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Contents

6 Introduction to Homological Algebra

6.1 Exact Sequences

In homological algebra we consider sequences

$$\cdots \longrightarrow F \xrightarrow{p} G \xrightarrow{q} H \xrightarrow{\cdots}$$

where F, G, H etc. are modules over some unital ring R and p, q etc. are R-module homomorphisms. We denote the trivial module $\{0\}$ by 0, and we denote by $0 \longrightarrow G$ and $G \longrightarrow 0$ the zero homomorphisms from 0 to G and from G to 0 respectively. (These zero homomorphisms are of course the only homomorphisms mapping out of and into the trivial module 0.)

Unless otherwise stated, all modules are considered to be left modules.

Definition Let R be a unital ring, let F, G and H be R-modules, and let $p: F \to G$ and $q: G \to H$ be R-module homomorphisms. The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of modules and homomorphisms is said to be *exact* at G if and only if $\mathrm{image}(p: F \to G) = \ker(q: G \to H)$. A sequence of modules and homomorphisms is said to be *exact* if it is exact at each module occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *monomorphism* is an injective homomorphism. An *epimorphism* is a surjective homomorphism. An *isomorphism* is a bijective homomorphism.

The following result follows directly from the relevant definitions.

Lemma 6.1 *let R be a unital ring, and let $h: G \to H$ be a homomorphism of R-modules. Then*

- *$h: G \to H$ is a monomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H$ is an exact sequence;*
- *$h: G \to H$ is an epimorphism if and only if $G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence;*
- *$h: G \to H$ is an isomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*

Let R be a unital ring, and let F be a submodule of an R-module G. Then the sequence

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} G/F \longrightarrow 0,$$

is exact, where G/F is the quotient module, $i: F \hookrightarrow G$ is the inclusion homomorphism, and $q: G \to G/F$ is the quotient homomorphism. Conversely, given any exact sequence of the form

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} H \longrightarrow 0,$$

we can regard F as a submodule of G (on identifying F with $i(F)$), and then H is isomorphic to the quotient module G/F. Exact sequences of this type are referred to as *short exact sequences.*

We now introduce the concept of a *commutative diagram.* This is a diagram depicting a collection of homomorphisms between various modules occurring on the diagram. The diagram is said to *commute* if, whenever there are two routes through the diagram from a module G to a module H, the homomorphism from G to H obtained by forming the composition of the homomorphisms along one route in the diagram agrees with that obtained by composing the homomorphisms along the other route. Thus, for example, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ D & \xrightarrow{h} & E & \xrightarrow{k} & F \end{array}$$

commutes if and only if $q \circ f = h \circ p$ and $r \circ g = k \circ q$.

Proposition 6.2 *Let R be a unital ring. Suppose that the following diagram of R-modules and R-module homomorphisms*

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_4} & & \downarrow{\scriptstyle \psi_5} \\ H_1 & \xrightarrow{\phi_1} & H_2 & \xrightarrow{\phi_2} & H_3 & \xrightarrow{\phi_3} & H_4 & \xrightarrow{\phi_4} & H_5 \end{array}$$

commutes and that both rows are exact sequences. Then the following results follow:

(i) *if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism,*

(ii) *if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism.*

Proof First we prove (i). Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove (ii). Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required. ∎

The following result is an immediate corollary of Proposition 6.2.

Lemma 6.3 (Five-Lemma) *Suppose that the rows of the commutative diagram of Proposition 6.2 are exact sequences and that ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms. Then ψ_3 is also an isomorphism.*

6.2 Chain Complexes

Definition A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of modules over some unital ring, together with homomorphisms $\partial_i \colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i \colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1} \colon C_{i+1} \to C_i$.

Note that if the modules C_* occuring in a chain complex C_* are modules over some unital ring R then the homology groups of the complex are also modules over this ring R.

Definition Let C_* and D_* be chain complexes. A *chain map* $f \colon C_* \to D_*$ is a sequence $f_i \colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

Note that a collection of homomorphisms $f_i \colon C_i \to D_i$ defines a chain map $f_* \colon C_* \to D_*$ if and only if the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & \\
\cdots \longrightarrow & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \longrightarrow \cdots
\end{array}$$

is commutative.

Let C_* and D_* be chain complexes, and let $f_* \colon C_* \to D_*$ be a chain map. Then $f_i(Z_i(C_*)) \subset Z_i(D_*)$ and $f_i(B_i(C_*)) \subset B_i(D_*)$ for all i. It follows from this that $f_i \colon C_i \to D_i$ induces a homomorphism $f_* \colon H_i(C_*) \to H_i(D_*)$ of homology groups sending $[z]$ to $[f_i(z)]$ for all $z \in Z_i(C_*)$, where $[z] = z + B_i(C_*)$, and $[f_i(z)] = f_i(z) + B_i(D_*)$.

Definition A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_* \colon A_* \to B_*$ and $q_* \colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

We see that $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \partial_{i+2} & & \downarrow \partial_{i+2} & & \downarrow \partial_{i+2} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{p_{i+1}} & B_{i+1} & \xrightarrow{q_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \\
0 & \longrightarrow & A_{i} & \xrightarrow{p_{i}} & B_{i} & \xrightarrow{q_{i}} & C_{i} & \longrightarrow & 0\,. \\
 & & \downarrow \partial_{i} & & \downarrow \partial_{i} & & \downarrow \partial_{i} & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{p_{i-1}} & B_{i-1} & \xrightarrow{q_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{i-1} & & \downarrow \partial_{i-1} & & \downarrow \partial_{i-1} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

Lemma 6.4 *Given any short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes, there is a well-defined homomorphism*

$$\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$$

which sends the homology class $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i \colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2} \colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) = \partial_i(b') = p_{i-1}(w'),$$

and $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}\colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$, as required. ∎

Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ and $0 \longrightarrow A'_* \xrightarrow{p'_*} B'_* \xrightarrow{q'_*} C'_* \longrightarrow 0$ be short exact sequences of chain complexes, and let $\lambda_*\colon A_* \to A'_*$, $\mu_*\colon B_* \to B'_*$ and $\nu_*\colon C_* \to C'_*$ be chain maps. For each integer i, let $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$ and $\alpha'_i\colon H_i(C'_*) \to H_{i-1}(A'_*)$ be the homomorphisms defined as described in Lemma 6.4. Suppose that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{p_*} & B_* & \xrightarrow{q_*} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_*} & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \nu_*} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{p'_*} & B'_* & \xrightarrow{q'_*} & C'_* & \longrightarrow & 0
\end{array}$$

commutes (i.e., $p'_i \circ \lambda_i = \mu_i \circ p_i$ and $q'_i \circ \mu_i = \nu_i \circ q_i$ for all i). Then the square

$$\begin{array}{ccc}
H_i(C_*) & \xrightarrow{\alpha_i} & H_{i-1}(A_*) \\
\downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \lambda_*} \\
H_i(C'_*) & \xrightarrow{\alpha'_i} & H_{i-1}(A'_*)
\end{array}$$

commutes for all $i \in \mathbb{Z}$ (i.e., $\lambda_* \circ \alpha_i = \alpha'_i \circ \nu_*$).

Proposition 6.5 *Let* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *be a short exact sequence of chain complexes. Then the (infinite) sequence*

$$\cdots \xrightarrow{\alpha_{i+1}} H_i(A_*) \xrightarrow{p_*} H_i(B_*) \xrightarrow{q_*} H_i(C_*) \xrightarrow{\alpha_i} H_{i-1}(A_*) \xrightarrow{p_*} H_{i-1}(B_*) \xrightarrow{q_*} \cdots$$

of homology groups is exact, where $\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$ *is the well-defined homomorphism that sends the homology class* $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_* \circ p_* = 0$. Thus the image of $p_* \colon H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_* \colon H_i(B_*) \to H_i(C_*)$. Let x be an element of $Z_i(B_*)$ for which $[x] \in \ker q_*$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1} \colon B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x - \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus a represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x - \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now

$$\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w],$$

where w is the unique element of $Z_i(A_*)$ satisfying $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let z be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*([b - p_i(a)]) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$.

Now let w be an element of $Z_{i-1}(A_*)$ for which $[w] \in \ker p_*$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_*)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_*)$, as required. ∎

7 Exact Sequences of Homology Groups

7.1 The Euler Characteristic of a Simplicial Complex

Lemma 7.1 *Let R be a field, and let*

$$0 \to U \xrightarrow{p} V \xrightarrow{q} W \to 0$$

be a short exact sequence of vector spaces over the field R. Then $\dim_R V = \dim_R U + \dim_R W$.

Proof The vector space W is isomorphic to the quotient space V/U. It follows from basic linear algebra that $\dim_R W = \dim_R V - \dim_R U$. The result follows. ∎

Definition Let K be a simplicial complex of dimension n. The *Euler characteristic* $\chi(K)$ of K is defined by the equation

$$\chi(K) = \sum_{q=0}^{n} (-1)^q \dim H_q(K;\mathbb{R})$$

Proposition 7.2 *Let K be a simplicial complex of dimension n, and, for each non-negative integer q, let m_q be the number of q-simplices of K. Then the Euler characteristic $\chi(K)$ of K satisfies the identity*

$$\chi(K) = \sum_{q=0}^{n} (-1)^q m_q.$$

Proof The groups $C_q(K;\mathbb{R})$, $Z_q(K;\mathbb{R})$, $B_q(K;\mathbb{R})$ of q-chains, q-cycles and q-boundaries of K with coefficients in the field $\mathbb{R}$ of real numbers are real vector spaces, as is the homology group $H_q(K;\mathbb{R})$. Moreover there are exact sequences

$$0 \to Z_q(K;\mathbb{R}) \to C_q(K;\mathbb{R}) \xrightarrow{\partial_q} B_{q-1}(K;\mathbb{R}) \to 0$$

and

$$0 \to B_q(K;\mathbb{R}) \to Z_q(K;\mathbb{R}) \to H_q(K;\mathbb{R}) \to 0.$$

It follows from Lemma 7.1 that

$$\dim Z_q(K;\mathbb{R}) = B_q(K;\mathbb{R}) + H_q(K;\mathbb{R})$$

for $q \geq 0$. Also $\dim Z_0(K;\mathbb{R}) = \dim C_0(K;\mathbb{R})$, and

$$\dim C_q(K;\mathbb{R}) = Z_q(K;\mathbb{R}) + B_{q-1}(K;\mathbb{R})$$

for $q > 0$. Now $\dim C_q(K;\mathbb{R}) = m_q$ for all non-negative integers q. Therefore

$$\begin{aligned}
\chi(K) &= \sum_{q=0}^{n} (-1)^q H_q(K;\mathbb{R}) \\
&= \sum_{q=0}^{n} (-1)^q Z_q(K;\mathbb{R}) - \sum_{q=0}^{n-1} (-1)^q B_q(K;\mathbb{R}) \\
&= \sum_{q=0}^{n} (-1)^q Z_q(K;\mathbb{R}) - \sum_{q=1}^{n} (-1)^{q-1} B_{q-1}(K;\mathbb{R}) \\
&= \sum_{q=0}^{n} (-1)^q Z_q(K;\mathbb{R}) + \sum_{q=1}^{n} (-1)^q \Big(C_q(K;\mathbb{R}) - Z_q(K;\mathbb{R}) \Big) \\
&= \sum_{q=0}^{n} (-1)^q C_q(K;\mathbb{R}) = \sum_{q=0}^{n} (-1)^q m_q,
\end{aligned}$$

as required. ∎

7.2 Homology Groups of Simplicial Pairs

A *simplicial pair* (K, L) consists of a simplicial complex K together with a subcomplex L.

Let (K, L) be a simplicial pair, and let R be an integral domain. The qth chain group $C_q(L; R)$ of the subcomplex L with coefficients in the integral domain R may be regarded as a module of the qth chain group $C_q(K; R)$ of the simplicial complex K, and the inclusion map $i\colon L \hookrightarrow K$ induces inclusion homomorphisms

$$i_q\colon C_q(L; R) \hookrightarrow C_q(K; R).$$

We define the qth chain group $C_q(K, L; R)$ of the simplicial pair to be the quotient group $C_q(K; R)/C_q(L; R)$. The boundary homomorphism

$$\partial_q\colon C_q(K; R) \to C_{q-1}(K; R)$$

maps the submodule $C_q(L; R)$ into $C_{q-1}(L; R)$, and therefore induces a homomorphism

$$\partial_q\colon C_q(K, L; R) \to C_{q-1}(K, L; R).$$

For each integer q, let

$$u_q\colon C_q(K; R) \to C_q(K, L; R)$$

be the quotient homomorphism from $C_q(K;R)$ to $C_q(K,L;R)$. Then $\partial_q \circ u_q = u_{q-1} \circ \partial_q$ for all integers q. (This is an immediate consequence of the fact that the homomorphism

$$\partial_q: C_q(K,L;R) \to C_{q-1}(K,L;R)$$

is by definition the homomorphism induced by the boundary homomorphism $\partial_q: C_q(K;R) \to C_{q-1}(K;R)$ of K.)

Now

$$\partial_{q-1} \circ \partial_q \circ u_q = \partial_{q-1} \circ u_{q-1} \circ \partial_q = u_{q-2} \circ \partial_{q-1} \circ \partial_q = 0.$$

Moreover the quotient homomorphism

$$u_q: C_q(K;R) \to C_q(K,L;R)$$

is surjective. It follows that the composition of the homomorphisms

$$\partial_q: C_q(K,L;R) \to C_{q-1}(K,L;R)$$

and

$$\partial_{q-1}: C_{q-1}(K,L;R) \to C_{q-2}(K,L;R)$$

is the zero homomorphism. Therefore the sequence of R-modules

$$(C_q(K,L;R) : q \in R)$$

and R-module homomorphisms

$$(\partial_q: C_q(K,L;R) \to C_{q-1}(K,L;R) : q \in R)$$

constitutes a chain complex $C_*(K,L;R)$, whose modules are the chain groups of the simplicial pair (K,L) with coefficients in the integral domain R. We shall refer to the R-module homomorphisms

$$\partial_q: C_q(K,L;R) \to C_{q-1}(K,L;R)$$

as the *boundary homomorphisms* of the simplicial pair (K,L).

The sequence of quotient homomorphisms

$$(u_q: C_q(K;R) \to C_q(K,L;R) : q \in R)$$

define a chain map

$$u_*: C_*(K;R) \to C_*(K,L;R)$$

between the chain complexes $C_*(K;R)$ and $C_*(K,L;R)$. The image $u_q(c)$ of a q-chain $c \in C_q(K;R)$ of K under the quotient homomorphism is the coset $c + C_q(L;R)$ of $C_q(L;R)$ in $C_q(K;R)$ that contains c. Moreover

$$\partial_q(c + C_q(L;R)) = \partial_q c + C_{q-1}(L;R).$$

We define

$$\begin{aligned} Z_q(K,L;R) &= \ker(\partial_q\colon C_q(K,L;R) \to C_{q-1}(K,L;R)) \\ &= \{c + C_q(L;R) : c \in C_q(K;R) \text{ and } \partial_q c \in C_{q-1}(L;R)\}, \\ B_q(K,L;R) &= \operatorname{image}(\partial_{q+1}\colon C_{q+1}(K,L;R) \to C_q(K,L;R)) \\ &= \{\partial_{q+1}(e) + C_q(L;R) : e \in C_{q+1}(K;R)\}. \end{aligned}$$

Then $B_q(K,L;R) \subset Z_q(K,L;R)$. We define

$$H_q(K,L;R) = Z_q(K,L;R)/B_q(K,L;R).$$

Let z be an element of $Z_q(K,L;R)$, and let c and c' be elements of $C_q(K;R)$ for which

$$z = c + C_q(L;R) = c' + C_q(L;R).$$

Then $c - c' \in C_q(L;R)$, $\partial_q c \in C_{q-1}(L;R)$ and $\partial_q c' \in C_{q-1}(L;R)$. But

$$\partial_{q-1}\partial_q c = \partial_{q-1}\partial_q c' = 0$$

and $\partial_q c - \partial_q c' = \partial_q(c - c')$. It follows that $\partial_q c \in Z_{q-1}(L)$, $\partial_q c' \in Z_{q-1}(L)$ and $\partial_q c - \partial_q c' \in B_{q-1}(L)$, and therefore $[\partial_q c] = [\partial_q c']$. It follows that there is a well-defined homomorphism from $Z_q(K,L;R)$ to $H_{q-1}(L;R)$ that maps $c + C_q(L;R)$ to $[\partial_q c]$. The submodule $B_q(K,L;R)$ is contained in the kernel of this homomorphism. The homomorphism therefore induces a homomorphism

$$\partial_*\colon H_q(K,L;R) \to H_{q-1}(L;R).$$

This homomorphism sends the homology class of $c + C_q(L;R)$ in $H_q(K,L;R)$ to the homology class of $\partial_q c$ in $C_q(L;R)$ for all $c \in C_q(K;R)$ satisfying $\partial_q c \in C_{q-1}(L;R)$.

Proposition 7.3 (The Homology Exact Sequence of a Simplicial Pair) *Let K be a simplicial complex, let L be a subcomplex of K, and let R be an integral domain. Then the sequence*

$$\begin{aligned} \cdots \xrightarrow{\partial_*} H_q(L;R) \xrightarrow{i_*} H_q(K;R) \xrightarrow{u_*} H_q(K,L;R) \\ \xrightarrow{\partial_*} H_{q-1}(L;R) \xrightarrow{i_*} H_{q-1}(K;R) \xrightarrow{u_*} \cdots \end{aligned}$$

of homology groups is exact, where $\partial_*\colon H_q(K,L;R) \to H_{q-1}(L;R))$ *is the homomorphism that sends the homology class of* $c + C_q(L;R)$ *in* $H_q(K,L;R)$ *to the homology class of* $\partial_q c$ *in* $H_{q-1}(L;R)$ *for all* $c \in C_q(K;R)$ *satisfying* $\partial_q c \in C_{q-1}(L;R)$.

Proof The sequence

$$0 \longrightarrow C_*(L;R) \xrightarrow{i_*} C_*(K;R) \xrightarrow{u_*} C_*(K,L;R) \longrightarrow 0$$

is a short exact sequence of chain complexes. It follows from Proposition 6.5 that there is a corresponding (infinite) sequence of homology groups. Moreover the homomorphism from $H_q(K,L;R)$ to $H_{q-1}(L;R)$ defined as in the statement of that proposition is the homomorphism $\partial_q\colon H_q(K,L;R) \to H_{q-1}(L;R)$ defined as described above. ∎

Corollary 7.4 *Let* K *be a simplicial complex, let* L *be a subcomplex of* K*, and let* R *be an integral domain. Suppose that*

$$H_{q+1}(K,L;R) = H_q(K,L;R) = 0$$

for some integer q*. Then* $i_*\colon H_q(L;R) \to H_q(K;R)$ *is an isomorphism.*

Corollary 7.5 *Let* K *be a simplicial complex, let* L *be a subcomplex of* K*, and let* R *be an integral domain. Suppose that*

$$H_q(K;R) = H_{q-1}(K;R) = 0$$

for some integer q*. Then* $\partial_*\colon H_q(K,L;R) \to H_{q-1}(L;R)$ *is an isomorphism.*

Corollary 7.6 *Let* K *be a simplicial complex, let* L *be a subcomplex of* K*, and let* R *be an integral domain. Suppose that*

$$H_q(L;R) = H_{q-1}(L;R) = 0$$

for some integer q*. Then* $u_*\colon H_q(K;R) \to H_q(K,L;R)$ *is an isomorphism.*

Example Let K be the simplicial complex consisting of all the faces of an n-dimensional simplex, and let L be the subcomplex consisting of all the proper faces of this simplex. Then $C_q(L;R) = C_q(K;R)$ when $q \neq n$, and therefore $C_q(K,L;R) = 0$ when $q \neq n$. Also $C_n(K,L;R) \cong R$. It follows that $H_n(K,L;R) \cong R$, $H_q(K,L;R) = 0$ when $q \neq n$. Also it follows from Proposition 4.8 that $H_q(K;R) = 0$ when $q > 0$.

Suppose that $n \geq 2$. It follows from Corollary 7.5 that $\partial_*\colon H_q(K, L; R) \to H_{q-1}(L; R)$ is an isomorphism for $q \geq 2$. Therefore $H_{n-1}(L; R) \cong R$, and $H_q(L; R) = 0$ for $q \neq 0, n-1$.

Now suppose that $n = 1$. We have an exact sequence

$$0 \longrightarrow H_1(K, L; R) \xrightarrow{\partial_*} H_0(L; R) \longrightarrow H_0(K; R) \longrightarrow 0.$$

Now $H_1(K, L; R) \cong R$ when $n = 1$. Also $H_0(K; R) \cong R$. From the exactness of the above sequence we can deduce that $H_0(L; R) \cong H_1(K, L; R) \oplus H_0(K; R) \cong R \oplus R$. This result is consistent with the fact that, in this case, L is a 0-dimensional simplicial complex consisting of two vertices.

7.3 The Excision Property

Lemma 7.7 *Let K be a simplicial complex, let L and M be subcomplexes of K, and let R be an integral domain. Suppose that $K = L \cup M$. Then*

$$C_q(K; R) = C_q(L; R) + C_q(M; R)$$

and

$$C_q(L \cap M; R) = C_q(L; R) \cap C_q(M; R).$$

Proof Proposition 4.3 ensures that there exists a free basis $\gamma_1, \gamma_2, \ldots, \gamma_k$ for $C_q(K; R)$, where these generators are in one-to-one correspondence with the q-simplices $\sigma_1, \sigma_2, \ldots, \sigma_k$ of K, and where

$$\gamma_j = \langle \mathbf{v}_0^{(j)}, \mathbf{v}_1^{(j)}, \ldots \mathbf{v}_q^{(j)} \rangle$$

for some chosen ordering $\mathbf{v}_0^{(j)}, \mathbf{v}_1^{(j)}, \ldots \mathbf{v}_q^{(j)}$ of the vertices of the corersponding q-simplex σ_j of K. Now any q-chain c of K with coefficients in the integral domain R can be represented uniquely as a linear combination of the form

$$c = \sum_{j=1}^{r} r_j \gamma_j.$$

The q-chain c then determines, and is determined by, the values of its coefficients $r_1, r_2, \ldots, r_k$. It follows that

$$\begin{aligned} c \in C_q(L; R) &\iff r_j = 0 \text{ whenever } \sigma_j \notin L, \\ c \in C_q(M; R) &\iff r_j = 0 \text{ whenever } \sigma_j \notin M, \\ c \in C_q(L \cap M; R) &\iff r_j = 0 \text{ whenever } \sigma_j \notin L \cap M, \end{aligned}$$

It follows that $C_q(L \cap M; R) = C_q(L; R) \cap C_q(M; R)$. Moreover each of the q-simplices σ_j of K belongs to one or other of the subcomplexes L and M. Thus if I is the set of those indices j between 1 and k for which $\sigma_j \in L$ then $\sum_{j \in I} r_j \gamma_j \in C_q(L; R)$ and $\sum_{j \notin I} r_j \gamma_j \in C_q(M; R)$, and therefore

$$c = \sum_{j \in I} r_j \gamma_j + \sum_{j \notin I} r_j \gamma_j \in C_q(L; R) + C_q(M; R).$$

This shows that $C_q(K; R) = C_q(L; R) + C_q(M; R)$, as required. ∎

Proposition 7.8 *Let K be a simplicial complex, let L and M be subcomplexes of K, and let R be an integral domain. Suppose that $K = L \cup M$. Then $H_q(M, L \cap M; R) \cong H_q(K, L; R)$ for all integers q.*

Proof It follows from Lemma 7.7 that

$$C_q(K; R) = C_q(L; R) + C_q(M; R)$$

and

$$C_q(L \cap M; R) = C_q(L; R) \cap C_q(M; R).$$

Let $w_q: C_q(M; R) \to C_q(K; L; R)$ be the R-module homomorphism from $C_q(M; R)$ to $C_q(K, L; R)$, where $C_q(K, L; R) = C_q(K; R)/C_q(L; R)$, defined such that $w_q(c) = C_q(L; R) + c$ for all $c \in C_q(M; R)$. Then the homomorphism w_q is surjective, because $C_q(K; R) = C_q(L; R) + C_q(M; R)$, and $\ker w_q = C_q(L \cap M; R)$, because $C_q(L \cap M; R) = C_q(L; R) \cap C_q(M; R)$. Now

$$C_q(M, L \cap M; R) = C_q(M; R)/C_q(L \cap M; R) = C_q(M; R)/ \ker w_q.$$

It follows that the homomorphism w_q induces an isomorphism

$$\hat{w}_q: C_q(M, L \cap M) \to C_q(K, L),$$

for each integer q, where

$$\hat{w}_q(C_q(L \cap M) + c) = w_q(c) = C_q(L; R) + c$$

for all $c \in C_q(M; R)$ (see Proposition 1.7). Moreover $\partial_q \circ \hat{w}_q = \hat{w}_{q-1} \circ \partial_q$ for each integer q. It follows that the isomorphisms

$$\hat{w}_q: C_q(M, L \cap M; R) \to C_q(K, M; R)$$

of chain groups induce corresponding isomorphisms

$$\hat{w}_{q*}: H_q(M, L \cap M; R) \to H_q(K, M; R)$$

of homology groups. The result follows. ∎

7.4 Homology Groups of some Closed Surfaces

Lemma 7.9 *Let K be a 2-dimensional simplicial complex, and let L and M be subcomplexes of L, where $K = L \cup M$. Suppose that M consists of a triangle of K, together with all its edges and vertices, and that $L \cap M$ matches one of the following descriptions:*

(i) *$L \cap M$ consists of a single vertex of the triangle;*

(ii) *$L \cap M$ consists of a single edge of the triangle together with the endpoints of that edge;*

(iii) *$L \cap M$ consists of two edges of the triangle together with the endpoints of those edges.*

Then $H_q(K, L; \mathbb{Z}) = 0$ for all integers q, and therefore the inclusion map $i: L \hookrightarrow K$ induces isomorphisms $i_*: H_q(L; \mathbb{Z}) \to H_q(K; \mathbb{Z})$ of homology groups.

Proof Let the triangle have vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, and let $\tau \in C_2(K; \mathbb{Z})$ and $\rho_0, \rho_1, \rho_2 \in C_1(K; \mathbb{Z})$ be defined by

$$\tau = \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle,$$

$$\rho_0 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad \rho_1 = \langle \mathbf{v}_2, \mathbf{v}_0 \rangle, \quad \rho_2 = \langle \mathbf{v}_0, \mathbf{v}_1 \rangle.$$

Then $\partial_2 \tau = \rho_0 + \rho_1 + \rho_2$ in $C_1(K; \mathbb{Z})$.

Consider first the case where $L \cap M$ is as described in (i). We label the vertices of the triangle so that $L \cap M$ consists of the single vertex $\mathbf{v}_0$. In this case

$$\begin{aligned}
C_2(K, L; \mathbb{Z}) &= \{n\tau + C_2(L; \mathbb{Z}) : n \in \mathbb{Z}\}, \\
C_1(K, L; \mathbb{Z}) &= \{n_0\rho_0 + n_1\rho_1 + n_2\rho_2 + C_1(L; \mathbb{Z}) : n_0, n_1 \in \mathbb{Z}\}, \\
C_0(K, L; \mathbb{Z}) &= \{r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + C_0(L; \mathbb{Z}) : r \in \mathbb{Z}\}.
\end{aligned}$$

Now $\partial_2 \tau \in \rho_0 + \rho_1 + \rho_2 + C_1(L; \mathbb{Z})$, and

$$\partial_1(n_0\rho_0 + n_1\rho_1 + n_2\rho_2) \in (n_2 - n_0)\langle \mathbf{v}_1 \rangle + (n_0 - n_1)\langle \mathbf{v}_2 \rangle + C_0(L; \mathbb{Z})$$

for all $n_0, n_1, n_2 \in \mathbb{Z}$. It follows that $B_2(K, L; \mathbb{Z}) = Z_2(K, L; \mathbb{Z}) = 0$,

$$Z_1(K, L; \mathbb{Z}) = B_1(K, L; \mathbb{Z}) = \{n(\rho_0 + \rho_1 + \rho_2) + C_1(L; \mathbb{Z}) : n \in \mathbb{Z}\},$$

and

$$Z_0(K, L; \mathbb{Z}) = B_0(K, L; \mathbb{Z}) = C_0(K, L; \mathbb{Z}).$$

Therefore $H_q(K, L; \mathbb{Z}) = 0$ for all integers q in the case when $L \cap M$ consists of a single vertex of the triangle.

Consider next the case where $L \cap M$ is as described in (ii). We label the vertices of the triangle so that $L \cap M$ consists of the single edge ρ_2, together with its endpoints $\mathbf{v}_0$ and $\mathbf{v}_1$. In this case

$$\begin{aligned} C_2(K, L; \mathbb{Z}) &= \{n\tau + C_2(L; \mathbb{Z}) : n \in \mathbb{Z}\}, \\ C_1(K, L; \mathbb{Z}) &= \{n_0\rho_0 + n_1\rho_1 + C_1(L; \mathbb{Z}) : n_0, n_1 \in \mathbb{Z}\}, \\ C_0(K, L; \mathbb{Z}) &= \{r\langle \mathbf{v}_2 \rangle + C_0(L; \mathbb{Z}) : r \in \mathbb{Z}\}. \end{aligned}$$

Now $\partial_2 \tau \in \rho_0 + \rho_1 + C_1(L; \mathbb{Z})$, and

$$\partial_1(n_0\rho_0 + n_1\rho_1) \in (n_0 - n_1)\langle \mathbf{v}_2 \rangle + C_0(L; \mathbb{Z})$$

for all $n_0, n_1 \in \mathbb{Z}$. It follows that $B_2(K, L; \mathbb{Z}) = Z_2(K, L; \mathbb{Z}) = 0$,

$$Z_1(K, L; \mathbb{Z}) = B_1(K, L; \mathbb{Z}) = \{n(\rho_0 + \rho_1) + C_1(L; \mathbb{Z}) : n \in \mathbb{Z}\},$$

and $Z_0(K, L; \mathbb{Z}) = B_0(K, L; \mathbb{Z}) = C_0(K, L; \mathbb{Z})$. Therefore $H_q(K, L; \mathbb{Z}) = 0$ for all integers q in the case when $L \cap M$ consists of a single edge of the triangle together with its endpoints.

Finally consider the case where $L \cap M$ is as described in (iii). We label the vertices of the triangle so that $L \cap M$ consists of the edges ρ_1 and ρ_2, together with the vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ of the triangle. In this case

$$\begin{aligned} C_2(K, L; \mathbb{Z}) &= \{n\tau + C_2(L; \mathbb{Z}) : n \in \mathbb{Z}\}, \\ C_1(K, L; \mathbb{Z}) &= \{n_0\rho_0 + C_1(L; \mathbb{Z}) : n_0 \in \mathbb{Z}\}, \\ C_0(K, L; \mathbb{Z}) &= 0. \end{aligned}$$

In this case

$$\partial_2 \colon C_2(K, L; \mathbb{Z}) \to C_1(K, L; \mathbb{Z})$$

is an isomorphism that sends $\tau + C_2(L; \mathbb{Z})$ to $\rho_0 + C_1(L; \mathbb{Z})$,

$$B_2(K, L; \mathbb{Z}) = Z_2(K, L; \mathbb{Z}) = 0, \quad B_1(K, L; \mathbb{Z}) = Z_1(K, L; \mathbb{Z}) = C_1(K, L; \mathbb{Z})$$

and

$$B_0(K, L; \mathbb{Z}) = Z_0(K, L; \mathbb{Z}) = C_0(K, L; \mathbb{Z}) = 0.$$

Therefore $H_q(K, L; \mathbb{Z}) = 0$ for all integers q in the case when $L \cap M$ consists of two edges of the triangle, together with the vertices of the triangle.

The exact sequence of homology groups of the simplicial pair (K, L) (Proposition 7.3) then ensures that the inclusion map $i \colon L \hookrightarrow K$ induces isomorphisms $i_* \colon H_q(L; \mathbb{Z}) \to H_q(K; \mathbb{Z})$ of homology groups, as required. ∎

Lemma 7.10 *Let K be a 2-dimensional simplicial complex, and let L and M be subcomplexes of L, where $K = L \cup M$. Suppose that M consists of a triangle of K, together with all its edges and vertices, and that $L \cap M$ consists of all the edges and vertices of this triangle. Then $H_2(K, L; \mathbb{Z}) \cong \mathbb{Z}$, and $H_q(K, L; \mathbb{Z}) = 0$ for all integers q satisfying $q \neq 2$. Moreover $H_0(L; \mathbb{Z}) \cong H_0(K; \mathbb{Z})$ and there are short exact sequences*

$$0 \longrightarrow H_2(L; \mathbb{Z}) \xrightarrow{i_*} H_2(K; \mathbb{Z}) \longrightarrow J \longrightarrow 0,$$

$$0 \longrightarrow I \longrightarrow H_1(L; \mathbb{Z}) \xrightarrow{i_*} H_1(K; \mathbb{Z}) \longrightarrow 0,$$

where $i_*\colon H_q(L; \mathbb{Z}) \to H_q(K; \mathbb{Z})$ is induced by the inclusion map $i\colon L \hookrightarrow K$ for all $q \in \mathbb{Z}$, and

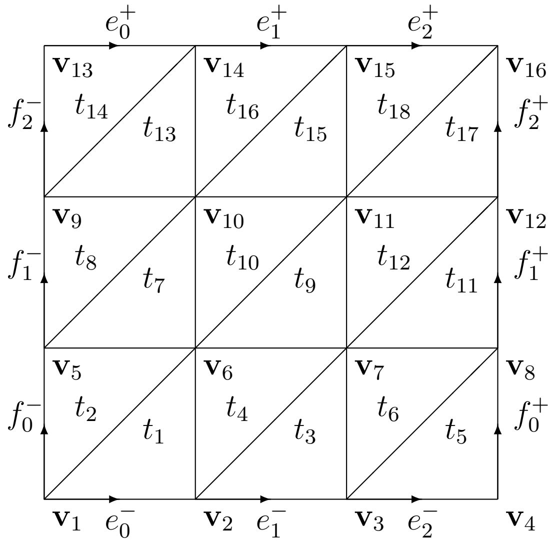
$$\begin{aligned} J &= \ker(\partial_*\colon H_2(K, L; \mathbb{Z}) \to H_1(L; \mathbb{Z})), \\ I &= \operatorname{image}(\partial_*\colon H_2(K, L; \mathbb{Z}) \to H_1(L; \mathbb{Z})). \end{aligned}$$

Proof Let the triangle have vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

$$C_2(K, L; \mathbb{Z}) = \{n\tau + C_2(L; \mathbb{Z}) : n \in \mathbb{Z}\},$$

where $\tau = \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle$, and therefore $C_2(K, L; \mathbb{Z}) \cong \mathbb{Z}$. Moreover $C_q(L; \mathbb{Z}) = C_q(K; \mathbb{Z})$ when $q \neq 2$, and thus $C_q(K, L; \mathbb{Z}) = 0$ when $q \neq 2$. It follows that $H_q(K, L; \mathbb{Z}) = 0$ when $q \neq 2$, and $H_2(K, L; \mathbb{Z}) \cong C_2(K, L; \mathbb{Z}) \cong \mathbb{Z}$. The exactness of the short exact sequences then follows from the exact sequence of homology groups of the simplicial pair (K, L) (Proposition 7.3). ∎

Example We calculate the homology groups $H_*(K_S, L_S; \mathbb{Z})$, where the simplicial complex K_S represents a square S, subdivided into eighteen triangles, and L_S is the subcomplex corresponding to the boundary of that square. We let $S = [0, 3] \times [0, 3]$, so that S is the square in the plane with corners at $(0, 0)$, $(3, 0)$, $(3, 3)$ and $(0, 3)$. The subdivision of this square into triangles is as depicted on the following diagram:



The vertices of this simplicial complex K_S are $\mathbf{v}_1, \ldots, \mathbf{v}_{16}$, where

$$\mathbf{v}_1 = (0,0), \quad \mathbf{v}_2 = (1,0), \quad \mathbf{v}_3 = (2,0), \quad \mathbf{v}_4 = (3,0),$$

$$\mathbf{v}_5 = (0,1), \quad \mathbf{v}_6 = (1,1), \quad \mathbf{v}_7 = (2,1), \quad \mathbf{v}_8 = (3,1),$$

$$\mathbf{v}_9 = (0,2), \quad \mathbf{v}_{10} = (1,2), \quad \mathbf{v}_{11} = (2,2), \quad \mathbf{v}_{12} = (3,2),$$

$$\mathbf{v}_{13} = (0,3), \quad \mathbf{v}_{14} = (1,3), \quad \mathbf{v}_{15} = (2,3), \quad \mathbf{v}_{16} = (3,3),$$

We label the exterior edges of the simplicial complex K_S as indicated on the diagram, so that

$$e_0^- = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad e_1^- = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle, \quad e_2^- = \langle \mathbf{v}_3, \mathbf{v}_4 \rangle,$$

$$e_0^+ = \langle \mathbf{v}_{13}, \mathbf{v}_{14} \rangle, \quad e_1^+ = \langle \mathbf{v}_{14}, \mathbf{v}_{15} \rangle, \quad e_2^+ = \langle \mathbf{v}_{15}, \mathbf{v}_{16} \rangle,$$

$$f_0^- = \langle \mathbf{v}_1, \mathbf{v}_5 \rangle, \quad f_1^- = \langle \mathbf{v}_5, \mathbf{v}_9 \rangle, \quad f_2^- = \langle \mathbf{v}_9, \mathbf{v}_{13} \rangle,$$

$$f_0^+ = \langle \mathbf{v}_4, \mathbf{v}_8 \rangle, \quad f_1^+ = \langle \mathbf{v}_8, \mathbf{v}_{12} \rangle, \quad f_2^+ = \langle \mathbf{v}_{12}, \mathbf{v}_{16} \rangle,$$

We also the vertices, triangles and exterior edges of the simplicial complex K_S as indicated on the diagram. Thus We give each triangle of the simplicial complex K_S the orientation determined by an anticlockwise ordering of its vertices. Then the oriented triangles of K_S are represented by $t_1, \ldots, t_{18}$, where

$$t_1 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_6 \rangle, \quad t_2 = \langle \mathbf{v}_1, \mathbf{v}_6, \mathbf{v}_5 \rangle, \quad t_3 = \langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7 \rangle,$$

$$t_4 = \langle \mathbf{v}_2, \mathbf{v}_7, \mathbf{v}_6 \rangle, \quad t_5 = \langle \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_8 \rangle, \quad t_6 = \langle \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_7 \rangle,$$

$$t_7 = \langle \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_{10} \rangle, \quad t_8 = \langle \mathbf{v}_5, \mathbf{v}_{10}, \mathbf{v}_9 \rangle, \quad t_9 = \langle \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_{11} \rangle,$$
$$t_{10} = \langle \mathbf{v}_6, \mathbf{v}_{11}, \mathbf{v}_{10} \rangle, \quad t_{11} = \langle \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{12} \rangle, \quad t_{12} = \langle \mathbf{v}_7, \mathbf{v}_{12}, \mathbf{v}_{11} \rangle,$$
$$t_{13} = \langle \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{14} \rangle, \quad t_{14} = \langle \mathbf{v}_9, \mathbf{v}_{14}, \mathbf{v}_{13} \rangle, \quad t_{15} = \langle \mathbf{v}_{10}, \mathbf{v}_{11}, \mathbf{v}_{15} \rangle,$$
$$t_{16} = \langle \mathbf{v}_{10}, \mathbf{v}_{15}, \mathbf{v}_{14} \rangle, \quad t_{17} = \langle \mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{16} \rangle, \quad t_{18} = \langle \mathbf{v}_{11}, \mathbf{v}_{16}, \mathbf{v}_{15} \rangle.$$

Let M_0 be the simplicial complex consisting of the single vertex $\mathbf{v}_1$ and, for each integer k between 1 and 18, let M_k be the subcomplex of K_S consisting of the triangles T_j represented by t_j for $1 \leq j \leq k$, together with all the edges and vertices of those triangles. Then $K_S = M_{18}$. Now examination of the diagrams shows that, for each integer k between 1 and 18, the intersection $T_k \cap \bigcup_{j<k} T_j$ is either a single vertex of T_k, or a single edge of T_k, or the union of two edges of T_k. It follows from Lemma 7.9 that the inclusion map $i_k \colon M_{k-1} \hookrightarrow M_k$ induces isomorphisms $i_{k*} \colon H_*(M_{k-1}; \mathbb{Z}) \to H_*(M_k; \mathbb{Z})$ of homology groups for all integers k satisfying $1 \leq k \leq 18$. Therefore $H_q(K_S; \mathbb{Z}) \cong H_q(M_0; \mathbb{Z})$ for all integers q, and thus $H_q(K_S; \mathbb{Z}) = 0$ when $q > 0$, and $H_0(K_S; \mathbb{Z}) \cong \mathbb{Z}$. Moreover $H_0(K_S; \mathbb{Z})$ is generated by the 0-dimensional homology class represented by the single vertex $\mathbf{v}_1$.

Let L_S denote the one-dimensional subcomplex of K_S consisting of all edges and vertices of K_S that are contained within the boundary of the square $[0,3] \times [0,3]$. Now $H_0(L_S; \mathbb{Z}) \cong \mathbb{Z}$, $H_1(L_S; \mathbb{Z}) \cong \mathbb{Z}$, and $H_q(L_S; \mathbb{Z}) = 0$ when $q > 1$. The group $H_0(L_S; \mathbb{Z})$ is generated by the homology class representing the vertex $\mathbf{v}_1$, and therefore the homomorphism $i_* \colon H_0(L_S; \mathbb{Z}) \to H_0(K_S; \mathbb{Z})$ induced by the inclusion map $i \colon L_S \hookrightarrow K_S$ is an isomorphism. The group $H_1(L_S; \mathbb{Z})$ is generated by the 1-cycle z_S, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

This generating 1-cycle z_S represents the sum of the edges of K_S that lie on the boundary of the square, where the orientation on each edge is consistent with an anticlockwise traversal of the boundary of the square S.

We can use the homology exact sequence of the simplicial pair (K_S, L_S) (Proposition 7.3) in order to evaluate the homology groups $H_*(K_S, L_S; \mathbb{Z})$. The sequence

$$H_1(K_S; \mathbb{Z}) \longrightarrow H_1(K_S, L_S; \mathbb{Z}) \xrightarrow{\partial_*} H_0(L_S; \mathbb{Z}) \xrightarrow{i_{S*}} H_0(K_S; \mathbb{Z})$$

is exact, where the homomorphism $i_{S*} \colon H_0(L_S; \mathbb{Z}) \to H_0(K_S; \mathbb{Z})$ is induced by the inclusion map $i_S \colon L_S \hookrightarrow K_S$. We have noted that this homomorphism is an isomorphism. It follows from the exactness of the above sequence that $\partial_* \colon H_1(K_S, L_S; \mathbb{Z}) \to H_0(L_S; \mathbb{Z})$ is the zero homomorphism. Therefore its kernel is the whole of $H_1(K_S, L_S; \mathbb{Z})$, and therefore the homomorphism from $H_1(K_S;\mathbb{Z})$ to $H_1(K_S, L_S;\mathbb{Z})$ is surjective. But we have shown that $H_q(K_S;\mathbb{Z}) = 0$ for $q > 0$. It follows that $H_1(K_S, L_S;\mathbb{Z}) = 0$.

The homology exact sequence of the pair also ensures that the homomorphism $\partial_*: H_2(K_S, L_S;\mathbb{Z}) \to H_1(L_S;\mathbb{Z})$ is an isomophism, since $H_2(K_S;\mathbb{Z}) = 0$ and $H_2(L_S;\mathbb{Z}) = 0$ (see Corollary 7.5). But $H_1(L_S;\mathbb{Z}) \cong \mathbb{Z}$. It follows that $H_2(K_S, L_S;\mathbb{Z}) \cong \mathbb{Z}$.

In fact $H_2(K_S, L_S;\mathbb{Z}) \cong Z_2(K_S, L_S)$ since $B_2(K_S, L_S) = 0$. Moreover

$$Z_2(K_S, L_S) = \{ny_S + C_2(L_S) : n \in \mathbb{Z}\},$$

where $y_S = \sum_{j=1}^{18} t_j$. Indeed let c be a 2-chain of K_S. Then there are integers $n_1, \ldots, n_{18}$ such that $c = \sum_{j=1}^{18} n_j t_j$. Now any edge belonging to $K_S \setminus L_S$ lies on the boundary of exactly two triangles T_j and $T_{j'}$ of K_S. Moreover the orientation on that edge determined by the anticlockwise ordering of the vertices of T_j is opposite to the orientation determined by the anticlockwise ordering of the vertices of $T_{j'}$, and therefore the coefficient of this edge in $\partial_2 c$ is $\pm(n_j - n_{j'})$. It follows that $\partial_2 c \in C_1(L;\mathbb{Z})$ if and only if $n_1 = n_2 = \cdots = n_{18}$, in which case $c_2 = ny_S$ for some integer n. It is then easy to verify that $\partial_2(y_S) = z_S$.

Example We shall make use of the above results to calculate the homology groups of a torus. The two-dimensional torus may be represented as the quotient space obtained from the square $[0,3] \times [0,3]$ by identifying the points $(x, 0)$ and $(x, 3)$ for all $x \in [0,3]$, and also identifying the points $(0, y)$ and $(3, y)$ for all $y \in [0,3]$. Thus each point on an edge of the square is identified with a corresponding point on the opposite edge of the square. The four corners of the square are identified together, so as to represent a single point of the torus.

Now there exists a simplicial complex K_T, and a simplicial map $p: K_S \to K_T$ where K_S is the simplicial complex triangulating the square $[0,3] \times [0,3]$ discussed in the previous example, where the polyhedron $|K_T|$ of K_T is homeomorphic to the torus, and where the induced map $p: |K_S| \to |K_T|$ between polyhedra is an identification map which identifies points on opposite edges of the square S as described above. Moreover this simplicial complex K_T has 18 triangles, 27 edges and 9 vertices. Throughout this example we shall use the notation developed in the previous example to describe the simplical complex K_S and its chain groups and homology groups.

Let the vertices of K_T be labelled as $\mathbf{w}_1, \ldots, \mathbf{w}_9$, where

$$\mathbf{w}_1 \;=\; p(\mathbf{v}_1) = p(\mathbf{v}_4) = p(\mathbf{v}_{13}) = p(\mathbf{v}_{16}),$$

$$\begin{aligned}
\mathbf{w}_2 &= p(\mathbf{v}_2) = p(\mathbf{v}_{14}),\\
\mathbf{w}_3 &= p(\mathbf{v}_3) = p(\mathbf{v}_{15}),\\
\mathbf{w}_4 &= p(\mathbf{v}_5) = p(\mathbf{v}_8),\\
\mathbf{w}_5 &= p(\mathbf{v}_9) = p(\mathbf{v}_{12}),\\
\mathbf{w}_6 &= p(\mathbf{v}_6),\\
\mathbf{w}_7 &= p(\mathbf{v}_7),\\
\mathbf{w}_8 &= p(\mathbf{v}_{10}),\\
\mathbf{w}_9 &= p(\mathbf{v}_{11}).
\end{aligned}$$

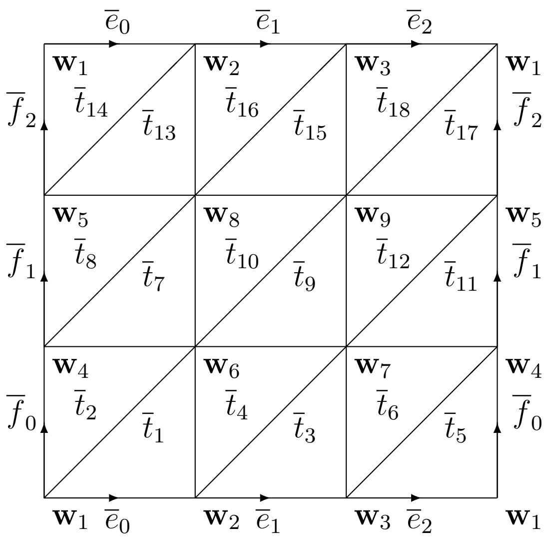
and let

$$\overline{e}_0 = \langle \mathbf{w}_1, \mathbf{w}_2\rangle, \quad \overline{e}_1 = \langle \mathbf{w}_2, \mathbf{w}_3\rangle, \quad \overline{e}_2 = \langle \mathbf{w}_3, \mathbf{w}_1\rangle,$$
$$\overline{f}_0 = \langle \mathbf{w}_1, \mathbf{w}_4\rangle, \quad \overline{f}_1 = \langle \mathbf{w}_4, \mathbf{w}_5\rangle, \quad \overline{f}_2 = \langle \mathbf{w}_5, \mathbf{w}_1\rangle,$$

Then

$$p_\#(e_0^+) = p_\#(e_0^-) = \overline{e}_0, \quad p_\#(e_1^+) = p_\#(e_1^-) = \overline{e}_1,$$
$$p_\#(e_2^+) = p_\#(e_2^-) = \overline{e}_2, \quad p_\#(f_0^+) = p_\#(f_0^-) = \overline{f}_0,$$
$$p_\#(f_1^+) = p_\#(f_1^-) = \overline{f}_1, \quad p_\#(f_2^+) = p_\#(f_2^-) = \overline{f}_2,$$

where $p_\#\colon C_1(K_S) \to C_1(K_T)$ is the homomorphism of chain groups induced by $p\colon K_S \to K_T$. Also let $\overline{t}_j = p_\#(t_j)$ for $j = 1, 2, \ldots, 18$, where $p_\#\colon C_2(K_S) \to C_2(K_T)$ is the homomorphism of chain groups induced by the simplicial map $p\colon K_S \to K_T$. Then the triangulation K_T of the torus may be represented by the following diagram:



Let $L_T = p(L_S)$. Then L_T is the subcomplex of K_T consisting of the five vertices $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ and $\mathbf{w}_5$, together with the six edges represented by $\overline{e}_0$, $\overline{e}_1$, $\overline{e}_3$, $\overline{f}_0$, $\overline{f}_1$ and $\overline{f}_3$. Now $H_1(L_T;\mathbb{Z}) \cong Z_1(L_T)$, since $B_1(L_T) = 0$. Moreover $Z_1(L_T) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed

$$Z_1(L_T) = \{n_1 z_1 + n_2 z_2 : n_1, n_2 \in \mathbb{Z}\},$$

where $z_1 = \overline{e}_0 + \overline{e}_1 + \overline{e}_2$ and $z_2 = \overline{f}_0 + \overline{f}_1 + \overline{f}_2$. The simplicial map $p\colon K_S \to K_T$ defines a bijection between the simplices of $K_S \setminus L_S$ and those of $K_T \setminus L_T$. It follows from this that the chain map $p_*\colon C_*(K_S, L_S) \to C_*(K_T, L_T)$ induced by the simplicial map is an isomorphism of chain complexes, and therefore induces isomorphisms

$$p_*\colon H_*(K_S, L_S;\mathbb{Z}) \to H_*(K_T, L_T;\mathbb{Z}).$$

We conclude that $H_2(K_T, L_T;\mathbb{Z}) \cong \mathbb{Z}$, and $H_q(K_T, L_T;\mathbb{Z}) = 0$ when $q \neq 2$. Moreover $H_2(K_T, L_T;\mathbb{Z})$ is generated by the homology class of z_T, where

$$y_T = \sum_{j=1}^{18} \overline{t}_j = p_\#(y_S).$$

Also $H_2(L_T;\mathbb{Z}) = 0$, because the simplicial complex L_T is one-dimensional.

We now determine the homomorphism $\partial_*\colon H_2(K_T, L_T;\mathbb{Z}) \to H_1(L_T;\mathbb{Z})$. Now the following diagram relating homology groups of the square and the torus is commutative:

$$\begin{array}{ccc} H_2(K_S, L_S;\mathbb{Z}) & \xrightarrow{\partial_*} & H_1(L_S;\mathbb{Z}) \\ \Big\downarrow{\scriptstyle p_*} & & \Big\downarrow{\scriptstyle p_*} \\ H_2(K_T, L_T;\mathbb{Z}) & \xrightarrow{\partial_*} & H_1(L_T;\mathbb{Z}) \end{array}$$

Moreover

$$H_1(L_S;\mathbb{Z}) \cong H_2(K_S, L_S;\mathbb{Z}) \cong H_2(K_T, L_T;\mathbb{Z}) \cong \mathbb{Z},$$

and the homomorphisms

$$\partial_*\colon H_2(K_S, L_S;\mathbb{Z}) \to H_1(L_S;\mathbb{Z})$$

and

$$p_*\colon H_2(K_S, L_S;\mathbb{Z}) \to H_2(K_T, L_T;\mathbb{Z})$$

are isomorphisms. Let μ_{K_S,L_S} and μ_{K_T,L_T} be the homology classes in the relative homology groups $H_2(K_S, L_S;\mathbb{Z})$ and $H_2(K_T, L_T;\mathbb{Z})$ respectively represented by y_S and y_T. Then $p_*(\mu_{K_S,L_S}) = \mu_{K_T,L_T}$. and $\partial_*(\mu_{K_S,L_S}) = [z_S]$, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

It follows that $\partial_*(\mu_{K_T,L_T}) = p_*(z_S)$.

We now calculate the image of z_S under the homomorphism

$$p_\#: C_1(L_S) \to C_1(L_T)$$

induced by the simplicial map $p: K_S \to K_T$. We find that

$$\begin{aligned} p_\#(z_S) &= p_\#(e_0^-) + p_\#(e_1^-) + p_\#(e_2^-) \\ &\quad + p_\#(f_0^+) + p_\#(f_1^+) + p_\#(f_2^+) \\ &\quad - p_\#(e_2^+) - p_\#(e_1^+) - p_\#(e_0^+) \\ &\quad - p_\#(f_2^-) - p_\#(f_1^-) - p_\#(f_0^-) \\ &= \overline{e}_0 + \overline{e}_1 + \overline{e}_2 \\ &\quad + \overline{f}_0 + \overline{f}_1 + \overline{f}_2 \\ &\quad - \overline{e}_2 - \overline{e}_1 - \overline{e}_0 \\ &\quad - \overline{f}_2 - \overline{f}_1 - \overline{f}_0 \\ &= z_1 + z_2 - z_1 - z_2 = 0. \end{aligned}$$

Therefore $\partial_*(\mu_{K_T,L_T}) = p_*(z_S) = 0$. We conclude from this that

$$\partial_*: H_2(K_T, L_T; \mathbb{Z}) \to H_1(L_T; \mathbb{Z})$$

is the zero homorphism.

We now have the information required in order to calculate the homology groups of the simplicial complex K_T. The homology exact sequence of the simplicial pair (K_T, L_T) gives rise to the following exact sequence:

$$0 \longrightarrow H_2(K_T; \mathbb{Z}) \longrightarrow H_2(K_T, L_T; \mathbb{Z}) \xrightarrow{\partial_*} H_1(L_T; \mathbb{Z}) \xrightarrow{i_{T*}} H_1(K_T; \mathbb{Z}) \longrightarrow 0.$$

Using the exactness of this sequence, together with the result that $\partial_* = 0$, we conclude that $H_2(K_T; \mathbb{Z}) \cong H_2(K_T, L_T; \mathbb{Z})$ and $H_1(K_T; \mathbb{Z}) \cong H_1(L_T; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed

$$H_2(K_T; \mathbb{Z}) = \{n[y_T] : n \in \mathbb{Z}\}$$

and

$$H_1(K_T; \mathbb{Z}) = \{n_1[z_1] + n_2[z_2] : n_1, n_2 \in \mathbb{Z}\},$$

where y_T, z_1 and z_2 are the 1-cycles of L_T defined above. Thus

$$H_0(K_T; \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K_T; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(K_T; \mathbb{Z}) \cong \mathbb{Z}.$$

Example We shall make use of the above results to calculate the homology groups of a Klein Bottle. The Klein Bottle may be represented as the quotient space obtained from the square $[0,3] \times [0,3]$ by identifying the points $(x, 0)$ and $(x, 3)$ for all $x \in [0,3]$, and also identifying the points $(0, y)$ and $(3, 3-y)$ for all $y \in [0,3]$. Thus each point on an edge of the square is identified with some other point on the opposite edge of the square. The four corners of the square are identified together, so as to represent a single point of the Klein Bottle.

Now there exists a simplicial complex K_{KlB}, and a simplicial map $r\colon K_S \to K_{KlB}$ where K_S is the simplicial complex triangulating the square $[0,3]\times[0,3]$ discussed in the previous example, where the polyhedron $|K_{KlB}|$ of K_{KlB} is homeomorphic to the Klein Bottle, and where the induced map $r\colon |K_S| \to |K_{KlB}|$ between polyhedra is an identification map which identifies points on opposite edges of the square S as described above. Moreover this simplicial complex K_{KlB} has 18 triangles, 27 edges and 9 vertices. Throughout this example we shall use the notation developed in a previous example to describe the simplical complex K_S and its chain groups and homology groups.

Let the vertices of K_{KlB} be labelled as $\mathbf{u}_1, \ldots, \mathbf{u}_9$, where

$$\begin{array}{rcl}
\mathbf{u}_1 & = & r(\mathbf{v}_1) = r(\mathbf{v}_4) = r(\mathbf{v}_{13}) = r(\mathbf{v}_{16}), \\
\mathbf{u}_2 & = & r(\mathbf{v}_2) = r(\mathbf{v}_{14}), \\
\mathbf{u}_3 & = & r(\mathbf{v}_3) = r(\mathbf{v}_{15}), \\
\mathbf{u}_4 & = & r(\mathbf{v}_9) = r(\mathbf{v}_8), \\
\mathbf{u}_5 & = & r(\mathbf{v}_5) = r(\mathbf{v}_{12}), \\
\mathbf{u}_6 & = & r(\mathbf{v}_6), \\
\mathbf{u}_7 & = & r(\mathbf{v}_7), \\
\mathbf{u}_8 & = & r(\mathbf{v}_{10}), \\
\mathbf{u}_9 & = & r(\mathbf{v}_{11}).
\end{array}$$

and let

$$\hat{e}_0 = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle, \quad \hat{e}_1 = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle, \quad \hat{e}_2 = \langle \mathbf{u}_3, \mathbf{u}_1 \rangle,$$

$$\hat{f}_0 = \langle \mathbf{u}_1, \mathbf{u}_4 \rangle, \quad \hat{f}_1 = \langle \mathbf{u}_4, \mathbf{u}_5 \rangle, \quad \hat{f}_2 = \langle \mathbf{u}_5, \mathbf{u}_1 \rangle,$$

Then

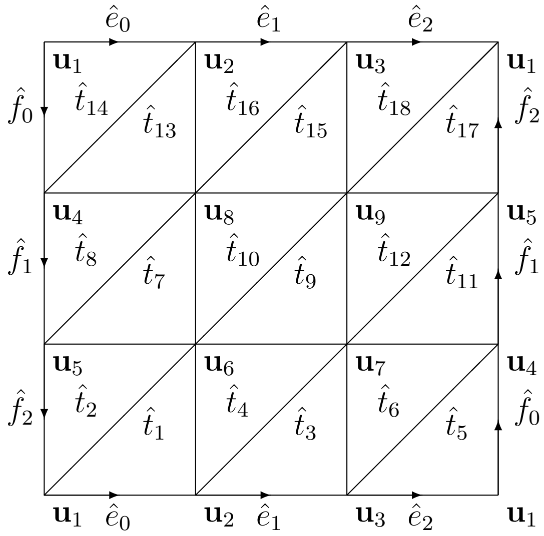
$$r_\#(e_0^+) = r_\#(e_0^-) = \hat{e}_0, \quad r_\#(e_1^+) = r_\#(e_1^-) = \hat{e}_1,$$

$$r_\#(e_2^+) = r_\#(e_2^-) = \hat{e}_2, \quad r_\#(f_0^+) = -r_\#(f_2^-) = \hat{f}_0,$$

$$r_\#(f_1^+) = -r_\#(f_1^-) = \hat{f}_1, \quad r_\#(f_2^+) = -r_\#(f_0^-) = \hat{f}_2,$$

where $r_\#\colon C_1(K_S) \to C_1(K_{KlB})$ is the homomorphism of chain groups induced by $r\colon K_S \to K_{KlB}$. Also let $\hat{t}_j = r_\#(t_j)$ for $j = 1, 2, \ldots, 18$, where

$r_\#: C_2(K_S) \to C_2(K_{KlB})$ is the homomorphism of chain groups induced by the simplicial map $r: K_S \to K_{KlB}$. Then the triangulation K_{KlB} of the Klein Bottle may be represented by the following diagram:



Let $L_{KlB} = r(L_S)$. Then L_{KlB} is the subcomplex of K_{KlB} consisting of the five vertices $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ and $\mathbf{u}_5$, together with the six edges represented by $\hat{e}_0$, $\hat{e}_1$, $\hat{e}_3$, $\hat{f}_0$, $\hat{f}_1$ and $\hat{f}_3$. Now $H_1(L_{KlB}; \mathbb{Z}) \cong Z_1(L_{KlB})$, since $B_1(L_{KlB}) = 0$. Moreover $Z_1(L_{KlB}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed

$$Z_1(L_{KlB}) = \{n_1 z_1 + n_2 z_2 : n_1, n_2 \in \mathbb{Z}\},$$

where $z_1 = \hat{e}_0 + \hat{e}_1 + \hat{e}_2$ and $z_2 = \hat{f}_0 + \hat{f}_1 + \hat{f}_2$. The simplicial map $r: K_S \to K_{KlB}$ defines a bijection between the simplices of $K_S \setminus L_S$ and those of $K_{KlB} \setminus L_{KlB}$. It follows from this that the chain map $r_*: C_*(K_S, L_S) \to C_*(K_{KlB}, L_{KlB})$ induced by the simplicial map is an isomorphism of chain complexes, and therefore induces isomorphisms

$$r_*: H_*(K_S, L_S; \mathbb{Z}) \to H_*(K_{KlB}, L_{KlB}; \mathbb{Z}).$$

We conclude that $H_2(K_{KlB}, L_{KlB}; \mathbb{Z}) \cong \mathbb{Z}$, and $H_q(K_{KlB}, L_{KlB}; \mathbb{Z}) = 0$ when $q \neq 2$. Moreover $H_2(K_{KlB}, L_{KlB}; \mathbb{Z})$ is generated by the homology class of z_{KlB}, where

$$y_{KlB} = \sum_{j=1}^{18} \hat{t}_j = r_\#(y_S).$$

Also $H_2(L_{KlB}; \mathbb{Z}) = 0$, because the simplicial complex L_{KlB} is one-dimensional.

We now determine the homomorphism

$$\partial_*\colon H_2(K_{KlB}, L_{KlB}; \mathbb{Z}) \to H_1(L_{KlB}; \mathbb{Z}).$$

Now the following diagram relating homology groups of the square and the Klein Bottle is commutative:

$$\begin{array}{ccc} H_2(K_S, L_S; \mathbb{Z}) & \xrightarrow{\partial_*} & H_1(L_S; \mathbb{Z}) \\ \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} \\ H_2(K_{KlB}, L_{KlB}; \mathbb{Z}) & \xrightarrow{\partial_*} & H_1(L_{KlB}; \mathbb{Z}) \end{array}$$

Moreover

$$H_1(L_S; \mathbb{Z}) \cong H_2(K_S, L_S; \mathbb{Z}) \cong H_2(K_{KlB}, L_{KlB}; \mathbb{Z}) \cong \mathbb{Z},$$

and the homomorphisms

$$\partial_*\colon H_2(K_S, L_S; \mathbb{Z}) \to H_1(L_S; \mathbb{Z})$$

and

$$r_*\colon H_2(K_S, L_S; \mathbb{Z}) \to H_2(K_{KlB}, L_{KlB}; \mathbb{Z})$$

are isomorphisms. Let μ_{K_S,L_S} and $\mu_{K_{KlB},L_{KlB}}$ be the homology classes in $H_2(K_S, L_S; \mathbb{Z})$ and $H_2(K_{KlB}, L_{KlB}; \mathbb{Z})$ respectively represented by y_S and y_{KlB}. Then $r_*(\mu_{K_S,L_S}) = \mu_{K_{KlB},L_{KlB}}$. and $\partial_*(\mu_{K_S,L_S}) = [z_S]$, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

It follows that $\partial_*(\mu_{K_{KlB},L_{KlB}}) = r_*(z_S)$.

We now calculate the image of z_S under the homomorphism

$$r_\#\colon C_1(L_S) \to C_1(L_{KlB})$$

induced by the simplicial map $r\colon K_S \to K_{KlB}$. We find that

$$\begin{aligned} r_\#(z_S) &= r_\#(e_0^-) + r_\#(e_1^-) + r_\#(e_2^-) \\ &\quad + r_\#(f_0^+) + r_\#(f_1^+) + r_\#(f_2^+) \\ &\quad - r_\#(e_2^+) - r_\#(e_1^+) - r_\#(e_0^+) \\ &\quad - r_\#(f_2^-) - r_\#(f_1^-) - r_\#(f_0^-) \\ &= \hat{e}_0 + \hat{e}_1 + \hat{e}_2 \\ &\quad + \hat{f}_0 + \hat{f}_1 + \hat{f}_2 \\ &\quad - \hat{e}_2 - \hat{e}_1 - \hat{e}_0 \\ &\quad + \hat{f}_0 + \hat{f}_1 + \hat{f}_2 \\ &= z_1 + z_2 - z_1 + z_2 = 2z_2. \end{aligned}$$

Therefore $\partial_*(\mu_{K_{KlB},L_{KlB}}) = r_*(z_S) = 2z_2$. We conclude from this that

$$\partial_* \colon H_2(K_{KlB}, L_{KlB}; \mathbb{Z}) \to H_1(L_{KlB}; \mathbb{Z})$$

is an injective homomorphism whose image is the subgroup of $H_1(KlB; \mathbb{Z})$ generated by $2[z_2]$.

We now have the information required in order to calculate the homology groups of the simplicial complex K_{KlB}. The homology exact sequence of the simplicial pair (K_{KlB}, L_{KlB}) gives rise to the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H_2(K_{KlB}; \mathbb{Z}) \longrightarrow H_2(K_{KlB}, L_{KlB}; \mathbb{Z}) &\xrightarrow{\partial_*} H_1(L_{KlB}; \mathbb{Z}) \\ &\xrightarrow{i_{T*}} H_1(K_{KlB}; \mathbb{Z}) \longrightarrow 0. \end{aligned}$$

Using the exactness of this sequence, together with the result that ∂_* is injective, we conclude that $H_2(K_{KlB}; \mathbb{Z}) = 0$. Also

$$H_1(K_{KlB}; \mathbb{Z}) \cong H_1(L_{KlB}; \mathbb{Z})/\partial_*(H_2(K_{KlB}, L_{KlB}; \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Indeed there is an isomorphism $\varphi \colon H_1(L_{KlB}; \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}$ which maps the homology classes of the cycles z_1 and z_2 to $(1, 0)$ and $(0, 1)$ respectively. Then $\varphi(\partial_*(H_2(K_{KlB}, L_{KlB}; \mathbb{Z})))$ is the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(0, 2)$, and the corresponding quotient group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. Thus

$$H_0(K_{KlB}; \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K_{KlB}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \quad H_2(K_{KlB}; \mathbb{Z}) \cong 0.$$

Example We shall make use of the above results to calculate the homology groups of a real projective plane. The real projective plane may be represented as the quotient space obtained from the square $[0, 3] \times [0, 3]$ by identifying the points $(x, 0)$ and $(3 - x, 3)$ for all $x \in [0, 3]$, and also identifying the points $(0, y)$ and $(3, 3 - y)$ for all $y \in [0, 3]$. Thus each point on an edge of the square is identified with some other point on the opposite edge of the square. Also each corner of the square is identified with the corner diagonally opposite, so as to represent a single point of the real projective plane.

Now there exists a simplicial complex K_{PP}, and a simplicial map $s \colon K_S \to K_{PP}$ where K_S is the simplicial complex triangulating the square $[0, 3] \times [0, 3]$ discussed in the previous example, where the polyhedron $|K_{PP}|$ of K_{PP} is homeomorphic to the real projective plane, and where the induced map $s \colon |K_S| \to |K_{PP}|$ between polyhedra is an identification map which identifies points on opposite edges of the square S as described above. Moreover this simplicial complex K_{PP} has 18 triangles, 27 edges and 10 vertices. Throughout this example we shall use the notation developed in a previous example to describe the simplical complex K_S and its chain groups and homology groups.

Let the vertices of K_{PP} be labelled as $\mathbf{q}_1, \ldots, \mathbf{q}_{10}$, where

$$\begin{aligned}
\mathbf{q}_1 &= s(\mathbf{v}_1) = s(\mathbf{v}_{16}), \\
\mathbf{q}_2 &= s(\mathbf{v}_2) = s(\mathbf{v}_{15}), \\
\mathbf{q}_3 &= s(\mathbf{v}_3) = s(\mathbf{v}_{14}), \\
\mathbf{q}_4 &= s(\mathbf{v}_4) = s(\mathbf{v}_{13}), \\
\mathbf{q}_5 &= s(\mathbf{v}_9) = s(\mathbf{v}_8), \\
\mathbf{q}_6 &= s(\mathbf{v}_5) = s(\mathbf{v}_{12}), \\
\mathbf{q}_7 &= s(\mathbf{v}_6), \\
\mathbf{q}_8 &= s(\mathbf{v}_7), \\
\mathbf{q}_9 &= s(\mathbf{v}_{10}), \\
\mathbf{q}_{10} &= s(\mathbf{v}_{11}).
\end{aligned}$$

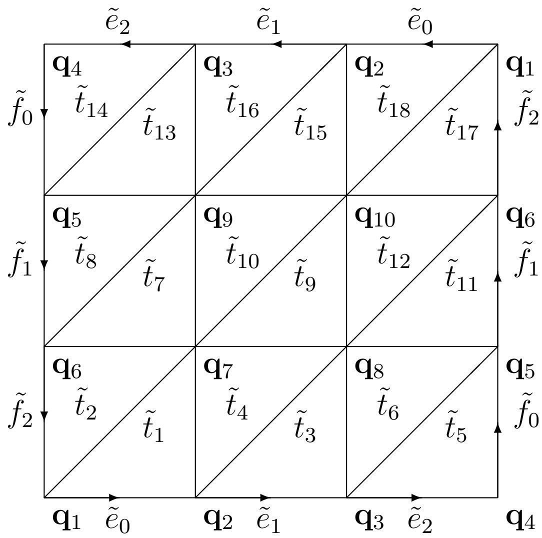
and let

$$\begin{aligned}
\tilde{e}_0 = \langle \mathbf{q}_1, \mathbf{q}_2 \rangle, \quad \tilde{e}_1 = \langle \mathbf{q}_2, \mathbf{q}_3 \rangle, \quad \tilde{e}_2 = \langle \mathbf{q}_3, \mathbf{q}_4 \rangle, \\
\tilde{f}_0 = \langle \mathbf{q}_4, \mathbf{q}_5 \rangle, \quad \tilde{f}_1 = \langle \mathbf{q}_5, \mathbf{q}_6 \rangle, \quad \tilde{f}_2 = \langle \mathbf{q}_6, \mathbf{q}_1 \rangle,
\end{aligned}$$

Then

$$\begin{aligned}
-s_\#(e_2^+) = s_\#(e_0^-) = \tilde{e}_0, \quad -s_\#(e_1^+) = s_\#(e_1^-) = \tilde{e}_1, \\
-s_\#(e_0^+) = s_\#(e_2^-) = \tilde{e}_2, \quad s_\#(f_0^+) = -s_\#(f_2^-) = \tilde{f}_0, \\
s_\#(f_1^+) = -s_\#(f_1^-) = \tilde{f}_1, \quad s_\#(f_2^+) = -s_\#(f_0^-) = \tilde{f}_2,
\end{aligned}$$

where $s_\#: C_1(K_S) \to C_1(K_{PP})$ is the homomorphism of chain groups induced by $s: K_S \to K_{PP}$. Also let $\tilde{t}_j = s_\#(t_j)$ for $j = 1, 2, \ldots, 18$, where $s_\#: C_2(K_S) \to C_2(K_{PP})$ is the homomorphism of chain groups induced by the simplicial map $s: K_S \to K_{PP}$. Then the triangulation K_{PP} of the real projective plane may be represented by the following diagram:



Let $L_{PP} = s(L_S)$. Then L_{PP} is the subcomplex of K_{PP} consisting of the six vertices $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$, $\mathbf{q}_5$ and $\mathbf{q}_6$, together with the six edges represented by $\tilde{e}_0$, $\tilde{e}_1$, $\tilde{e}_3$, $\tilde{f}_0$, $\tilde{f}_1$ and $\tilde{f}_3$. Now $H_1(L_{PP};\mathbb{Z}) \cong Z_1(L_{PP})$, since $B_1(L_{PP}) = 0$. Moreover $Z_1(L_{PP}) \cong \mathbb{Z}$. Indeed

$$Z_1(L_{PP}) = \{nz_0 : n \in \mathbb{Z}\},$$

where

$$z_0 = \tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 + \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2.$$

The simplicial map $s\colon K_S \to K_{PP}$ defines a bijection between the simplices of $K_S \setminus L_S$ and those of $K_{PP} \setminus L_{PP}$. It follows from this that the chain map $s_*\colon C_*(K_S, L_S) \to C_*(K_{PP}, L_{PP})$ induced by the simplicial map is an isomorphism of chain complexes, and therefore induces isomorphisms

$$s_*\colon H_*(K_S, L_S;\mathbb{Z}) \to H_*(K_{PP}, L_{PP};\mathbb{Z}).$$

We conclude that $H_2(K_{PP}, L_{PP};\mathbb{Z}) \cong \mathbb{Z}$, and $H_q(K_{PP}, L_{PP};\mathbb{Z}) = 0$ when $q \neq 2$. Moreover $H_2(K_{PP}, L_{PP};\mathbb{Z})$ is generated by the homology class of z_{PP}, where

$$y_{PP} = \sum_{j=1}^{18} \tilde{t}_j = s_\#(y_S).$$

Also $H_2(L_{PP};\mathbb{Z}) = 0$, because the simplicial complex L_{PP} is one-dimensional.

We now determine the homomorphism

$$\partial_*\colon H_2(K_{PP}, L_{PP};\mathbb{Z}) \to H_1(L_{PP};\mathbb{Z}).$$

Now the following diagram relating homology groups of the square and the real projective plane is commutative:

$$\begin{array}{ccc} H_2(K_S, L_S; \mathbb{Z}) & \xrightarrow{\partial_*} & H_1(L_S; \mathbb{Z}) \\ \Big\downarrow{\scriptstyle s_*} & & \Big\downarrow{\scriptstyle s_*} \\ H_2(K_{PP}, L_{PP}; \mathbb{Z}) & \xrightarrow{\partial_*} & H_1(L_{PP}; \mathbb{Z}) \end{array}$$

Moreover

$$H_1(L_S; \mathbb{Z}) \cong H_2(K_S, L_S; \mathbb{Z}) \cong H_2(K_{PP}, L_{PP}; \mathbb{Z}) \cong \mathbb{Z},$$

and the homomorphisms

$$\partial_*: H_2(K_S, L_S; \mathbb{Z}) \to H_1(L_S; \mathbb{Z})$$

and

$$s_*: H_2(K_S, L_S; \mathbb{Z}) \to H_2(K_{PP}, L_{PP}; \mathbb{Z})$$

are isomorphisms. Let μ_{K_S,L_S} and $\mu_{K_{PP},L_{PP}}$ be the homology classes in $H_2(K_S, L_S; \mathbb{Z})$ and $H_2(K_{PP}, L_{PP}; \mathbb{Z})$ respectively represented by y_S and y_{PP}. Then $s_*(\mu_{K_S,L_S}) = \mu_{K_{PP},L_{PP}}$. and $\partial_*(\mu_{K_S,L_S}) = [z_S]$, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

It follows that $\partial_*(\mu_{K_{PP},L_{PP}}) = s_*(z_S)$.

We now calculate the image of z_S under the homomorphism

$$s_\#: C_1(L_S) \to C_1(L_{PP})$$

induced by the simplicial map $s: K_S \to K_{PP}$. We find that

$$\begin{aligned} s_\#(z_S) &= s_\#(e_0^-) + s_\#(e_1^-) + s_\#(e_2^-) \\ &\quad + s_\#(f_0^+) + s_\#(f_1^+) + s_\#(f_2^+) \\ &\quad - s_\#(e_2^+) - s_\#(e_1^+) - s_\#(e_0^+) \\ &\quad - s_\#(f_2^-) - s_\#(f_1^-) - s_\#(f_0^-) \\ &= \tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 \\ &\quad + \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2 \\ &\quad + \tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 \\ &\quad + \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2 \\ &= 2z_0. \end{aligned}$$

Therefore $\partial_*(\mu_{K_{PP},L_{PP}}) = s_*(z_S) = 2z_0$. We conclude from this that

$$\partial_*: H_2(K_{PP}, L_{PP}; \mathbb{Z}) \to H_1(L_{PP}; \mathbb{Z})$$

is an injective homomorphism whose image is the subgroup of $H_1(PP;\mathbb{Z})$ generated by $2[z_0]$.

We now have the information required in order to calculate the homology groups of the simplicial complex K_{PP}. The homology exact sequence of the simplicial pair (K_{PP}, L_{PP}) gives rise to the following exact sequence:

$$0 \longrightarrow H_2(K_{PP};\mathbb{Z}) \longrightarrow H_2(K_{PP}, L_{PP};\mathbb{Z}) \xrightarrow{\partial_*} H_1(L_{PP};\mathbb{Z}) \xrightarrow{i_{T*}} H_1(K_{PP};\mathbb{Z}) \longrightarrow 0.$$

Using the exactness of this sequence, together with the result that ∂_* is injective, we conclude that $H_2(K_{PP};\mathbb{Z}) = 0$. Also

$$H_1(K_{PP};\mathbb{Z}) \cong H_1(L_{PP};\mathbb{Z})/\partial_*(H_2(K_{PP}, L_{PP};\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

Thus

$$H_0(K_{PP};\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K_{PP};\mathbb{Z}) \cong \mathbb{Z}_2, \quad H_2(K_{PP};\mathbb{Z}) \cong 0.$$

7.5 The Mayer-Vietoris Sequence

Let K be a simplicial complex, let L and M be subcomplexes of K such that $K = L \cup M$, and let R be an integral domain. Let

$$i_q: C_q(L \cap M) \to C_q(L;R), \qquad j_q: C_q(L \cap M) \to C_q(M;R),$$

$$u_q: C_q(L;R) \to C_q(K;R), \qquad v_q: C_q(M;R) \to C_q(K;R)$$

be the inclusion homomorphisms induced by the inclusion maps $i: L \cap M \hookrightarrow L$, $j: L \cap M \hookrightarrow M$, $u: L \hookrightarrow K$ and $v: M \hookrightarrow K$. Then

$$0 \longrightarrow C_*(L \cap M) \xrightarrow{k_*} C_*(L;R) \oplus C_*(M;R) \xrightarrow{w_*} C_*(K;R) \longrightarrow 0$$

is a short exact sequence of chain complexes, where

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M)$, $c' \in C_q(L;R)$ and $c'' \in C_q(M;R)$. It follows from Lemma 6.4 that there is a well-defined homomorphism $\alpha_q: H_q(K;R) \to H_{q-1}(L \cap M;R)$ such that $\alpha_q([z]) = [\partial_q(c')] = -[\partial_q(c'')]$ for any $z \in Z_q(K)$, where c' and c'' are any q-chains of L and M respectively satisfying $z = c' + c''$. (Note that $\partial_q(c') \in Z_{q-1}(L \cap M)$ since $\partial_q(c') \in Z_{q-1}(L)$, $\partial_q(c'') \in Z_{q-1}(M)$

and $\partial_q(c') = -\partial_q(c'')$.) It now follows immediately from Proposition 6.5 that the infinite sequence

$$\begin{aligned}\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M; R) \xrightarrow{k_*} H_q(L; R) \oplus H_q(M; R) \xrightarrow{w_*} H_q(K; R)\\ \xrightarrow{\alpha_q} H_{q-1}(L \cap M; R) \xrightarrow{k_*} \cdots,\end{aligned}$$

of homology groups is exact. This long exact sequence of homology groups is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.

8 The Topological Invariance of Simplicial Homology Groups

8.1 Contiguous Simplicial Maps

Definition Two simplicial maps $s\colon K \to L$ and $t\colon K \to L$ between simplicial complexes K and L are said to be *contiguous* if, given any simplex σ of K, there exists a simplex τ of L such that $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ.

Lemma 8.1 *Let K and L be simplicial complexes, and let $s\colon K \to L$ and $t\colon K \to L$ be simplicial approximations to some continuous map $f\colon |K| \to |L|$. Then the simplicial maps s and t are contiguous.*

Proof Let $\mathbf{x}$ be a point in the interior of some simplex σ of K. Then $f(\mathbf{x})$ belongs to the interior of a unique simplex τ of L, and moreover $s(\mathbf{x}) \in \tau$ and $t(\mathbf{x}) \in \tau$, since s and t are simplicial approximations to the map f. But $s(\mathbf{x})$ and $t(\mathbf{x})$ are contained in the interior of the simplices $s(\sigma)$ and $t(\sigma)$ of L. It follows that $s(\sigma)$ and $t(\sigma)$ are faces of τ, and hence $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ, as required. ∎

Proposition 8.2 *Let $s\colon K \to L$ and $t\colon K \to L$ be simplicial maps between simplicial complexes K and L, and let R be an integral domain. Suppose that s and t are contiguous. Then the homomorphisms $s_*\colon H_q(K;R) \to H_q(L;R)$ and $t_*\colon H_q(K;R) \to H_q(L;R)$ coincide for all q.*

Proof Choose an ordering of the vertices of K. Then there are well-defined homomorphisms $D_q\colon C_q(K;R) \to C_{q+1}(L;R)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) = \sum_{j=0}^{q} (-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle.$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_q$ are the vertices of a q-simplex of K listed in increasing order (with respect to the chosen ordering of the vertices of K). Then

$$\partial_1(D_0(\langle \mathbf{v}\rangle)) = \partial_1(\langle s(\mathbf{v}), t(\mathbf{v})\rangle) = \langle t(\mathbf{v})\rangle - \langle s(\mathbf{v})\rangle,$$

and thus $\partial_1 \circ D_0 = t_0 - s_0$. Also

$$D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q\rangle))$$

$$
\begin{aligned}
&= \sum_{i=0}^{q}(-1)^i D_{q-1}(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q\rangle) \\
&= \sum_{i=0}^{q}\sum_{j=0}^{i-1}(-1)^{i+j}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q)\rangle \\
&\quad + \sum_{i=0}^{q}\sum_{j=i+1}^{q}(-1)^{i+j-1}\langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \ldots \mathbf{v}_q\rangle)) \\
&\quad = \sum_{j=0}^{q}(-1)^j \partial_{q+1}(\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle) \\
&\quad = \sum_{j=0}^{q}\sum_{i=0}^{j-1}(-1)^{i+j}\langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle \\
&\qquad + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle + \sum_{j=1}^{q}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle \\
&\qquad - \sum_{j=0}^{q-1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle \\
&\qquad + \sum_{j=0}^{q}\sum_{i=j+1}^{q}(-1)^{i+j+1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q)\rangle \\
&\quad = -D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q\rangle)) + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle
\end{aligned}
$$

and thus

$$\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$$

for all $q > 0$. It follows that $t_q(z) - s_q(z) = \partial_{q+1}(D_q(z))$ for any q-cycle z of K, and therefore $s_*([z]) = t_*([z])$. Thus $s_* = t_*$ as homomorphisms from $H_q(K; R)$ to $H_q(L; R)$, as required. ∎

8.2 The Homology of Barycentric Subdivisions

We shall show that the homology groups of a simplicial complex are isomorphic to those of its first barycentric subdivision.

We recall that the vertices of the first barycentric subdivision K' of a simplicial complex K are the barycentres $\hat{\sigma}$ of the simplices σ of K, and that K' consists of the simplices spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_q$, where $\sigma_0, \sigma_1, \ldots, \sigma_q \in K$ and σ_{j-1} is a proper face of σ_j for $j = 1, 2, \ldots, q$.

Lemma 8.3 *Let K' be the first barycentric subdivision of a simplicial complex K. Then a function $\zeta\colon \mathrm{Vert}\, K' \to \mathrm{Vert}\, K$ from the vertices of K' to those of K represents a simplicial approximation to the identity map of $|K|$ if and only if it sends the barycentre of any simplex of K to some vertex of that simplex.*

Proof If ζ represents a simplicial approximation to the identity map of $|K|$ then $\zeta(\hat{\sigma}) \in \sigma$ for any $\sigma \in K$, and hence $\zeta(\hat{\sigma})$ is a vertex of σ.

Conversely suppose that the function ζ sends the barycentre of any simplex of K to a vertex of that simplex. Let τ be a simplex of K'. Then it follows from the definition of K' that the interior of τ is contained in the interior of some simplex σ of K, and the vertices of τ are barycentres of faces of σ. Then ζ must map the vertices of τ to vertices of σ, and hence ζ represents a simplicial map from K' to K. Moreover this simplicial map is a simplicial approximation to the identity map, since the interior of τ is contained in σ and ζ maps the interior of τ into σ. ∎

It follows from Lemma 8.3 that there exist simplicial approximations $\zeta\colon K' \to K$ to the identity map of $|K|$: such a simplicial approximation can be obtained by choosing, for each $\sigma \in K$, a vertex $\mathbf{v}_\sigma$ of σ, and defining $\zeta(\hat{\sigma}) = \mathbf{v}_\sigma$.

Suppose that $\zeta\colon K' \to K$ and $\theta\colon K' \to K$ are both simplicial approximations to the identity map of $|K|$. Then ζ and θ are contiguous (Lemma 8.1), and therefore the homomorphisms ζ_* and θ_* of homology groups induced by ζ and θ must coincide. It follows that, given any integral domain R and any integer q, there is a well-defined natural homomorphism $\nu_K\colon H_q(K'; R) \to H_q(K; R)$ of homology groups which coincides with ζ_* for any simplicial approximation $\zeta\colon K' \to K$ to the identity map of $|K|$.

Theorem 8.4 *Let K be a simplicial complex, and let R be an integral domain. The natural homomorphism $\nu_K\colon H_q(K'; R) \to H_q(K; R)$ is an isomorphism for any simplicial complex K.*

Proof Let M be the simplicial complex consisting of some simplex σ together with all of its faces. Then $H_0(M; R) \cong R$, $H_0(M'; R) \cong R$, and $H_q(M; R) = 0 = H_q(M'; R)$ for all $q > 0$ (see Proposition 4.8 and the following example). Let $\mathbf{v}$ be a vertex of M. If $\theta\colon M' \to M$ is any simplicial approximation to the identity map of $|M|$ then $\theta(\mathbf{v}) = \mathbf{v}$. But the homology class of $\langle \mathbf{v} \rangle$ generates both $H_0(M;R)$ and $H_0(M';R)$. It follows that $\theta_*: H_0(M';R) \to H_0(M;R)$ is an isomorphism, and thus $\nu_M: H_q(M';R) \to H_q(M;R)$ is an isomorphism for all q.

We now use induction on the number of simplices in K to prove the theorem in the general case. It therefore suffices to prove that the required result holds for a simplicial complex K under the additional assumption that the result is valid for all proper subcomplexes of K.

Let σ be a simplex of K whose dimension equals the dimension of K. Then σ is not a face of any other simplex of K, and therefore $K \setminus \{\sigma\}$ is a subcomplex of K. Let M be the subcomplex of K consisting of the simplex σ, together with all of its faces. We have already proved the result in the special case when $K = M$. Thus we only need to verify the result in the case when M is a proper subcomplex of K. In that case $K = L \cup M$, where $L = K \setminus \{\sigma\}$.

Let $\zeta: K' \to K$ be a simplicial approximation to the identity map of $|K|$. Then the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively. Therefore the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_q(L'\cap M';R) \arrow[r] \arrow[d, "\zeta|L'\cap M'"] & C_q(L';R)\oplus C_q(M';R) \arrow[r] \arrow[d, "(\zeta|L')\oplus(\zeta|M')"] & C_q(K';R) \arrow[r] \arrow[d, "\zeta"] & 0 \\
0 \arrow[r] & C_q(L\cap M;R) \arrow[r] & C_q(L;R)\oplus C_q(M;R) \arrow[r] & C_q(K;R) \arrow[r] & 0
\end{tikzcd}$$

commutes, and its rows are short exact sequences. But the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively, and therefore induce the natural homomorphisms ν_L, ν_M and $\nu_{L\cap M}$. We therefore obtain a commutative diagram

$$\begin{tikzcd}
H_q(L'\cap M';R) \arrow[r] \arrow[d, "\nu_{L\cap M}"] & H_q(L';R)\oplus H_q(M';R) \arrow[r] \arrow[d, "\nu_L\oplus\nu_M"] & H_q(K';R) \arrow[r, "\alpha_q"] \arrow[d, "\nu_K"] & H_{q-1}(L'\cap M';R) \arrow[r] \arrow[d, "\nu_{L\cap M}"] & H_{q-1}(L';R)\oplus H_{q-1}(M';R) \arrow[d, "\nu_L\oplus\nu_M"] \\
H_q(L\cap M;R) \arrow[r] & H_q(L;R)\oplus H_q(M;R) \arrow[r] & H_q(K;R) \arrow[r, "\alpha_q"] & H_{q-1}(L\cap M;R) \arrow[r] & H_{q-1}(L;R)\oplus H_{q-1}(M;R)
\end{tikzcd}$$

in which the rows are exact sequences, and are the Mayer-Vietoris sequences corresponding to the decompositions $K = L \cup M$ and $K' = L' \cup M'$ of K and K'. But the induction hypothesis ensures that the homomorphisms ν_L, ν_M and $\nu_{L\cap M}$ are isomorphisms, since L, M and $L \cap M$ are all proper subcomplexes of K. It now follows directly from the Five-Lemma (Lemma 6.3) that $\nu_K: H_q(K';R) \to H_q(K;R)$ is also an isomorphism, as required. ∎

We refer to the isomorphism $\nu_K: H_q(K';R) \to H_q(K;R)$ as the *canonical isomorphism* from the qth homology group of K' to that of K.

For each $j > 0$, we define the canonical isomorphism $\nu_{K,j}\colon H_q(K^{(j)}; R) \to H_q(K; R)$ from the homology groups of the jth barycentric subdivision $K^{(j)}$ of K to those of K itself to be the composition of the natural isomorphisms

$$H_q(K^{(j)}; R) \to H_q(K^{(j-1)}; R) \to \cdots \to H_q(K'; R) \to H_q(K; R)$$

induced by appropriate simplicial approximations to the identity map of $|K|$. Note that if $i \leq j$ then $\nu_{K,i}^{-1} \circ \nu_{K,j}$ is induced by a composition of simplicial approximations to the identity map of $|K|$. But any composition of simplicial approximations to the identity map is itself a simplicial approximation to the identity map (Corollary 3.16). We deduce the following result.

Lemma 8.5 *Let K be a simplicial complex, let i and j be positive integers satisfying $i \leq j$. Then $\nu_{K,j} = \nu_{K,i} \circ \zeta_*$ for some simplicial approximation $\zeta\colon K^{(j)} \to K^{(i)}$ to the identity map of $|K|$.*

8.3 Continuous Maps and Induced Homomorphisms

Proposition 8.6 *Let K and L be simplicial complexes, and let R be an integral domain. Then any continuous map $f\colon |K| \to |L|$ between the polyhedra of K and L induces a well-defined homomorphism $f_*\colon H_q(K; R) \to H_q(L; R)$ of homology groups such that $f_* = s_* \circ \nu_{K,i}^{-1}$ for any simplicial approximation $s\colon K^{(i)} \to L$ to the map f, where $s_*\colon H_q(K^{(i)}; R) \to H_q(L; R)$ is the homomorphism induced by the simplicial map s and $\nu_{K,i}\colon H_q(K^{(i)}; R) \to H_q(K; R)$ is the canonical isomorphism.*

Proof The Simplicial Approximation Theorem (Theorem 3.17) guarantees the existence of a simplicial approximation $s\colon K^{(i)} \to L$ to the map f defined on the ith barycentric subdivision $K^{(i)}$ of K for some sufficiently large i. Thus it only remains to verify that if $s\colon K^{(i)} \to L$ and $t\colon K^{(j)} \to L$ are both simplicial approximations to the map f then $s_* \circ \nu_{K,i}^{-1} = t_* \circ \nu_{K,j}^{-1}$.

Suppose that $i \leq j$. Then $\nu_{K,i}^{-1}\nu_{K,j} = \zeta_*$ for some simplicial approximation $\zeta\colon K^{(j)} \to K^{(i)}$ to the identity map of $|K|$ (Lemma 8.5). Thus $s_* \circ \nu_{K,i}^{-1} = s_* \circ \zeta_* \circ \nu_{K,j}^{-1} = (s \circ \zeta)_* \circ \nu_{K,j}^{-1}$. Moreover $\zeta\colon K^{(j)} \to K^{(i)}$ and $s\colon K^{(i)} \to L$ are simplicial approximations to the identity map of $|K|$ and to $f\colon |K| \to |L|$ respectively, and therefore $s \circ \zeta\colon K^{(j)} \to L$ is a simplicial approximation to $f\colon |K| \to |L|$ (Corollary 3.16). But then $s \circ \zeta$ and t are simplicial approximations to the same continuous map, and thus are contiguous simplicial maps from $K^{(j)}$ to L (Lemma 8.1). It follows that $(s \circ \zeta)_*$ and t_* coincide as homomorphisms from $H_q(K^{(j)}; R)$ to $H_q(L; R)$ (Lemma 8.2), and therefore $s_* \circ \nu_{K,i}^{-1} = t_* \circ \nu_{K,j}^{-1}$, as required. ∎

Proposition 8.7 *Let K, L and M be simplicial complexes and let $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$ be continuous maps. Then the homomorphisms f_*, g_* and $(g \circ f)_*$ of homology groups induced by the maps f, g and $g \circ f$ satisfy $(g \circ f)_* = g_* \circ f_*$.*

Proof Let $t\colon L^{(m)} \to M$ be a simplicial approximation to g and let $s\colon K^{(j)} \to L^{(m)}$ be a simplicial approximation to f. Now the canonical isomorphism $\nu_{L,m}$ from $H_q(L^{(m)}; R)$ to $H_q(L; R)$ is induced by some simplicial approximation to the identity map of $|L|$. It follows that $\nu_{L,m} \circ s_*$ is induced by some simplicial approximation to f (Corollary 3.16), and therefore $f_* = \nu_{L,m} \circ s_* \circ \nu_{K,j}^{-1}$. Also $g_* = t_* \circ \nu_{L,m}^{-1}$. It follows that $g_* \circ f_* = t_* \circ s_* \circ \nu_{K,j}^{-1} = (t \circ s)_* \circ \nu_{K,j}^{-1}$. But $t \circ s\colon K^{(j)} \to M$ is a simplicial approximation to $g \circ f$ (Corollary 3.16). Thus $(g \circ f)_* = g_* \circ f_*$, as required. ∎

Corollary 8.8 *If the polyhedra $|K|$ and $|L|$ of simplicial complexes K and L are homeomorphic then the homology groups of K and L are isomorphic.*

Proof Let $h\colon |K| \to |L|$ be a homeomorphism, and let R be an integral domain. Then $h_*\colon H_q(K; R) \to H_q(L; R)$ is an isomorphism whose inverse is $(h^{-1})_*\colon H_q(L; R) \to H_q(K; R)$. ∎

One can make use of induced homomorphisms in homology theory in order to prove the Brouwer Fixed Point Theorem in all dimensions. The Brouwer Fixed Point Theorem is a consequence of the fact that there is no continuous map $r\colon \Delta \to \partial\Delta$ from an n-simplex Δ to its boundary $\partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$. Such a continuous map would induce homomorphisms $r_*\colon H_q(\Delta; \mathbb{Z}) \to H_q(\partial\Delta; \mathbb{Z})$ of homology groups for all non-negative integers q, and $r_* \circ i_*$ would be the identity automorphism of $H_q(\partial\Delta; \mathbb{Z})$ for all q, where $i_*\colon H_q(\partial\Delta; \mathbb{Z}) \to H_q(\Delta; \mathbb{Z})$ is induced by the inclusion map $i\colon \partial\Delta \hookrightarrow \Delta$. But this would imply that $r_*\colon H_q(\Delta; \mathbb{Z}) \to H_q(\partial\Delta; \mathbb{Z})$ was surjective for all non-negative integers q, which is impossible, since $H_{n-1}(\Delta) = 0$ and $H_{n-1}(\partial\Delta) \cong \mathbb{Z}$ when $n \geq 2$ (and $H_{n-1}(\Delta) \cong \mathbb{Z}$ and $H_{n-1}(\partial\Delta) \cong \mathbb{Z} \oplus \mathbb{Z}$ when $n = 1$). We conclude therefore that there is no continuous map $r\colon \Delta \to \partial\Delta$ that fixes all points of $\partial\Delta$, and therefore the Brouwer Fixed Point Theorem is satisfied in all dimensions.

We next show that homotopic maps between the polyhedra of simplicial complexes induce the same homomorphisms of homology groups. For this we require the following result.

Lemma 8.9 *For any simplicial complex L there is some $\varepsilon > 0$ with the following property: given continuous maps $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$*

defined on the polyhedron of some simplicial complex K, where $f(\mathbf{x})$ is within a distance ε of $g(\mathbf{x})$ for all $\mathbf{x} \in |K|$, there exists a simplicial map defined on $K^{(i)}$ for some sufficiently large i which is a simplicial approximation to both f and g.

Proof An application of the Lebesgue Lemma shows that there exists $\varepsilon > 0$ such that the open ball of radius 2ε about any point of $|L|$ is contained in $\mathrm{st}_L(\mathbf{b})$ for some vertex $\mathbf{b}$ of L. Let $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ be continuous maps. Suppose that $f(\mathbf{x})$ is within a distance ε of $g(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Another application of the Lebesgue Lemma (to the open cover of $|K|$ by preimages of open balls of radius ε) shows that there exists $\delta > 0$ such that any subset S of $|K|$ whose diameter is less than δ is mapped by f into an open ball of radius ε about some point of $|L|$, and is therefore mapped by g into an open ball of of radius 2ε about that point. But then $f(S) \subset \mathrm{st}_L(\mathbf{b})$ and $g(S) \subset \mathrm{st}_L(\mathbf{b})$ for some vertex $\mathbf{b}$ of L. Now choose i such that $\mu(K^{(i)}) < \frac{1}{2}\delta$. As in the proof of the Simplicial Approximation Theorem (Theorem 3.17) we see that, for every vertex $\mathbf{a}$ of $K^{(i)}$, the diameter of $\mathrm{st}_{K^{(i)}}(\mathbf{a})$ is less than δ, and hence $f(\mathrm{st}_{K^{(i)}}(\mathbf{a})) \subset \mathrm{st}_L(s(\mathbf{a}))$ and $g(\mathrm{st}_{K^{(i)}}(\mathbf{a})) \subset \mathrm{st}_L(s(\mathbf{a}))$ for some vertex $s(\mathbf{a})$ of L. It then follows from Proposition 3.15 that the function $s\colon \mathrm{Vert}\, K^{(i)} \to \mathrm{Vert}\, L$ constructed in this manner is the required simplicial approximation to f and g. ∎

Theorem 8.10 *Let K and L be simplicial complexes, let R be an integral domain, and let $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ be continuous maps from $|K|$ to $|L|$. Suppose that f and g are homotopic. Then the induced homomorphisms f_* and g_* from $H_q(K; R)$ to $H_q(L; R)$ are equal for all q.*

Proof Let $F\colon |K| \times [0,1] \to |L|$ be a homotopy with $F(\mathbf{x}, 0) = f(\mathbf{x})$ and $F(\mathbf{x}, 1) = g(\mathbf{x})$, and let $\varepsilon > 0$ be given. Using the well-known fact that continuous functions defined on compact metric spaces are uniformly continuous (which is easily proved using the Lebesgue Lemma), we see that there exists some $\delta > 0$ such that if $|s - t| < \delta$ then the distance from $F(\mathbf{x}, s)$ to $F(\mathbf{x}, t)$ is less than ε. Let $f_i(\mathbf{x}) = F(\mathbf{x}, t_i)$ for $i = 0, 1, \ldots, r$, where $t_0, t_1, \ldots, t_r$ have been chosen such that $0 = t_0 < t_1 < \cdots < t_r = 1$ and $t_i - t_{i-1} < \delta$ for all i. Then $f_{i-1}(\mathbf{x})$ is within a distance ε of $f_i(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Using Lemma 8.9, we see that the maps f_{i-1} and f_i from $|K|$ to $|L|$ have a common simplicial approximation, and thus f_{i-1} and f_i induce the same homomorphisms of homology groups, provided that $\varepsilon > 0$ has been chosen sufficiently small. It follows that the maps f and g induce the same homomorphisms of homology groups, as required. ∎

8.4 Homotopy Equivalence

Let X and Y be topological spaces. Continuous maps $f\colon X \to Y$ and $g\colon X \to Y$ from X to Y are said to be *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Definition Let X and Y be topological spaces. A continuous map $f\colon X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $h\colon Y \to X$ such that $h \circ f$ is homotopic to the identity map of X and $f \circ h$ is homotopic to the identity map of Y. The spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence from X to Y.

Lemma 8.11 *A composition of homotopy equivalences is itself a homotopy equivalence.*

Proof Let X, Y and Z be topological spaces, and let $f\colon X \to Y$ and $h\colon Y \to Z$ be homotopy equivalences. Then there exist continuous maps $g\colon Y \to X$ and $k\colon Z \to Y$ such that $g \circ f \simeq i_X$, $f \circ g \simeq i_Y$, $k \circ h \simeq i_Y$ and $h \circ k \simeq i_Z$, where i_X, i_Y and i_Z denote the identity maps of the spaces X, Y, Z. Then $(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq g \circ i_Y \circ f = g \circ f \simeq i_X$ and $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ i_Y \circ k = h \circ k \simeq i_Z$. Thus $h \circ f\colon X \to Z$ is a homotopy equivalence from X to Z. ∎

Lemma 8.12 *Let K and L be simplicial complexes, and let R be an integral domain. Let $f\colon |K| \to |L|$ be a homotopy equivalence between the polyhedra of K and L. Then, for each non-negative integer q, the induced homomorphism $f_*\colon H_q(K;R) \to H_q(L;R)$ of homology groups is an isomorphism.*

Proof There exists a continuous map $g\colon |L| \to |K|$ such that $g \circ f$ is homotopic to the identity map of $|K|$ and $f \circ g$ is homotopic to the identity map of $|L|$. It follows that the induced homomorphisms $(g \circ f)_*\colon H_q(K;R) \to H_q(K;R)$ and $(f \circ g)_*\colon H_q(L;R) \to H_q(L;R)$ are the identity automorphisms of $H_q(K;R)$ and $H_q(L;R)$ for each q. But $(g \circ f)_* = g_* \circ f_*$ and $(f \circ g)_* = f_* \circ g_*$. It follows that $f_*\colon H_q(K;R) \to H_q(L;R)$ is an isomorphism with inverse $g_*\colon H_q(L;R) \to H_q(K;R)$. ∎

Definition A subset A of a topological space X is said to be a *deformation retract* of X if there exists a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) \in A$ for all $x \in X$ and $H(a,1) = a$ for all $a \in A$.

Thus a subset A of a topological space X is a deformation retract of X if and only if there exists a function $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$ and r is homotopic in X to the identity map of X.

Example The unit sphere S^{n-1} in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For if $H(\mathbf{x}, t) = (1 - t + t/|\mathbf{x}|)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0, 1]$ then $H(\mathbf{x}, 0) = \mathbf{x}$ and $H(\mathbf{x}, 1) \in S^{n-1}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $H(\mathbf{x}, 1) = \mathbf{x}$ when $\mathbf{x} \in S^{n-1}$.

If A is a deformation retract of a topological space X then the inclusion map $i: A \hookrightarrow X$ is a homotopy equivalence.

Theorem 8.13 *The spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.*

Proof Let S^{m-1} and S^{n-1} denote the unit spheres in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then S^{m-1} and S^{n-1} are homeomorphic to the polyhedra of simplicial complexes K and L respectively. Let $i_m: S^{m-1} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$ be the inclusion map and let $r_n: \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ be the map that sends $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ to $(1/|\mathbf{x}|)\mathbf{x}$. Then both $i_m: S^{m-1} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $r_n: \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ are homotopy equivalences.

Suppose that there were to exist a homeomorphism $h: \mathbb{R}^m \to \mathbb{R}^n$. Let $f(\mathbf{x}) = h(\mathbf{x}) - h(\mathbf{0})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. Then $f: \mathbb{R}^m \setminus \{\mathbf{0}\} \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ would also be a homeomorphism, and therefore $r_n \circ f \circ i_m: S^{m-1} \to S^{n-1}$ would be a homotopy equivalence. Thus if $\mathbb{R}^m$ and $\mathbb{R}^n$ were homeomorphic then S^{m-1} and S^{n-1} would be homotopy equivalent, and therefore the homology groups of the simplicial complexes K and L would be isomorphic. But $H_q(K; \mathbb{Z}) \cong \mathbb{Z}$ when $q = 0$ and $q = m - 1$ and $H_q(K; \mathbb{Z}) = 0$ for all other values of q, whereas $H_q(L; \mathbb{Z}) \cong \mathbb{Z}$ when $q = 0$ and $q = n - 1$ and $H_q(L; \mathbb{Z}) = 0$ for all other values of q. Thus if $m \neq n$ then the homology groups of the simplicial complexes K and L are not isomorphic, and therefore $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic. ∎