > 1$. It follows that $H_q(K; R) = 0$ when $q > 1$.

Let c be a 1-chain of the simplicial complex K with coefficients in the ring R. Then there exist uniquely determined elements $r_1, r_2, \ldots, r_n$ of the coefficient ring R such that

$$c = r_1 \langle \mathbf{v}_n\,\mathbf{v}_1 \rangle + \sum_{i=2}^{n} r_i \langle \mathbf{v}_{i-1}\,\mathbf{v}_i \rangle.$$

Then

$$\begin{aligned}
\partial_1 c &= r_1 \partial_1(\langle \mathbf{v}_n\,\mathbf{v}_1 \rangle) + \sum_{i=2}^{n} r_i \partial_1(\langle \mathbf{v}_{i-1}\,\mathbf{v}_i \rangle) \\
&= r_1(\langle \mathbf{v}_1 \rangle - \langle \mathbf{v}_n \rangle) + \sum_{i=2}^{n} r_i(\langle \mathbf{v}_i \rangle - \langle \mathbf{v}_{i-1} \rangle) \\
&= \sum_{i=1}^{n-1} (r_i - r_{i+1}) \langle \mathbf{v}_i \rangle + (r_n - r_1) \langle \mathbf{v}_n \rangle.
\end{aligned}$$

Thus if $\partial_1 c = 0$ then $r_i = r_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $r_n = r_1$. It follows that $\partial_1 c = 0$ if and only if

$$r_1 = r_2 = \cdots = r_n,$$

and therefore

$$Z_1(K; R) = \{r\gamma : r \in R\},$$

where

$$\gamma = \sum_{i=2}^{n} \langle \mathbf{v}_{i-1}\, \mathbf{v}_i \rangle + \langle \mathbf{v}_n \mathbf{v}_1 \rangle.$$

Now $B_1(K; R) = 0$ because a one-dimensional simplicial complex cannot have any non-zero 1-boundaries. It follows that $H_1(K; R) = Z_1(K; R) \cong R$.

Let z be a 0-chain of the simplicial complex K with coefficients in the ring R. Then there exist elements $s_1, s_2, \ldots, s_n$ of the coefficient ring R such that $z = \sum\limits_{i=1}^{n} s_i \langle \mathbf{v}_i \rangle$. Let

$$c = r_1 \langle \mathbf{v}_n\, \mathbf{v}_1 \rangle + \sum_{i=2}^{n} r_i \langle \mathbf{v}_{i-1}\, \mathbf{v}_i \rangle,$$

where $r_1, r_2, \ldots, r_n \in R$. The calculation in the previous paragraph ensures that $z = \partial_1 c$ if and only if $s_i = r_i - r_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $s_n = r_n - r_1$. It then follows that if $z = \partial_1 c$ then $\sum\limits_{i=1}^{n} s_i = 0_R$. Conversely if $\sum\limits_{i=1}^{n} s_i = 0_R$, then $r_1, r_2, \ldots, r_n$ can be determined such that $r_1 = 0_R$ and $r_i = -\sum\limits_{j=1}^{i-1} s_j$ for $i = 2, 3, \ldots, n$. Then $r_i - r_{i+1} = s_i$ for $i = 1, 2, \ldots, n-1$ and

$$r_n - r_1 = -\sum_{j=1}^{n-1} s_j = s_n,$$

and therefore $z = \partial_1 c$. It follows that $z \in B_0(K; R)$ if and only if $\sum\limits_{i=1}^{n} s_i = 0_R$.

Now $Z_0(K; R) = C_0(K; R)$, and therefore $H_0(K; R) \cong C_0(K; R)/B_0(K; R)$. Let $\varepsilon\colon C_0(K; R) \to R$ be the R-module homomorphism defined such that

$$\varepsilon\left(\sum_{i=1}^{n} s_i \langle \mathbf{v}_i \rangle\right) = \sum_{i=1}^{n} s_i.$$

Then $\ker \varepsilon = B_0(K; R)$. It follows that

$$H_0(K; R) = C_0(K; R)/B_0(K; R) = C_0(K; R)/\ker \varepsilon \cong R,$$

as required. ∎

7.3 The Two-Dimensional Homology Group of a Simplicial Complex triangulating a Region of the Plane

Proposition 7.3 *Let K be a 2-dimensional simplicial complex whose polyhedron $|K|$ is a closed bounded region of the plane, and let R be a unital ring. Then $Z_2(K;R) = 0$, and thus $H_2(K;R) = 0$.*

Proof Let c be a non-zero 2-chain of K with coefficients in the ring R. Then c is expressible in the form

$$c = \sum_{i=1}^{m} r_i \langle \mathbf{v}_0^{(i)}\, \mathbf{v}_1^{(i)}\, \mathbf{v}_2^{(i)} \rangle$$

so as to satisfy the following conditions: the coefficient r_i is a non-zero element of the coefficient ring R for $i = 1, 2, \ldots, m$; the vertices $\mathbf{v}_0^{(i)}$, $\mathbf{v}_1^{(i)}$ and $\mathbf{v}_2^{(i)}$ are distinct and span a 2-simplex (or triangle) τ_i of K for $i = 1, 2, \ldots, m$; the 2-simplices $\tau_1, \tau_2, \ldots, \tau_m$ determined in this fashion are distinct.

Let $\rho_1, \rho_2, \ldots, \rho_n$ denote the edges of the triangles $\tau_1, \tau_2, \ldots, \tau_m$, where $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, and let $\mathbf{w}_0^{(j)}$ and $\mathbf{w}_1^{(j)}$ be the endpoints of the edge ρ_j for $j = 1, 2, \ldots, n$. Then there exist uniquely-determined elements $s_1, s_2, \ldots, s_n$ of the coefficient ring R such that

$$\partial_2 c = \sum_{j=1}^{n} s_j \langle \mathbf{w}_0^{(j)}\, \mathbf{w}_1^{(j)} \rangle.$$

The coefficients $s_1, s_2, \ldots, s_n$ of $\partial_2 c$ in this expression need not all be non-zero, but we shall show that at least one of these coefficients is non-zero.

Now the fact that the triangles of K are all contained in the plane ensures that no edge of K can form part of the boundary of more than two of the triangles $\tau_1, \tau_2, \ldots, \tau_m$. Moreover the union of these triangles is a closed bounded set in the plane and therefore has a non-empty boundary that incorporates at least three of the edges $\rho_1, \rho_2, \ldots, \rho_n$. Suppose that ρ_j is contained in the boundary of $\tau_1 \cup \tau_2 \cup \cdots \cup \tau_m$. Then this edge is an edge of exactly one of the triangles $\tau_1, \tau_2, \ldots, \tau_m$. Suppose that ρ_j is an edge of τ_i. Then $s_j = \pm r_i$, and therefore $s_j \neq 0$. We have thus shown that if c is a non-zero 2-chain of K with coefficients in R then $\partial_2 c$ is a non-zero 1-chain of K. Therefore $Z_2(K;R) = 0$, and thus $H_2(K;R) = 0$, as required. ∎

7.4 Attaching Triangles to Two-Dimensional Simplicial Complexes

Lemma 7.4 *Let K be a 2-dimensional simplicial complex, and let R be a unital ring. Let τ be a triangle of K, and let L be the subcomplex of K con-*

sisting of all the other triangles of K, together with their edges and vertices. Suppose that $\tau \cap |L|$ consists of the union of two edges of the triangle τ. Then the homomorphism $i_*: H_q(L;R) \to H_q(K;R)$ of homology groups induced by the inclusion map $i: L \hookrightarrow K$ is an isomorphism for all non-negative integers q.

Proof Let the vertices of the triangle τ be $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{u}\,\mathbf{v}$ and $\mathbf{u}\,\mathbf{w}$ are the two edges of τ that belong to the subcomplex L.

If $q > 2$ then $H_q(L;R) = 0$ and $H_q(K;R) = 0$, because the simplicial complexes K and L are of dimension at most 2, and thus there is nothing to prove.

We show first that $Z_2(K;R) = Z_2(L;R)$. Given any 2-cycle z_2 of K with coefficients in R, there exists a 2-chain $\hat{c}_2$ of L with coefficients in R and a uniquely-determined element of R such that $z_2 = \hat{c}_2 + r\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle$. Now $\partial_2 z_2 = 0$. It follows that

$$r\langle \mathbf{v}\,\mathbf{w}\rangle = r\partial_2(\langle\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle) + r\langle \mathbf{u}\,\mathbf{w}\rangle - r\langle \mathbf{u}\,\mathbf{v}\rangle = -\partial_2\hat{c}_2 + r\langle \mathbf{u}\,\mathbf{w}\rangle - r\langle \mathbf{u}\,\mathbf{v}\rangle.$$

Moreover $\partial_2\hat{c}_2 \in C_1(L;R)$ and $\langle \mathbf{u}\,\mathbf{v}\rangle, \langle \mathbf{u}\,\mathbf{w}\rangle \in C_1(L;R)$, and therefore

$$r\langle \mathbf{v}\,\mathbf{w}\rangle \in C_1(L;R).$$

But $\mathbf{v}\,\mathbf{w}$ is not an edge of L. It follows that $r = 0$, and thus $z_2 \in Z_2(L;R)$. Thus $Z_2(K;R) = Z_2(L;R)$. Now $B_2(L;R) = 0$ and $B_2(K;R) = 0$, and therefore $H_2(L;R) = Z_2(L;R)$ and $H_2(K;R) = Z_2(K;R)$. It follows that $i_*: H_2(L;R) \to H_2(K;R)$ is an isomorphism.

We now show that the homomorphism $i_*: H_1(L;R) \to H_1(K;R)$ is injective. Let $\hat{z}$ be a 1-cycle of L with coefficients in R, and let $[\hat{z}]_L$ denote the homology class of $\hat{z}$ in $H_1(L;R)$. Suppose that $i_*([\hat{z}]_L) = 0$. Then $\hat{z}_L$ is a 1-boundary of the larger simplicial complex K, and thus there exists some 2-chain c_2 of K with coefficients in R such that $\hat{z} = \partial_2 c_2$. Moreover there then exists some element r of R and a 2-chain $\hat{c}_2$ of L such that $c_2 = \hat{c}_2 + r\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle$. But then

$$\hat{z} = \partial_2 c_2 = \partial_2\hat{c}_2 + r\langle \mathbf{u}\,\mathbf{v}\rangle + r\langle \mathbf{v}\,\mathbf{w}\rangle - r\langle \mathbf{u}\ \mathbf{w}\rangle,$$

Now $\partial_2\hat{c}_2 \in C_1(L;R)$ and $\langle \mathbf{u}\ \mathbf{v}\rangle, \langle \mathbf{u}\ \mathbf{w}\rangle \in C_1(L;R)$, and therefore

$$r\langle \mathbf{v}\,\mathbf{w}\rangle \in C_1(L;R).$$

But $\mathbf{v}\,\mathbf{w}$ is not an edge of L, and therefore $r\langle \mathbf{v}\,\mathbf{w}\rangle$ cannot be a 1-chain of L unless $r = 0$. Therefore $c_2 = \hat{c}_2$. But then $\hat{z} = \partial_2\hat{c}_2$, where $\hat{c}_2 \in C_2(L;R)$, and therefore $\hat{z} \in B_1(L;R)$ and thus $[\hat{z}]_L = 0$ in $H_1(L;R)$. We conclude from this that the homomorphism $i_*: H_1(L;R) \to H_1(K;R)$ is injective.

We now show that the homomorphism $i_*: H_1(L; R) \to H_1(K; R)$ is surjective. Let z be a 1-cycle of K with coefficients in the ring R. Then there exist a 1-chain c_1 of L with coefficients in R and a uniquely-determined element r of the coefficient ring R such that

$$z = c_1 + r\langle \mathbf{v}\,\mathbf{w} \rangle.$$

But then $z = \hat{z} + r\partial_2(\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w} \rangle)$ where

$$\hat{z} = c_1 - r\langle \mathbf{u}\,\mathbf{v} \rangle + r\langle \mathbf{u}\,\mathbf{w} \rangle.$$

Then $c_1 \in C_1(L; R)$ and $\langle \mathbf{u}\,\mathbf{v} \rangle, \langle \mathbf{u}\,\mathbf{w} \rangle \in C_1(L; R)$, and therefore $\hat{z} \in C_1(L; R)$. Also

$$\partial_1 \hat{z} = \partial_1 z - r\partial_1(\partial_2(\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w} \rangle)) = 0.$$

It follows that $\hat{z} \in Z_1(L; R)$. Also $z - \hat{z} \in B_1(K; R)$, and therefore $[z]_K = [\hat{z}]_K$, where $[z]_K$ and $[\hat{z}]_K$ denote the homology classes of z and $\hat{z}$ respectively in $H_1(K; R)$. Now $[\hat{z}]_K = i_*([\hat{z}]_L)$, where $[\hat{z}]_L$ denotes the homology class of $\hat{z}$ in $H_1(L; R)$. It follows that $[z]_K \in i_*(H_1(L; R))$. We have thus proved that the homomorphism $i_*: H_1(L; R) \to H_1(K; R)$ is surjective. This homomorphism was earlier shown to be injective. Therefore it is an isomorphism.

It remains to prove that $i_*: H_0(L; R) \to H_0(K; R)$ is an isomorphism. Now every vertex of K is a vertex of L. It follows that $C_0(K; R) = C_0(L; R)$. Let c_1 be a 1-chain of K with coefficients in R. Then the exists a 1-chain $\hat{c}_1$ of L with coefficients in R and an element r of R such that $c_1 = \hat{c}_1 + r\langle \mathbf{v}\,\mathbf{w} \rangle$. Let

$$\tilde{c}_1 = \hat{c}_1 + r\langle \mathbf{u}\,\mathbf{w} \rangle - r\langle \mathbf{u}\,\mathbf{v} \rangle.$$

Then $\tilde{c}_1 \in C_1(L; R)$ and

$$\partial_1 \tilde{c}_1 = \partial_1 \hat{c}_1 + r\langle \mathbf{w} \rangle - r\langle \mathbf{v} \rangle = \partial_1 c_1.$$

It follows that $\partial c_1 \in B_1(L; R)$. We conclude that $B_0(K; R) = B_0(L; R)$. Now $H_0(L; R) = C_0(L; R)/B_0(L; R)$, because $Z_0(L; R) = C_0(L; R)$, and similarly $H_0(K; R) = C_0(K; R)/B_0(K; R)$. It follows that the homomorphism $i_*: H_0(L; R) \to H_0(K; R)$ is an isomorphism. This completes the proof. ∎

Lemma 7.5 *Let K be a 2-dimensional simplicial complex, and let R be a unital ring. Let τ be a triangle of K, and let L be the subcomplex of K consisting of all the other triangles of K, together with their edges and vertices. Suppose that $\tau \cap |L|$ consists of a single edge of the triangle τ. Then the homomorphism $i_*: H_q(L; R) \to H_q(K; R)$ of homology groups induced by the inclusion map $i: L \hookrightarrow K$ is an isomorphism for all non-negative integers q.*

Proof Let the vertices of the triangle τ be $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{u}$ and $\mathbf{w}$ are the endpoints of the edge of τ that belongs to the subcomplex L. Then the vertex $\mathbf{v}$ does not belong to L.

If $q > 2$ then $H_q(L; R) = 0$ and $H_q(K; R) = 0$, because the simplicial complexes K and L are of dimension at most 2, and thus there is nothing to prove.

We show first that $Z_2(K; R) = Z_2(L; R)$. Given any 2-cycle z_2 of K with coefficients in R, there exists a 2-chain $\hat{c}_2$ of L with coefficients in R and a uniquely-determined element of R such that $z_2 = \hat{c}_2 + r\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle$. Now $\partial_2 z_2 = 0$. It follows that

$$r\langle \mathbf{u}\,\mathbf{v}\rangle + r\langle \mathbf{v}\,\mathbf{w}\rangle = r\partial_2(\langle\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle) + r\langle \mathbf{u}\,\mathbf{w}\rangle = -\partial_2\hat{c}_2 + r\langle \mathbf{u}\,\mathbf{w}\rangle.$$

Moreover $\partial_2\hat{c}_2 \in C_1(L; R)$ and $\langle \mathbf{u}\,\mathbf{w}\rangle \in C_1(L; R)$, and therefore

$$r\langle \mathbf{u}\,\mathbf{v}\rangle + r\langle \mathbf{v}\,\mathbf{w}\rangle \in C_1(L; R).$$

But $\mathbf{u}\,\mathbf{v}$ and $\mathbf{v}\,\mathbf{w}$ are not edges of L. It follows that $r = 0$, and thus $z_2 \in Z_2(L; R)$. Thus $Z_2(K; R) = Z_2(L; R)$. Now $B_2(L; R) = 0$ and $B_2(K; R) = 0$, and therefore $H_2(L; R) = Z_2(L; R)$ and $H_2(K; R) = Z_2(K; R)$. It follows that $i_*: H_2(L; R) \to H_2(K; R)$ is an isomorphism.

Next we show that the homomorphism $i_*: H_1(L; R) \to H_1(K; R)$ is injective. Let $\hat{z}$ be a 1-cycle of L with coefficients in R, and let $[\hat{z}]_L$ denote the homology class of $\hat{z}$ in $H_1(L; R)$. Suppose that $i_*([\hat{z}]_L) = 0$. Then $\hat{z}_L$ is a 1-boundary of the larger simplicial complex K, and thus there exists some 2-chain c_2 of K with coefficients in R such that $\hat{z} = \partial_2 c_2$. Moreover there then exists some element r of R and a 2-chain $\hat{c}_2$ of L such that $c_2 = \hat{c}_2 + r\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle$. But then

$$\hat{z} = \partial_2 c_2 = \partial_2\hat{c}_2 + r\langle \mathbf{u}\,\mathbf{v}\rangle + r\langle \mathbf{v}\,\mathbf{w}\rangle - r\langle \mathbf{u}\ \mathbf{w}\rangle,$$

Now $\partial_2\hat{c}_2 \in C_1(L; R)$ and $\langle \mathbf{u}\ \mathbf{w}\rangle \in C_1(L; R)$, and therefore

$$r\langle \mathbf{u}\,\mathbf{v}\rangle + r\langle \mathbf{v}\,\mathbf{w}\rangle \in C_1(L; R).$$

But $\mathbf{u}\,\mathbf{v}$ and $\mathbf{v}\,\mathbf{w}$ are not edges of L, and therefore $r\langle \mathbf{u}\,\mathbf{v}\rangle + r\langle \mathbf{v}\,\mathbf{w}\rangle$ cannot be a 1-chain of L unless $r = 0$. Therefore $c_2 = \hat{c}_2$. But then $\hat{z} = \partial_2\hat{c}_2$, where $\hat{c}_2 \in C_2(L; R)$, and therefore $\hat{z} \in B_1(L; R)$ and thus $[\hat{z}]_L = 0$ in $H_1(L; R)$. We conclude from this that the homomorphism $i_*: H_1(L; R) \to H_1(K; R)$ is injective.

We now show that the homomorphism $i_*: H_1(L; R) \to H_1(K; R)$ is surjective. Let z be a 1-cycle of K with coefficients in the ring R. Then there exist a 1-chain c_1 of L with coefficients in R and uniquely-determined elements r_1 and r_2 of the coefficient ring R such that

$$z = c_1 + r_1\langle \mathbf{u}\,\mathbf{v}\rangle + r_2\langle \mathbf{v}\,\mathbf{w}\rangle.$$

Then

$$0 = \partial_1 z = \partial_1 c_1 - r_1\langle \mathbf{u}\rangle + r_2\langle \mathbf{w}\rangle + (r_1 - r_2)\langle \mathbf{v}\rangle.$$

Now $\partial_1 c_1 \in C_0(L;R)$ and $\langle \mathbf{u}\rangle, \langle \mathbf{w}\rangle \in C_0(L:R)$. It follows that $(r_1-r_2)\langle \mathbf{v}\rangle \in C_0(L;R)$. But $\mathbf{v}$ is not itself a vertex of L. It follows that $(r_1 - r_2)\langle \mathbf{v}\rangle = 0$, and therefore $r_1 = r_2$. Let $r = r_1 = r_2$. Then $z = \hat{z} + r\partial_2(\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle)$ where $\hat{z} = c_1 + r\langle \mathbf{u}\,\mathbf{w}\rangle$. Now $\hat{z} \in C_1(L;R)$ and

$$\partial_1\hat{z} = \partial_1 z - r\partial_1(\partial_2(\langle \mathbf{u}\,\mathbf{v}\,\mathbf{w}\rangle)) = 0.$$

It follows that $\hat{z} \in Z_1(L;R)$. Also $z - \hat{z} \in B_1(K;R)$, and therefore $[z]_K = [\hat{z}]_K$, where $[z]_K$ and $[\hat{z}]_K$ denote the homology classes of z and $\hat{z}$ respectively in $H_1(K;R)$. Now $[\hat{z}]_K = i_*([\hat{z}]_L)$, where $[\hat{z}]_L$ denotes the homology class of $\hat{z}$ in $H_1(L;R)$. It follows that $[z]_K \in i_*(H_1(L;R))$. We have thus proved that the homomorphism $i_*\colon H_1(L;R) \to H_1(K;R)$ is surjective. This homomorphism was earlier shown to be injective. Therefore it is an isomorphism.

It remains to prove that $i_*\colon H_0(L;R) \to H_0(K;R)$ is an isomorphism. First we prove that this homomorphism is injective. Now $Z_0(L;R) = C_0(L;R)$ and $Z_0(K;R) = C_0(K;R)$. Let $\hat{z}_0$ be a 0-chain of L with coefficients in R, and let $[\hat{z}_0]_L$ and $[\hat{z}_0]_K$ denote the homology classes of $\hat{z}_0$ in $H_0(L;R)$ and $H_0(K;R)$ respectively. Then $i_*([\hat{z}_0]_L) = [\hat{z}_0]_K$.

Suppose that $i_*([\hat{z}_0]_L) = 0$. Then $[\hat{z}_0]_K = 0$, and therefore $\hat{z}_0 \in B_0(K;R)$. Then there exists a 1-chain c_1 of K such that $\hat{z}_0 = \partial_1 c_1$. Moreover there exist a 1-chain $\hat{c}_1$ of L and elements r_1 and r_2 of R such that

$$c_1 = \hat{c}_1 + r_1\langle \mathbf{u}\,\mathbf{v}\rangle + r_2\langle \mathbf{v}\,\mathbf{w}\rangle.$$

Then

$$\hat{z}_0 = \partial_1 c_1 = \partial_1\hat{c}_1 - r_1\langle \mathbf{u}\rangle - (r_2 - r_1)\langle \mathbf{v}\rangle + r_2\langle \mathbf{w}\rangle.$$

But then

$$(r_2 - r_1)\langle \mathbf{v}\rangle = -\hat{z}_0 + \partial_1\hat{c}_1 - r_1\langle \mathbf{u}\rangle + r_2\langle \mathbf{w}\rangle.$$

and therefore $(r_2 - r_1)\langle \mathbf{v}\rangle \in C_0(L;R)$. But the vertex $\mathbf{v}$ does not belong to the subcomplex L. It follows that $r_1 = r_2$. But then

$$\hat{z}_0 = \partial_1\hat{c}_1 + r_1\langle \mathbf{w}\rangle - r_1\langle \mathbf{u}\rangle = \partial_1\left(\hat{c}_1 + r_1\langle \mathbf{u}\,\mathbf{w}\rangle\right),$$

and therefore $\hat{z}_0 \in B_0(L;R)$. It follows that $[\hat{z}_0]_L = 0$. We conclude that $i_*\colon H_0(L;R) \to H_0(K;R)$ is injective.

Now let z_0 be a 0-chain of K with coefficients in R, and let $[z_0]_K$ denote the homology class of z_0 in $H_0(K;R)$. Then there exists a 0-chain $\hat{z}_0$ of L with coefficients in R and an element r of R such that $z = \hat{z}_0 + r\langle \mathbf{v}\rangle$. Let $\tilde{z}_0 = \hat{z}_0 + r\langle \mathbf{u}\rangle$. Then $\tilde{z}_0 \in C_0(L;R)$, and $z = \tilde{z} + r\,\partial_1(\langle \mathbf{u}\,\mathbf{v}\rangle)$. It follows that $[z]_K = [\tilde{z}_0]_K = i_*([\tilde{z}_0]_L)$. This shows that $i_*: H_0(L;R) \to H_0(K;R)$ is surjective. We have already shown that this homomorphism is injective. It follows that the homomorphism is an isomorphism. This completes the proof. ∎

7.5 Homology of a Planar Region bounded by a Simple Polygon

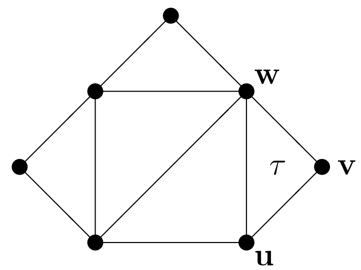
The next proposition enables us to prove results about 2-dimensional simplicial complexes triangulating regions of the plane bounded by simple polygons by induction on the number of triangles in the complex.

Proposition 7.6 *Let K be a 2-dimensional simplicial complex with more than one triangle whose polyhedron $|K|$ is a closed bounded region of the plane bounded by a simple polygon. Then there exists a triangle τ of K and a subcomplex L of K such that the following conditions are satisfied:*

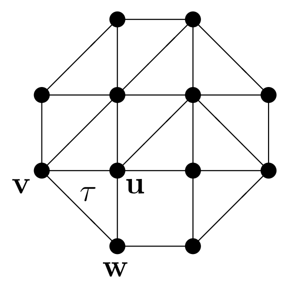
- (i) *the simplicial complex K consists of the simplices of the subcomplex L, the triangle τ, and the edges and vertices of τ;*
- (ii) *The polyhedron $|L|$ of the subcomplex L is bounded by a simple polygon;*
- (iii) *the intersection $\tau \cap |L|$ of τ with the polyhedron of L is either a single edge of τ or else is the union of two edges of τ.*

Proof We say that a vertex $\mathbf{v}$ of K is an *boundary vertex* of K if it belongs to the bounding polygon of $|K|$, and we say that an edge $\mathbf{v}\,\mathbf{w}$ of K is an *boundary edge* of K if it is contained in the bounding polygon of $|K|$. Vertices of K that are not boundary vertices are said to be *interior vertices*, and edges of K that are not boundary edges are said to be *interior edges* of K. The requirement that the boundary of $|K|$ is a simple polygon ensures that each boundary vertex of K is an endpoint of exactly two boundary edges of K. Also every interior edge of K is an edge of exactly two triangles of K, and every boundary edge of K is an edge of exactly one triangle of K. No more than two edges of any triangle of K can be boundary edges of K, because the simplicial complex K contains more than one triangle.

First consider the special case where two edges of some triangle τ of K are boundary edges of K. Let the vertex $\mathbf{v}$ of τ be the common endpoint of the two boundary edges, and let $\mathbf{u}$ and $\mathbf{w}$ be the other two vertices of τ. Also let L be the subcomplex of K consisting of all triangles of K other than the triangle τ, together with all the edges and vertices of these triangles. Then the polyhedron $|L|$ of the subcomplex L is bounded by the simple polygon obtained from the bounding polygon of $|K|$ by replacing the two edges $\mathbf{u}\,\mathbf{v}$ and $\mathbf{v}\,\mathbf{w}$ of this polygon by the single edge $\mathbf{u}\,\mathbf{w}$, thereby excluding the triangle τ from the interior of the resulting polygon. Moreover $\tau \cap |L|$ coincides with the edge $\mathbf{u}\,\mathbf{w}$ of the triangle τ. The conclusions of the proposition are therefore true in this special case.



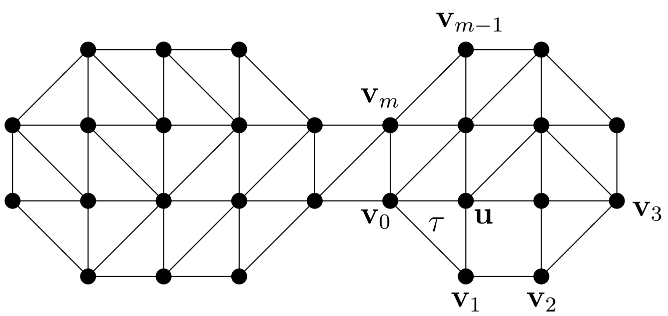
Next we consider the special case when at least one triangle τ of K contains both an boundary edge of K and an internal vertex of K. Let $\mathbf{u}$ denote the vertex of τ that is an internal vertex of K, and let $\mathbf{v}$ and $\mathbf{w}$ denote the vertices of τ that are endpoints of an boundary edge of K. Let L be the subcomplex of K that is the union of the triangles of K other than τ, together with all the vertices and edges of those triangles. Then the polyhedron $|L|$ of the subcomplex L is bounded by the simple polygon obtained from the bounding polygon of $|K|$ by replacing the edge $\mathbf{v}\,\mathbf{w}$ of this bounding polygon by the two edges $\mathbf{v}\,\mathbf{u}$ and $\mathbf{u}\,\mathbf{w}$, thereby excluding the triangle τ from the interior of the resulting polygon. Moreover $\tau \cap |L|$ in this case coincides with the union of the two edges $\mathbf{v}\,\mathbf{u}$ and $\mathbf{u}\,\mathbf{w}$ of the triangle τ. The conclusions of the proposition are therefore true in this special case also.



We complete the proof by showing that, for all simplicial complexes K satisfying the conditions of the proposition, one or other of the special cases already considered is applicable to the simplicial complex K. For this purpose, we consider separately the case when no internal edge of K has endpoints that are both boundary vertices of K and the remaining case when at least one internal edge of K has endpoints that are both boundary points of K.

Thus suppose that no internal edge of K has endpoints that are both boundary vertices of K. The endpoints $\mathbf{v}$ and $\mathbf{w}$ of some boundary edge of K are vertices of a triangle τ of K. Let $\mathbf{u}$ be the third vertex of this triangle. The three edges of the triangle τ cannot all be boundary edges of K, because the simplicial complex K contains more than one triangle. Therefore at least one of the edges of τ must be an internal edge of K. In the case under consideration the endpoints of this internal edge cannot both be boundary vertices of K. It follows that the vertex $\mathbf{u}$ must be an internal vertex of K, and thus the simplicial complex K contains a triangle τ that has both a boundary edge $\mathbf{v}\,\mathbf{w}$ of K and an internal vertex $\mathbf{u}$ of K. It then follows from a case previously considered that the conclusions of the proposition are true in the case under consideration.

It only remains to prove that the conclusions of the proposition are true in the case when at least one internal edge of K has endpoints that are both boundary vertices of K. In this case there exists a positive integer m which is the smallest positive integer for which there exists a finite sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ consisting of $m+1$ boundary vertices of K, where the the edge $\mathbf{v}_{i-1}\,\mathbf{v}_i$ is a boundary edge of K for $i = 1, 2, \ldots, m$ and the edge $\mathbf{v}_0\,\mathbf{v}_m$ is an interior edge of K. There then exists a unique triangle τ of K whose vertices include both $\mathbf{v}_0$ and $\mathbf{v}_1$. Let $\mathbf{u}$ be the third vertex of the triangle τ.



The criterion that determines the value of m ensures that there cannot exist any integer i satisfying $3 \leq i \leq m$ for which $\mathbf{v}_1$ and $\mathbf{v}_i$ are the endpoints of an interior edge of K. It follows that the vertex $\mathbf{u}$ of the triangle τ cannot coincide with $\mathbf{v}_i$ for any integer i satisfying $3 \leq i \leq m$. Also the vertex $\mathbf{u}$ of the triangle τ cannot coincide with either of the vertices $\mathbf{v}_0$ or $\mathbf{v}_1$. Thus if $\mathbf{u} = \mathbf{v}_i$ for some integer i satisfing $0 \leq i \leq m$ then $i = 2$ and thus $\mathbf{u} = \mathbf{v}_2$. But then the vertices of the triangle τ are $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, and therefore two of the edges of the triangle τ are boundary edges of K. A case previously considered therefore ensures that the conclusions of the proposition are true in the case when $\mathbf{u} = \mathbf{v}_2$.

Now the interior of the triangle τ lies inside the simple polygon whose vertices are $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ and whose edges are the edges $\mathbf{v}_{i-1}\,\mathbf{v}_i$ for $i = 1, 2, \ldots, m$ together with the edge $\mathbf{v}_m\,\mathbf{v}_0$, whereas all boundary vertices of K apart from $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ lie outside this simple polygon. It follows that the vertex $\mathbf{u}$ cannot coincide with any boundary vertex of K other than the vertex $\mathbf{v}_2$. Thus if $\mathbf{u} \neq \mathbf{v}_2$ then $\mathbf{u}$ must be an interior vertex of K. But then the simplicial complex K has a triangle τ with both an boundary edge and an interior vertex, and a case previously considered establishes that the conclusions of the proposition are true in this case also. We have therefore established that the conclusions of the proposition are true in all possible cases, as required. ∎

Theorem 7.7 *Let K be a 2-dimensional simplicial complex whose polyhedron $|K|$ is a closed bounded region of the plane bounded by a simple polygon, and let R be a unital ring. Then $H_0(K; R) \cong R$, $H_1(K; R) = 0$ and $H_2(K; R) = 0$.*

Proof We prove the result by induction on the number of triangles in K.

First consider the case when K consists of a single triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{array}{rcl} Z_0(K;R) &=& C_0(K;R) \\ B_0(K;R) &=& \{r_1\langle\mathbf{u}\rangle + r_2\langle\mathbf{v}\rangle + r_3\langle\mathbf{w}\rangle \in C_0(K;R) : r_1 + r_2 + r_3 = 0_R\}. \\ B_1(K;R) &=& Z_1(K;R) = \{r\,(\langle\mathbf{v}\,\mathbf{w}\rangle - \langle\mathbf{u}\,\mathbf{w}\rangle + \langle\mathbf{u}\,\mathbf{v}\rangle) : r \in R\}, \\ B_2(K;R) &=& Z_2(K;R) = 0. \end{array}$$

Let $\varepsilon\colon C_0(K;R) \to R$ be the homomorphism of R-modules defined such that

$$\varepsilon(r_1\langle\mathbf{u}\rangle + r_2\langle\mathbf{v}\rangle + r_3\langle\mathbf{w}\rangle) = r_1 + r_2 + r_3$$

for all $r_1, r_2, r_3 \in R$. Then the homomorphism ε is surjective, and $\ker\varepsilon = B_0(K;R)$. It follows that

$$H_0(K;R) = C_0(K;R)/B_0(K;R) = C_0(K;R)/\ker\varepsilon \cong R.$$

Also $H_1(K;R) = 0$, because $B_1(K;R) = Z_1(K;R)$, $H_2(K;R) = 0$, because $B_2(K;R) = Z_2(K;R) = 0$, and $H_q(K;R) = 0$ for all $q > 2$ because the simplicial complex K is two-dimensional. This proves the result in the case when the simplicial complex K consists of a single triangle together with all its vertices and edges.

Suppose therefore as our induction hypothesis that the simplicial complex K satisfying the conditions of the proposition has more than one triangle, and that the result holds for all simplicial complexes that satisfy the conditions of the proposition and that have fewer triangles than the simplicial complex K. It follows from Proposition 7.6 that there exists a triangle τ of K and a subcomplex L of K such that the following conditions are satisfied:

(i) the simplicial complex K consists of the simplices of the subcomplex L, the triangle τ, and the edges and vertices of τ;

(ii) The polyhedron $|L|$ of the subcomplex L is bounded by a simple polygon;

(iii) the intersection $\tau \cap |L|$ of τ with the polyhedron of L is either a single edge of τ or else is the union of two edges of τ.

It then follows from the induction hypothesis that $H_1(L; R) = 0$. Now if $\tau \cap |L|$ is the union of two edges of τ then Lemma 7.4 ensures that $i_*: H_q(L; R) \to H_q(K; R)$ is an isomorphism for all non-negative integers q in this case. Otherwise $\tau \cap |L|$ is a single edge of τ and Lemma 7.5 ensures that $i_*: H_q(L; R) \to H_q(K; R)$ is an isomorphism for all non-negative integers q in this case also. The result therefore follows by induction on the number of triangles in the simplicial complex K. ∎

8 General Theorems concerning the Homology of Simplical Complexes

8.1 The Homology of Cone-Shaped Simplicial Complexes

Proposition 8.1 *Let K be a simplicial complex, and let R be an unital ring. Suppose that there exists a vertex $\mathbf{w}$ of K with the following property:*

- *if vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K then so do $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.*

Then $H_0(K; R) \cong R$, and $H_q(K; R)$ is the zero module for all $q > 0$.

Proof Using Lemma 4.2, we see that there is a well-defined R-module homomorphism $D_q\colon C_q(K; R) \to C_{q+1}(K; R)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\langle \mathbf{v} \rangle)) = \langle \mathbf{v} \rangle - \langle \mathbf{w} \rangle$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{k=1}^{s} r_k \langle \mathbf{v}_k \rangle - \left(\sum_{k=1}^{s} r_k \right) \langle \mathbf{w} \rangle = \sum_{k=1}^{s} r_k (\langle \mathbf{v}_k \rangle - \langle \mathbf{w} \rangle) \in B_0(K; R)$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K. It follows that

$$z - \varepsilon(z) \langle w \rangle \in B_0(K; R)$$

for all $z \in C_0(K; R)$, where $\varepsilon\colon C_0(K; R) \to R$ is the R-module homomorphism from $C_0(K; R)$ to R defined such that

$$\varepsilon \left(\sum_{k=1}^{s} r_k \langle \mathbf{v}_k \rangle \right) = \sum_{k=1}^{s} r_k$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K. It follows that $\ker \varepsilon \subset B_0(K; R)$. But

$$\varepsilon(\partial_1(\langle \mathbf{u}, \mathbf{v} \rangle)) = \varepsilon(\langle \mathbf{v} \rangle - \langle \mathbf{u} \rangle) = 0$$

for all edges $\mathbf{u}\,\mathbf{v}$ of K, and therefore $B_0(K; R) \subset \ker \varepsilon$. We conclude therefore that $B_0(K; R) = \ker \varepsilon$.

Now $Z_0(K;R) = C_0(K;R)$ (because $\partial_0: C_0(K;R) \to C_{-1}(K;R)$ is defined to be the zero homomorphism from $C_0(K;R)$ to the zero module $C_{-1}(K;R)$), and therefore

$$H_0(K;R) = C_0(K;R)/B_0(K;R),$$

where $B_0(K;R) = \ker \varepsilon$. It follows that the R-module homomorphism $\varepsilon: C_0(K;R) \to R$ induces a well-defined isomorphism from $H_0(K;R)$ to the coefficient ring R that sends the homology class of $\sum_{k=1}^{s} r_k \langle \mathbf{v}_k \rangle$ to $\sum_{k=1}^{s} r_k$ for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K (see Corollary 1.4).

Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\
&\qquad = \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K;R)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K;R)$, and hence $Z_q(K;R) = B_q(K;R)$. It follows that $H_q(K;R)$ is the zero group for all $q > 0$, as required. ∎

Remark Let K be a simplicial complex. Suppose that there exists a vertex $\mathbf{w}$ of K with the property described in the statement of Proposition 8.1 so that, if vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K span a simplex of K then so do $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Let L be the subcomplex of K consisting of all simplices of K that do not have $\mathbf{v}$ as a vertex, and let $|L|$ be the polyhedron of L. Then the polyhedron $|K|$ is the union of all line segments with one endpoint at $\mathbf{w}$ and the other endpoint in the polyhedron $|L|$ of L. Thus the polyhedron $|K|$ K has the form of a cone with apex $\mathbf{w}$ whose base is the polyhedron $|L|$ of the subcomplex L.

Corollary 8.2 *Let σ be a simplex, let K_σ be the simplicial complex consisting of the simplex σ together with all of its faces, and let R be an unital ring. Then $H_0(K_\sigma;R) \cong R$, and $H_q(K_\sigma;R)$ is the zero module for all $q > 0$.*

Proof The hypotheses of Proposition 8.1 are satisfied for the complex K_σ. ∎

8.2 Simplicial Maps and Induced Homomorphisms

Let K and L be simplicial complexes, and let R be an unital ring. It follows from Proposition 4.5 that any simplicial map $\varphi: K \to L$ between the simplicial complexes K and L induces well-defined homomorphisms $\varphi_q: C_q(K; R) \to C_q(L; R)$ of chain groups, where

$$\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Now $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$ for each integer q. Therefore

$$\varphi_q(Z_q(K; R)) \subset Z_q(L; R) \quad \text{and} \quad \varphi_q(B_q(K; R)) \subset B_q(L; R)$$

for all integers q. It follows that any simplicial map $\varphi: K \to L$ induces well-defined homomorphisms

$$\varphi_*: H_q(K; R) \to H_q(L; R)$$

of homology groups, where $\varphi_*([z]) = [\varphi_q(z)]$ for all q-cycles $z \in Z_q(K; R)$. It is a trivial exercise to verify that if K, L and M are simplicial complexes and if $\varphi: K \to L$ and $\psi: L \to M$ are simplicial maps then the induced homomorphisms of homology groups satisfy $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

8.3 Connectedness and $H_0(K; R)$

Lemma 8.3 *Let K be a simplicial complex. Then K can be partitioned into pairwise disjoint subcomplexes $K_1, K_2, \ldots, K_s$ whose polyhedra are the connected components of the polyhedron $|K|$ of K.*

Proof Let $X_1, X_2, \ldots, X_s$ be the connected components of the polyhedron of K, and, for each j, let K_j be the collection of all simplices σ of K for which $\sigma \subset X_j$. If a simplex belongs to K_j for all j then so do all its faces. Therefore $K_1, K_2, \ldots, K_s$ are subcomplexes of K. These subcomplexes are pairwise disjoint since the connected components $X_1, X_2, \ldots, X_s$ of $|K|$ are pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some j, since σ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup \cdots \cup K_s$ and $|K| = |K_1| \cup |K_2| \cup \cdots \cup |K_s|$, as required. ∎

Let R be an unital ring. The *direct sum* $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ of R-modules $M_1, M_2, \ldots, M_k$ is defined to be the R-module consisting of all k-tuples $(x_1, x_2, \ldots, x_k)$ with $x_i \in M_i$ for $i = 1, 2, \ldots, k$, where

$$(x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$$

and

$$r(x_1, x_2, \ldots, x_k) = (rx_1, rx_2, \ldots, rx_k)$$

for all elements $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of $M_1 \oplus M_2 \oplus \cdots \oplus M_k$, and for all $r \in R$.

Lemma 8.4 *Let K be a simplicial complex, and let R be an unital ring. Suppose that $K = K_1 \cup K_2 \cup \cdots \cup K_s$, where $K_1, K_2, \ldots K_s$ are pairwise disjoint. Then*

$$H_q(K; R) \cong H_q(K_1; R) \oplus H_q(K_2; R) \oplus \cdots \oplus H_q(K_s; R)$$

for all integers q.

Proof We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K; R) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any q-chain c of K with coefficients in the unital ring R can be expressed uniquely as a sum of the form $c = c_1 + c_2 + \cdots + c_s$, where c_j is a q-chain of K_j for $j = 1, 2, \ldots, s$. It follows that

$$C_q(K; R) \cong C_q(K_1; R) \oplus C_q(K_2; R) \oplus \cdots \oplus C_q(K_s; R).$$

Now let $z \in Z_q(K; R)$. We can express z uniquely in the form $z = z_1 + z_2 + \cdots + z_s$, where $z_j \in C_q(K_j; R)$ for $j = 1, 2, \ldots, s$. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + \cdots + \partial_q(z_s),$$

and $\partial_q(z_j)$ is a $(q-1)$-chain of K_j for $j = 1, 2, \ldots, s$. It follows that $\partial_q(z_j) = 0$ for $j = 1, 2, \ldots, s$. Hence each z_j is a q-cycle of K_j, and thus

$$Z_q(K; R) \cong Z_q(K_1; R) \oplus Z_q(K_2; R) \oplus \cdots \oplus Z_q(K_s; R).$$

Now let $b \in B_q(K; R)$. Then $b = \partial_{q+1}(c)$ for some $c \in C_{q+1}(K; R)$. Moreover $c = c_1 + c_2 + \cdots c_s$, where $c_j \in C_{q+1}(K_j; R)$ for $j = 1, 2, \ldots, s$. Thus $b = b_1 + b_2 + \cdots b_s$, where $b_j = \partial_{q+1} c_j$ for $j = 1, 2, \ldots, s$. Moreover $b_j \in B_q(K_j; R)$ for $j = 1, 2, \ldots, s$. We deduce that

$$B_q(K; R) \cong B_q(K_1; R) \oplus B_q(K_2; R) \oplus \cdots \oplus B_q(K_s; R).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu\colon H_q(K_1; R) \oplus H_q(K_2; R) \oplus \cdots \oplus H_q(K_s; R) \to H_q(K; R)$$

which maps $([z_1], [z_2], \ldots, [z_s])$ to $[z_1 + z_2 + \cdots + z_s]$, where $[z_j]$ denotes the homology class of a q-cycle z_j of K_j for $j = 1, 2, \ldots, s$. ∎

Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

Lemma 8.5 *The polyhedron $|K|$ of a simplicial complex K is a connected topological space if and only if any two vertices of K can be joined by an edge path.*

Proof It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required. ∎

Theorem 8.6 *Let K be a simplicial complex and let R be an unital ring. Suppose that the polyhedron $|K|$ of K is connected. Then $H_0(K; R) \cong R$.*

Proof Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ be the vertices of the simplicial complex K. Every 0-chain of K with coefficients in R can be expressed uniquely as a formal sum of the form

$$r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + \cdots + r_s\langle \mathbf{v}_s \rangle$$

for some $r_1, r_2, \ldots, r_s \in R$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K; R) \to R$ defined such that

$$\varepsilon\left(r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + \cdots + r_s\langle \mathbf{v}_s \rangle\right) = r_1 + r_2 + \cdots + r_s.$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K.

Now $\varepsilon(\partial_1(\langle \mathbf{u}, \mathbf{v} \rangle)) = \varepsilon(\langle \mathbf{v} \rangle - \langle \mathbf{u} \rangle) = 0$ whenever $\mathbf{u}$ and $\mathbf{v}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and therefore $B_0(K; R) \subset \ker \varepsilon$.

Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ be vertices of K determining an edge path. Then $\mathbf{w}_{j-1} \mathbf{w}_j$ is an edge of K for $j = 1, 2, \ldots, m$, and

$$\langle \mathbf{w}_m \rangle - \langle \mathbf{w}_0 \rangle = \sum_{j=1}^{m} \Big(\langle \mathbf{w}_j \rangle - \langle \mathbf{w}_{j-1} \rangle \Big) = \partial_1 \left(\sum_{j=1}^{m} \langle \mathbf{w}_{j-1}, \mathbf{w}_j \rangle \right) \in B_0(K; R).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path (Lemma 8.5). We deduce that $\langle \mathbf{v} \rangle - \langle \mathbf{u} \rangle \in B_0(K; R)$ for all vertices $\mathbf{u}$ and $\mathbf{v}$ of K.

Choose a vertex $\mathbf{u} \in K$. Then

$$\sum_{j=1}^{s} r_j \langle \mathbf{v}_j \rangle = \sum_{j=1}^{s} r_j (\langle \mathbf{v}_j \rangle - \langle \mathbf{u} \rangle) + \left(\sum_{j=1}^{s} r_j \right) \langle \mathbf{u} \rangle \in B_0(K; R) + \left(\sum_{j=1}^{s} r_j \right) \langle \mathbf{u} \rangle$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K, and therefore

$$z - \varepsilon(z) \langle \mathbf{u} \rangle \in B_0(K; R)$$

for all $z \in C_0(K; R)$. It follows that $\ker \varepsilon \subset B_0(K; R)$. But we have already shown that $B_0(K; R) \subset \ker \varepsilon$. It follows that $\ker \varepsilon = B_0(K; R)$.

Now the homomorphism $\varepsilon\colon C_0(K; R) \to R$ is surjective and its kernel is $B_0(K; R)$. Moreover $Z_0(K; R) = C_0(K; R)$ (because $\partial_0\colon C_0(K; R) \to C_{-1}(K; R)$ is defined to be the zero homomorphism from $C_0(K; R)$ to the zero module $C_{-1}(K; R)$), and therefore

$$H_0(K; R) = Z_0(K; R)/B_0(K; R) = C_0(K; R)/B_0(K; R).$$

It follows that the homomorphism ε induces an isomorphism from $H_0(K; R)$ to R (see Corollary 1.4), and therefore $H_0(K; R) \cong R$, as required. ∎

On combining Theorem 8.6 with Lemmas 8.3 and 8.4 we obtain immediately the following result.

Corollary 8.7 *Let K be a simplicial complex, and let R be an unital ring. Then $H_0(K; R) \cong R^s$, where s is the number of connected components of $|K|$.*

8.4 The Homology Groups of the Boundary of a Simplex

Proposition 8.8 *Let K be the simplicial complex consisting of all the proper faces of an $(n + 1)$-dimensional simplex σ, where $n > 0$. Then*

$$H_0(K; \mathbb{Z}) \cong \mathbb{Z}, \quad H_n(K; \mathbb{Z}) \cong \mathbb{Z}, \quad H_q(K; \mathbb{Z}) = 0 \text{ when } q \neq 0, n.$$

Proof Let M be the simplicial complex consisting of the $(n+1)$-dimensional simplex σ, together with all its faces. Then K is a subcomplex of M, and $C_q(K;\mathbb{Z}) = C_q(M;\mathbb{Z})$ when $q \leq n$.

It follows from Proposition 8.1 that $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$ and $H_q(M;\mathbb{Z}) = 0$ when $q > 0$. (Here 0 denotes the zero group.) Now $Z_q(K;\mathbb{Z}) = Z_q(M;\mathbb{Z})$ when $q \leq n$, and $B_q(K;\mathbb{Z}) = B_q(M;\mathbb{Z})$ when $q < n$. It follows that $H_q(K;\mathbb{Z}) = H_q(M;\mathbb{Z})$ when $q < n$. Thus $H_0(K;\mathbb{Z}) \cong \mathbb{Z}$ and $H_q(K;\mathbb{Z}) = 0$ when $0 < q < n$. Also $H_q(K;\mathbb{Z}) = 0$ when $q > n$, since the simplicial complex K is of dimension n. Thus, to determine the homology of the complex K, it only remains to find $H_n(K;\mathbb{Z})$.

Let the $(n+1)$-dimensional simplex σ have vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$. Then

$$C_{n+1}(M;\mathbb{Z}) = \{n\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\rangle : n \in \mathbb{Z}\}.$$

and therefore $B_n(M;\mathbb{Z}) = \{nz : n \in \mathbb{Z}\}$, where

$$z = \partial_{n+1}\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\rangle\right).$$

Now $H_n(M;\mathbb{Z}) = 0$ (Proposition 8.1). It follows that $Z_n(M;\mathbb{Z}) = B_n(M;\mathbb{Z})$. But $Z_n(K;\mathbb{Z}) = Z_n(M;\mathbb{Z})$, since $C_n(K;\mathbb{Z}) = C_n(M;\mathbb{Z})$ and the definition of the boundary homomorphism on $C_n(K;\mathbb{Z})$ is consistent with the definition of the boundary homomorphism on $C_n(M;\mathbb{Z})$. Also $B_n(K;\mathbb{Z}) = 0$, because the simplicial complex K is of dimension n, and therefore has no non-zero n-boundaries. It follows that

$$H_n(K;\mathbb{Z}) \cong Z_n(K;\mathbb{Z}) = Z_n(M;\mathbb{Z}) = B_n(M;\mathbb{Z}) \cong \mathbb{Z}.$$

Indeed $H_n(K;\mathbb{Z}) = \{n[z] : n \in \mathbb{Z}\}$, where $[z]$ denotes the homology class of the n-cycle z of K defined above. ∎

Remark Note that the n-cycle z is an n-cycle of the simplicial complex K, since it is a linear combination, with integer coefficients, of oriented n-simplices of K. The n-cycle z is an n-boundary of the large simplicial complex M. However it is not an n-boundary of K. Indeed the n-dimensional simplicial complex K has no non-zero $(n+1)$-chains, therefore has no non-zero n-boundaries. Therefore z represents a non-zero homology class $[z]$ of $H_n(K;\mathbb{Z})$. This homology class generates the homology group $H_n(K;\mathbb{Z})$.

Remark The boundary of a 1-simplex consists of two points. Thus if K is the simplicial complex representing the boundary of a 1-simplex then $H_0(K;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ (Corollary 8.7), and $H_q(K;\mathbb{Z}) = 0$ when $q > 0$.

8.5 The Reduced Homology of a Simplicial Complex

Lemma 8.9 *Let K be a non-empty simplicial complex, and let R be a unital ring with multiplicative identity element 1_R. Let $\varepsilon\colon C_0(K;R) \to R$ be the homomorphism defined such that*

$$\varepsilon(r_1\langle \mathbf{v}_1\rangle + r_2\langle \mathbf{v}_2\rangle + \cdots + r_k\langle \mathbf{v}_k\rangle) = r_1 + r_2 + \cdots + r_k$$

for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of K and coefficients $r_1, r_2, \ldots, r_k$ belonging to the coefficient ring R. Let

$$\tilde{H}_0(K;R) = \ker\varepsilon/\partial_1(C_1(K;R)).$$

Then $\tilde{H}_0(K;R)$ is a subgroup of $H_0(K;R)$, and

$$H_0(K;R) \cong \tilde{H}_0(K;R) \oplus R.$$

Moreover $\tilde{H}_0(K;R)$ is the kernel of the homomorphism $\varepsilon_*\colon H_0(K;R) \to R$ defined such that $\varepsilon_*([\langle \mathbf{v}\rangle]) = 1_R$ for all vertices $\mathbf{v}$ of K.

Proof The definition of the homomorphisms ε and ∂_1 ensure that

$$\varepsilon(\partial_1(\langle \mathbf{v}_0, \mathbf{v}_1\rangle)) = \varepsilon(\langle \mathbf{v}_1\rangle - \langle \mathbf{v}_0\rangle)) = 0$$

whenever $\mathbf{v}_0$ and $\mathbf{v}_1$ are the endpoints of an edge of K. It follows that $B_0(K;R) \subset \ker\varepsilon$. Now $Z_0(K;R) = C_0(K;R)$, and therefore $H_0(K;R) = C_0(K;R)/B_0(K;R)$. It follows that the quotient of the subgroup $\ker\varepsilon$ of $C_0(K;R)$ by the subgroup $B_0(K;R)$ is a well-defined subgroup $\tilde{H}_0(K;R)$ of $H_0(K;R)$. Moreover the surjective homomorphism $\varepsilon\colon C_0(K;R) \to R$ induces a well-defined homomorphism $\varepsilon_*\colon H_0(K;R) \to R$, and $\tilde{H}_0(K;R) = \ker\varepsilon_*$.

Choose a vertex $\mathbf{w}$ of K. Then there is a well-defined homomorphism $\mu_*\colon H_0(K;R) \to \tilde{H}_0(K;R) \oplus R$ defined such that

$$\mu_*(\eta) = \Big(\eta - \varepsilon_*(\eta)[\langle \mathbf{w}\rangle],\, \varepsilon_*(\eta)\Big)$$

for all $\eta \in H_0(K;R)$. This homomorphism is an isomorphism whose inverse sends (η, r) to $\eta + r[\langle \mathbf{w}\rangle]$ for all $\eta \in \tilde{H}_0(K;R)$ and $r \in R$. The result follows. ∎

Definition Let K be a simplicial complex, let R be a unital ring with identity element 1_R, and let $\varepsilon_*\colon H_0(K;R) \to R$ be the homomorphism from $H_0(K;R)$ to R characterized by the requirement that $\varepsilon_*([\langle \mathbf{v}\rangle]) = 1_R$ for all vertices $\mathbf{v}$ of K. The *reduced homology groups* $\tilde{H}_q(K;R)$ of K are defined such that

$$\tilde{H}_q(K;R) = \begin{cases} \ker\varepsilon_* & \text{if } q = 0; \\ H_q(K;R) & \text{if } q > 0; \\ 0 & \text{if } q < 0. \end{cases}$$

Lemma 8.10 *Let K and L be simplicial complexes, let R be a unital ring, and let $\varphi\colon K \to L$ be a simplicial map from K to L. Then the induced homomorphism $\varphi_*\colon H_0(K;R) \to H_0(L;R)$ of homology groups in dimension zero maps the reduced homology group $\tilde{H}_0(K;R)$ into the reduced homology group $\tilde{H}_1(L;R)$ of L. Moreover*

$$\ker(\varphi_*\colon H_0(K;R) \to H_0(L;R)) \subset \tilde{H}_0(K;R)$$

and

$$\varphi_*(H_0(K;R)) \cap \tilde{H}_0(L;R) = \varphi_*(\tilde{H}_0(K;R)).$$

Proof The relevant definitions ensure that

$$\varepsilon_*(\varphi_*([\langle \mathbf{v} \rangle])) = \varepsilon_*([\langle \varphi(\mathbf{v}) \rangle]) = 1_R = \varepsilon_*([\langle \mathbf{v} \rangle]).$$

for all vertices $\mathbf{v}$ of K. The homology group $H_0(K;R)$ is generated by the homology classes of the vertices of K. It follows that $\varepsilon_*(\varphi_*(\eta)) = \varepsilon_*(\eta)$ for all $\eta \in H_0(K;R)$. It follows that an element η of $H_0(K;R)$ belongs to the reduced homology group $\tilde{H}_0(K;R)$ if and only if $\varphi_*(\eta)$ belongs to the reduced homology group $\tilde{H}_0(L;R)$. Therefore φ_* maps the kernel of $\varepsilon_*\colon H_0(K;R) \to R$ into the kernel of $\varepsilon_*\colon H_0(L;R) \to R$, and thus $\varphi_*(\tilde{H}_0(K;R)) \subset \tilde{H}_0(L;R)$. Also the kernel of $\varphi_*\colon H_0(K;R) \to H_0(L;R)$ must be contained in $\tilde{H}_0(K;R)$, and

$$\varphi_*(H_0(K;R)) \cap \tilde{H}_0(L;R) = \varphi_*(\tilde{H}_0(K;R)),$$

as required. ∎