# Module MA3428: Algebraic Topology II
# Hilary Term 2011
# Part II (Sections 3, 4 and 5)

D. R. Wilkins



## Contents

# 3 Simplicial Complexes

## 3.1 Geometrical Independence

**Definition** Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *geometrically independent* (or *affine independent*) if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum\limits_{j=0}^{q} \lambda_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum\limits_{j=0}^{q} \lambda_j & = & 0 \end{array} \right.$$

is the trivial solution $\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0$.

**Lemma 3.1** *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension $k$. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.*

**Proof** Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent. Let $\lambda_1, \lambda_2, \ldots, \lambda_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} \lambda_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum\limits_{j=0}^{q} \lambda_j \mathbf{v}_j = \mathbf{0}$ and $\sum\limits_{j=0}^{q} \lambda_j = 0$, where $\lambda_0 = -\sum\limits_{j=1}^{q} \lambda_j$, and therefore

$$\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_q$ be real numbers which satisfy the equations $\sum\limits_{j=0}^{q} \lambda_j \mathbf{v}_j = \mathbf{0}$ and $\sum\limits_{j=0}^{q} \lambda_j = 0$. Then $\lambda_0 = -\sum\limits_{j=1}^{q} \lambda_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} \lambda_j \mathbf{v}_j = \lambda_0 \mathbf{v}_0 + \sum_{j=1}^{q} \lambda_j \mathbf{v}_j = \sum_{j=1}^{q} \lambda_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_q = 0.$$

But then $\lambda_0 = 0$ also, because $\lambda_0 = -\sum_{j=1}^{q} \lambda_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent, as required. ∎

It follows from Lemma 3.1 that any set of geometrically independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of geometrically independent points in $\mathbb{R}^k$, then so does every subset of that set.

## 3.2 Simplices

**Definition** A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$. The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are referred to as the *vertices* of the simplex. The non-negative integer $q$ is referred to as the *dimension* of the simplex.

**Example** A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

**Example** A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let $\lambda$ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned} \lambda &= \{s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1\} \\ &= \{(1-t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1\}, \end{aligned}$$

and thus $\lambda$ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

**Example** A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let $\tau$ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1\}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1-r}$. Moreover $0 < r \leq 1$ and $0 \leq p \leq 1$. Moreover the above formula determines a point of the 2-simplex $\tau$ for each pair of real numbers $r$ and $p$ satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$. Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as $p$ increases from 0 to 1. It follows that $\tau$ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

**Example** A 3-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least three is a tetrahedron on that space. Indeed let $\mathbf{x}$ be a point of a 3-simplex $\sigma$ in $\mathbb{R}^3$ with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then there exist non-negative real numbers $s$, $t$, $u$ and $v$ such that

$$\mathbf{x} = s\,\mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c} + v\,\mathbf{d},$$

and $s+t+u+v = 1$. These real numbers $s$, $t$, $u$ and $v$ all have values between 0 and 1, and moreover $0 \leq t \leq 1-s$, $0 \leq u \leq 1-s$ and $0 \leq v \leq 1-s$. Suppose that $\mathbf{x} \neq \mathbf{a}$. Then $0 \leq s < 1$ and $\mathbf{x} = s\,\mathbf{a} + (1-s)\mathbf{y}$, where

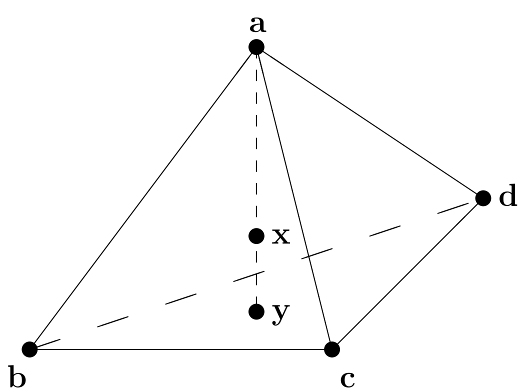
$$\mathbf{y} = \frac{t}{1-s}\,\mathbf{b} + \frac{u}{1-s}\,\mathbf{c} + \frac{v}{1-s}\,\mathbf{d}.$$

Moreover $\mathbf{y}$ is a point of the triangle $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$, because

$$0 \leq \frac{t}{1-s} \leq 1, \quad 0 \leq \frac{u}{1-s} \leq 1, \quad 0 \leq \frac{v}{1-s} \leq 1$$

and

$$\frac{t}{1-s} + \frac{u}{1-s} + \frac{v}{1-s} = 1.$$



It follows that the point $\mathbf{x}$ lies on a line segment with one endpoint at the vertex $\mathbf{a}$ of the 3-simplex and the other at some point $\mathbf{y}$ of the triangle $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$. Thus the 3-simplex $\sigma$ has the form of a tetrahedron (i.e., it has the form of a pyramid on a triangular base $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ with apex $\mathbf{a}$).

A simplex of dimension $q$ in $\mathbb{R}^k$ determines a subset of $\mathbb{R}^k$ that is a translate of a $q$-dimensional vector subspace of $\mathbb{R}^k$. Indeed let the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a $q$-dimensional simplex $\sigma$ in $\mathbb{R}^k$. Then these points are geometrically independent. It follows from Lemma 3.1 that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent. These vectors therefore span a $k$-dimensional vector subspace $V$ of $\mathbb{R}^k$. Now, given any point $\mathbf{x}$ of $\sigma$, there exist real numbers $t_0, t_1, \ldots, t_q$ such that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$ and $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$. Then

$$\mathbf{x} = \left( \sum_{j=0}^{q} t_j \right) \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows that

$$\sigma = \left\{ \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) : 0 \leq t_j \leq 1 \text{ for } j = 1, 2, \ldots, q \text{ and } \sum_{j=1}^{q} t_j \leq 1 \right\},$$

and therefore $\sigma \subset \mathbf{v_0} + V$. Moreover the $q$-dimensional vector subpace $V$ of $\mathbb{R}^k$ is the unique $k$-dimensional vector subpace of $\mathbb{R}^k$ that contains the displacement vectors between each pair of points belonging to the simplex $\sigma$.

## 3.3 Barycentric Coordinates

Let $\sigma$ be a $q$-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of $\sigma$ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^{q} s_j \mathbf{v}_j = \sum_{j=0}^{q} t_j \mathbf{v}_j$ and $\sum_{j=0}^{q} s_j = \sum_{j=0}^{q} t_j = 1$, then $\sum_{j=0}^{q} (t_j - s_j) \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, and therefore $t_j - s_j = 0$ for $j = 0, 1, \ldots, q$, because the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent.

**Definition** Let $\sigma$ be a $q$-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $\mathbf{x} \in \sigma$. The *barycentric coordinates* of the point $\mathbf{x}$ (with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$) are the unique real numbers $t_0, t_1, \ldots, t_q$ for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1.$$

The barycentric coordinates $t_0, t_1, \ldots, t_q$ of a point of a $q$-simplex satisfy the inequalities $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$.

**Example** Consider the triangle $\tau$ in $\mathbb{R}^3$ with vertices at $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0) \quad \text{and} \quad \mathbf{k} = (0, 0, 1).$$

Then

$$\tau = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1 \text{ and } x + y + z = 1\}.$$

The barycentric coordinates on this triangle $\tau$ then coincide with the Cartesian coordinates $x$, $y$ and $z$, because

$$(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$$

for all $(x, y, z) \in \tau$.

**Example** Consider the triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$. This triangle is the set

$$\{(x, y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } x + y \leq 1.\}.$$

The barycentric coordinates of a point $(x, y)$ of this triangle are $t_0$, $t_1$ and $t_2$, where

$$t_0 = 1 - x - y, \quad t_1 = x \quad \text{and} \quad t_2 = y.$$

**Example** Consider the triangle in $\mathbb{R}^2$ with vertices at $(1, 2)$, $(3, 3)$ and $(4, 5)$. Let $t_0$, $t_1$ and $t_2$ be the barycentric coordinates of a point $(x, y)$ of this triangle. Then $t_0$, $t_1$, $t_2$ are non-negative real numbers, and $t_0 + t_1 + t_2 = 1$. Moreover

$$(x, y) = (1 - t_1 - t_2)(1, 2) + t_1(3, 3) + t_2(4, 5),$$

and thus

$$x = 1 + 2t_1 + 3t_2 \quad \text{and} \quad y = 2 + t_1 + 3t_2.$$

It follows that

$$t_1 = x - y + 1 \quad \text{and} \quad t_2 = \tfrac{1}{3}(x - 1 - 2t_1) = \tfrac{2}{3}y - \tfrac{1}{3}x - 1,$$

and therefore

$$t_0 = 1 - t_1 - t_2 = \tfrac{1}{3}y - \tfrac{2}{3}x + 1.$$

In order to verify these formulae it suffices to note that $(t_0, t_1, t_2) = (1, 0, 0)$ when $(x, y) = (1, 2)$, $(t_0, t_1, t_2) = (0, 1, 0)$ when $(x, y) = (3, 3)$ and $(t_0, t_1, t_2) = (0, 0, 1)$ when $(x, y) = (4, 5)$.

**Lemma 3.2** *Let $q$ be a non-negative integer, let $\sigma$ be a $q$-simplex in $\mathbb{R}^m$, and let $\tau$ be a $q$-simplex in $\mathbb{R}^n$, where $m \geq q$ and $n \geq q$. Then $\sigma$ and $\tau$ are homeomorphic.*

**Proof** Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of $\sigma$, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of $\tau$. The required homeomorphism $h: \sigma \to \tau$ is given by

$$h\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \mathbf{w}_j$$

for all $t_0, t_1, \ldots, t_q$ satisfying $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum\limits_{j=0}^{q} t_j = 1$. ∎

A homeomorphism between two $q$-simplices defined as in the above proof is referred to as a *simplicial homeomorphism*.

It follows from Lemma 3.2 that every $q$-simplex is homeomorphic to the *standard $q$-simplex* in $\mathbb{R}^{q+1}$ whose vertices are the points

$$(1, 0, 0, \ldots, 0),\ \ (0, 1, 0, \ldots, 0), \ldots,\ \ (0, 0, 0, \ldots, 1).$$

This standard $q$-simplex is the subset of $\mathbb{R}^{q+1}$ consisting of those points $(t_0, t_1, \ldots, t_q)$ of $\mathbb{R}^{q+1}$ which satisfy $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum\limits_{j=0}^{q} t_j = 1$.

**Example** Let $\sigma$ be the triangle in $\mathbb{R}^2$ with vertices at $(1, 2)$, $(3, 3)$ and $(4, 5)$ discussed in a previous example, and let the map $h: \sigma \to \mathbb{R}^3$ from $\sigma$ to $\mathbb{R}^3$ be defined such that

$$h(x, y) = \left(\tfrac{1}{3}y - \tfrac{2}{3}x + 1,\ \ x - y + 1,\ \ \tfrac{2}{3}y - \tfrac{1}{3}x - 1\right).$$

We have already verified that the components of this map $h$ are the barycentric coordinate functions on the triangle $\sigma$. It follows that $h$ maps this triangle homeomorphically onto the triangle in $\mathbb{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

## 3.4 Simplicial Complexes in Euclidean Spaces

**Definition** Let $\sigma$ and $\tau$ be simplices in $\mathbb{R}^k$. We say that $\tau$ is a *face* of $\sigma$ if the set of vertices of $\tau$ is a subset of the set of vertices of $\sigma$. A face of $\sigma$ is said to be a *proper face* if it is not equal to $\sigma$ itself. An $r$-dimensional face of $\sigma$ is referred to as an *$r$-face* of $\sigma$. A 1-dimensional face of $\sigma$ is referred to as an *edge* of $\sigma$.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

**Definition** A finite collection $K$ of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if $\sigma$ is a simplex belonging to $K$ then every face of $\sigma$ also belongs to $K$,
- if $\sigma_1$ and $\sigma_2$ are simplices belonging to $K$ then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both $\sigma_1$ and $\sigma_2$.

The *dimension* of a simplicial complex $K$ is the greatest non-negative integer $n$ with the property that $K$ contains an $n$-simplex. The union of all the simplices of $K$ is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the *polyhedron* of $K$. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

**Example** Let $K_\sigma$ consist of some $n$-simplex $\sigma$ together with all of its faces. Then $K_\sigma$ is a simplicial complex of dimension $n$, and $|K_\sigma| = \sigma$.

**Lemma 3.3** *Let $K$ be a simplicial complex, and let $X$ be a topological space. A function $f\colon |K| \to X$ is continuous on the polyhedron $|K|$ of $K$ if and only if the restriction of $f$ to each simplex of $K$ is continuous on that simplex.*

**Proof** If a topological space can be expressed as a finite union of closed subsets, then a function is continuous on the whole space if and only if its restriction to each of the closed subsets is continuous on that closed set. The required result is a direct application of this general principle. ∎

We shall denote by $\operatorname{Vert} K$ the set of vertices of a simplicial complex $K$ (i.e., the set consisting of all vertices of all simplices belonging to $K$). A collection of vertices of $K$ is said to *span* a simplex of $K$ if these vertices are the vertices of some simplex belonging to $K$.

**Definition** Let $K$ be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of $K$ is a collection $L$ of simplices belonging to $K$ with the following property:—

- if $\sigma$ is a simplex belonging to $L$ then every face of $\sigma$ also belongs to $L$.

Note that every subcomplex of a simplicial complex $K$ is itself a simplicial complex.

**Definition** Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a $q$-simplex $\sigma$ in some Euclidean space $\mathbb{R}^k$. We define the *interior* of the simplex $\sigma$ to be the set of all points of $\sigma$ that are of the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_j > 0$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$.

**Lemma 3.4** *Let $\sigma$ be a $q$-simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$. Then a point of $\sigma$ belongs to the interior of $\sigma$ if and only if it does not belong to any proper face of $\sigma$.*

**Proof** Every proper face of the $q$-dimensional simplex $\sigma$ is contained in one of the $(q-1)$-dimensional proper faces of $\sigma$ whose vertex set omits exactly one vertex of $\sigma$. Let $\mathbf{x}$ be a point of $\sigma$ with barycentric coordinates $t_0, t_1, \ldots, t_q$ with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of $\sigma$. Then

$$\mathbf{x} = t_0\, \mathbf{v}_0 + t_1\, \mathbf{v}_1 + \cdots + t_q \mathbf{v}_q,$$

where $t_j \geq 0$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The barycentric coordinates $t_0, t_1, \ldots, t_q$ of $\mathbf{x}$ are uniquely determined by the point $\mathbf{x}$. It follows that $\mathbf{x}$ belongs to the $(q-1)$-dimensional proper face of $\sigma$ whose vertex set omits the vertex $\mathbf{v}_j$ of $\sigma$ if and only if $t_j = 0$. It follows that $\mathbf{x}$ belongs to some proper face of $\sigma$ if and only if $t_j = 0$ for at least one integer $j$ between 0 and $q$. Thus a necessary and sufficient condition to ensure that a point $\mathbf{x}$ of the simplex $\sigma$ belongs to no proper face of the simplex is that the barycentric coordinates $t_1, t_2, \ldots, t_q$ of that point must all be strictly positive. It therefore follows from the definition of the interior of a simplex that a point of that simplex belongs to the interior of the simplex if and only if it does not belong to any proper face of the simplex. ∎

**Example** A 0-simplex consists of a single vertex $\mathbf{v}$. The interior of that 0-simplex is the vertex $\mathbf{v}$ itself.

**Example** A 1-simplex is a line segment. The interior of a line segment in a Euclidean space $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$ is

$$\{(1-t)\,\mathbf{v} + t\,\mathbf{w} : 0 < t < 1\}.$$

Thus the interior of the line segment consists of all points of the line segement that are not endpoints of the line segment.

**Example** A 2-simplex is a triangle. The interior of a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is the set

$$\{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 < r, s, t < 1 \text{ and } r + s + t = 1\}.$$

The interior of this triangle consists of all points of the triangle that do not lie on any edge of the triangle.

**Remark** Let $\sigma$ be a $q$-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$. If $k > q$ then the interior of the simplex (defined according to the definition given above) will not coincide with the topological interior determined by the usual topology on $\mathbb{R}^k$. Consider for example a triangle embedded in three-dimensional Euclidean space $\mathbb{R}^3$. The interior of the triangle (defined according to the definition given above) consists of all points of the triangle that do not lie on any edge of the triangle. But of course no three-dimensional ball of positive radius centred on any point of that triangle is wholly contained within the triangle. It follows that the topological interior of the triangle is the empty set when that triangle is considered as a subset of three-dimensional space $\mathbb{R}^3$.

**Lemma 3.5** *Any point of a simplex belongs to the interior of a unique face of that simplex.*

**Proof** let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a simplex $\sigma$, and let $\mathbf{x} \in \sigma$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are the barycentric coordinates of the point $\mathbf{x}$. Moreover $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of $\sigma$ containing $\mathbf{x}$ in its interior is then the face spanned by those vertices $\mathbf{v}_j$ for which $t_j > 0$. ∎

**Lemma 3.6** *Let $K$ be a finite collection of simplices in some Euclidean space $\mathbb{R}^k$, and let $|K|$ be the union of all the simplices in $K$. Then $K$ is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:—*

- *$K$ contains the faces of its simplices,*
- *every point of $|K|$ belongs to the interior of a unique simplex of $K$.*

**Proof** Suppose that $K$ is a simplicial complex. Then $K$ contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of $K$. Let $\mathbf{x} \in |K|$. Then $\mathbf{x} \in \rho$ for some simplex $\rho$ of $K$.

It follows from Lemma 3.5 that there exists a unique face $\sigma$ of $\rho$ such that the point $\mathbf{x}$ belongs to the interior of $\sigma$. But then $\sigma \in K$, because $\rho \in K$ and $K$ contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of $K$.

Suppose that $\mathbf{x}$ were to belong to the interior of two distinct simplices $\sigma$ and $\tau$ of $K$. Then $\mathbf{x}$ would belong to some common face $\sigma \cap \tau$ of $\sigma$ and $\tau$ (since $K$ is a simplicial complex). But this common face would be a proper face of one or other of the simplices $\sigma$ and $\tau$ (since $\sigma \neq \tau$), contradicting the fact that $\mathbf{x}$ belongs to the interior of both $\sigma$ and $\tau$. We conclude that the simplex $\sigma$ of $K$ containing $\mathbf{x}$ in its interior is uniquely determined, as required.

Conversely, we must show that if $K$ is some finite collection of simplices in some Euclidean space, if $K$ contains the faces of all its simplices, and if every point of the union $|K|$ of those simplices belongs the the interior of a unique simplex in the collection, then that collection is a simplicial complex. To achieve this, we must prove that if $\sigma$ and $\tau$ are simplices belonging to the collection $K$, and if $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a common face of $\sigma$ and $\tau$.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex $\omega$ belonging to the collection $K$. However any point of $\sigma$ or $\tau$ belongs to the interior of a unique face of that simplex, and all faces of $\sigma$ and $\tau$ belong to $K$. It follows that $\omega$ is a common face of $\sigma$ and $\tau$, and thus the vertices of $\omega$ are vertices of both $\sigma$ and $\tau$. It follows that the simplices $\sigma$ and $\tau$ have vertices in common.

Let $\rho$ be the simplex whose vertex set is the intersection of the vertex sets of $\sigma$ and $\tau$. Then $\rho$ is a common face of both $\sigma$ and $\tau$, and therefore $\rho \in K$. Moreover if $\mathbf{x} \in \sigma \cap \tau$ and if $\omega$ is the unique simplex of $K$ whose interior contains the point $\mathbf{x}$, then (as we have already shown), all vertices of $\omega$ are vertices of both $\sigma$ and $\tau$. But then the vertex set of $\omega$ is a subset of the vertex set of $\rho$, and thus $\omega$ is a face of $\rho$. Thus each point $\mathbf{x}$ of $\sigma \cap \tau$ belongs to $\rho$, and therefore $\sigma \cap \tau \subset \rho$. But $\rho$ is a common face of $\sigma$ and $\tau$ and therefore $\rho \subset \sigma \cap \tau$. It follows that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of $\sigma$ and $\tau$. This completes the proof that the collection $K$ of simplices satisfying the given conditions is a simplicial complex. ∎

**Definition** A *triangulation* $(K, h)$ of a topological space $X$ consists of a simplicial complex $K$ in some Euclidean space, together with a homeomorphism $h\colon |K| \to X$ mapping the polyhedron $|K|$ of $K$ onto $X$.

The polyhedron of a simplicial complex is a compact Hausdorff space. Thus if a topological space admits a triangulation then it must itself be a compact Hausdorff space.

**Lemma 3.7** *Let $X$ be a Hausdorff topological space, let $K$ be a simplicial complex, and let $h\colon |K| \to X$ be a bijection mapping $|K|$ onto $X$. Suppose that the restriction of $h$ to each simplex of $K$ is continuous on that simplex. Then the map $h\colon |K| \to X$ is a homeomorphism, and thus $(K, h)$ is a triangulation of $X$.*

**Proof** Each simplex of $K$ is a closed subset of $|K|$, and the number of simplices of $K$ is finite. It follows from Lemma 3.3 that $h\colon |K| \to X$ is continuous. Also the polyhedron $|K|$ of $K$ is a compact topological space. But every continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism. Thus $(K, h)$ is a triangulation of $X$. ∎

## 3.5 Simplicial Maps

**Definition** A *simplicial map* $\varphi\colon K \to L$ between simplicial complexes $K$ and $L$ is a function $\varphi\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ from the vertex set of $K$ to that of $L$ such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to $L$ whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$.

Note that a simplicial map $\varphi\colon K \to L$ between simplicial complexes $K$ and $L$ can be regarded as a function from $K$ to $L$: this function sends a simplex $\sigma$ of $K$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of $L$ spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

A simplicial map $\varphi\colon K \to L$ also induces in a natural fashion a continuous map $\varphi\colon |K| \to |L|$ between the polyhedra of $K$ and $L$, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum\limits_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$. The continuity of this map follows immediately from a straightforward application of Lemma 3.3. Note that the interior of a simplex $\sigma$ of $K$ is mapped into the interior of the simplex $\varphi(\sigma)$ of $L$.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

## 3.6 Barycentric Subdivision of a Simplicial Complex

Let $\sigma$ be a $q$-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The *barycentre* of $\sigma$ is defined to be the point

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

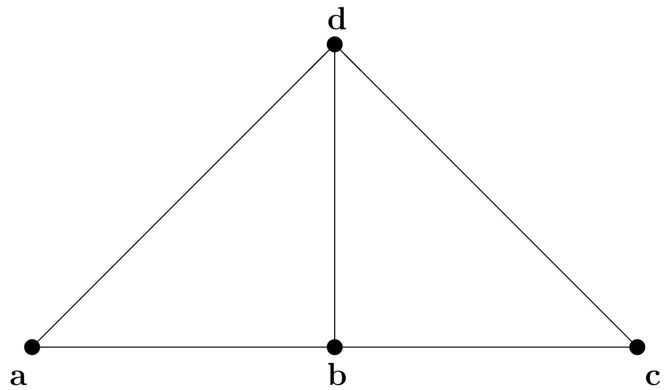
Let $\sigma$ and $\tau$ be simplices in some Euclidean space. If $\sigma$ is a proper face of $\tau$ then we denote this fact by writing $\sigma < \tau$.

A simplicial complex $K_1$ is said to be a *subdivision* of a simplicial complex $K$ if $|K_1| = |K|$ and each simplex of $K_1$ is contained in a simplex of $K$.

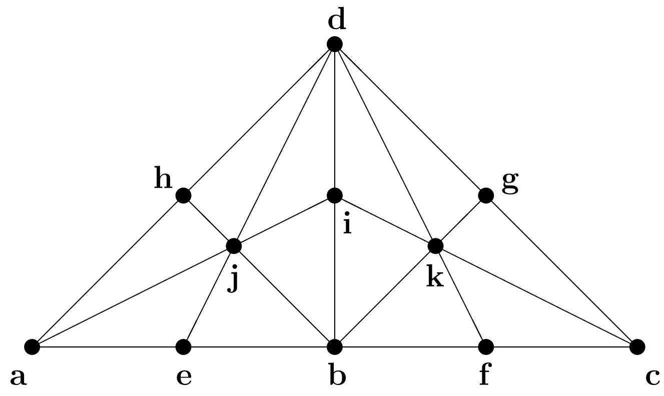
**Definition** Let $K$ be a simplicial complex in some Euclidean space $\mathbb{R}^k$. The *first barycentric subdivision* $K'$ of $K$ is defined to be the collection of simplices in $\mathbb{R}^k$ whose vertices are $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$ for some sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of simplices of $K$ with $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. Thus the set of vertices of $K'$ is the set of all the barycentres of all the simplices of $K$.

Note that every simplex of $K'$ is contained in a simplex of $K$. Indeed if $\sigma_0, \sigma_1, \ldots, \sigma_r \in K$ satisfy $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ then the simplex of $K'$ spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$, is contained in the simplex $\sigma_r$ of $K$.

**Example** Let $K$ be the simplicial complex consisting of two triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ that intersect along a common edge $\mathbf{b}\,\mathbf{d}$, together with all the edges and vertices of the two triangles, as depicted in the following diagram:




The barycentric subdivision $K'$ of this simplicial complex is then as depicted in the following diagram:




We see that $K'$ consists of 12 triangles, together with all the edges and vertices of those triangles. Of the 11 vertices of $K'$, the vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the vertices of the original complex $K$, the vertices $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{i}$ are the barycentres of the edges $\mathbf{a}\,\mathbf{b}$, $\mathbf{b}\,\mathbf{c}$, $\mathbf{c}\,\mathbf{d}$, $\mathbf{a}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{d}$ respectively, and are located at the midpoints of those edges, and the vertices $\mathbf{j}$ and $\mathbf{k}$ are the barycentres of the triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ of $K$. Thus $\mathbf{e} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{f} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$, etc., and $\mathbf{j} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{d}$ and $\mathbf{k} = \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{d}$.

**Proposition 3.8** *Let $K$ be a simplicial complex in some Euclidean space, and let $K'$ be the first barycentric subdivision of $K$. Then $K'$ is itself a simplicial complex, and $|K'| = |K|$.*

**Proof** We prove the result by induction on the number of simplices in $K$. The result is clear when $K$ consists of a single simplex, since that simplex must then be a point and therefore $K' = K$. We prove the result for a simplicial complex $K$, assuming that it holds for all complexes with fewer simplices.

It is clear from the definition of the barycentric subdivision $K'$ that any face of a simplex of $K'$ must itself belong to $K'$. We must verify that any two simplices of $K'$ are disjoint or else intersect in a common face.

Choose a simplex $\sigma$ of $K$ for which $\dim \sigma = \dim K$, and let $L = K \setminus \{\sigma\}$. Then $L$ is a subcomplex of $K$, since $\sigma$ is not a proper face of any simplex of $K$. Now $L$ has fewer simplices than $K$. It follows from the induction hypothesis that $L'$ is a simplicial complex and $|L'| = |L|$. Also it follows from the definition of $K'$ that $K'$ consists of the following simplices:—

- the simplices of $L'$,
- the barycentre $\hat{\sigma}$ of $\sigma$,

- simplices $\hat{\sigma}\rho$ whose vertex set is obtained by adjoining $\hat{\sigma}$ to the vertex set of some simplex $\rho$ of $L'$, where the vertices of $\rho$ are barycentres of proper faces of $\sigma$.

By checking all possible intersections of simplices of the above types, it is easy to verify that any two simplices of $K'$ intersect in a common face. Indeed any two simplices of $L'$ intersect in a common face, since $L'$ is a simplicial complex. If $\rho_1$ and $\rho_2$ are simplices of $L'$ whose vertices are barycentres of proper faces of $\sigma$, then $\rho_1 \cap \rho_2$ is a common face of $\rho_1$ and $\rho_2$ which is of this type, and $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2 = \hat{\sigma}(\rho_1 \cap \rho_2)$. Thus $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2$ is a common face of $\hat{\sigma}\rho_1$ and $\hat{\sigma}\rho_2$. Also any simplex $\tau$ of $L'$ is disjoint from the barycentre $\hat{\sigma}$ of $\sigma$, and $\hat{\sigma}\rho \cap \tau = \rho \cap \tau$. We conclude that $K'$ is indeed a simplicial complex.

It remains to verify that $|K'| = |K|$. Now $|K'| \subset |K|$, since every simplex of $K'$ is contained in a simplex of $K$. Let $\mathbf{x}$ be a point of the chosen simplex $\sigma$. Then there exists a point $\mathbf{y}$ belonging to a proper face of $\sigma$ and some $t \in [0,1]$ such that $\mathbf{x} = (1-t)\hat{\sigma} + t\,\mathbf{y}$. But then $\mathbf{y} \in |L|$, and $|L| = |L'|$ by the induction hypothesis. It follows that $\mathbf{y} \in \rho$ for some simplex $\rho$ of $L'$ whose vertices are barycentres of proper faces of $\sigma$. But then $\mathbf{x} \in \hat{\sigma}\rho$, and therefore $\mathbf{x} \in |K'|$. Thus $|K| \subset |K'|$, and hence $|K'| = |K|$, as required. ∎

We define (by induction on $j$) the $j$th barycentric subdivision $K^{(j)}$ of $K$ to be the first barycentric subdivision of $K^{(j-1)}$ for each $j > 1$.

**Lemma 3.9** *Let $\sigma$ be a $q$-simplex and let $\tau$ be a face of $\sigma$. Let $\hat{\sigma}$ and $\hat{\tau}$ be the barycentres of $\sigma$ and $\tau$ respectively. If all the 1-simplices (edges) of $\sigma$ have length not exceeding $d$ for some $d > 0$ then*

$$|\hat{\sigma} - \hat{\tau}| \leq \frac{qd}{q+1}.$$

**Proof** Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of $\sigma$. Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\sigma$. We can write $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_i \leq 1$ for $i = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. Now

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= \left| \sum_{i=0}^{q} t_i (\mathbf{x} - \mathbf{v}_i) \right| \leq \sum_{i=0}^{q} t_i |\mathbf{x} - \mathbf{v}_i| \\ &\leq \text{maximum}\left( |\mathbf{x} - \mathbf{v}_0|, |\mathbf{x} - \mathbf{v}_1|, \ldots, |\mathbf{x} - \mathbf{v}_q| \right). \end{aligned}$$

Applying this result with $\mathbf{x} = \hat{\sigma}$ and $\mathbf{y} = \hat{\tau}$, we find that

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum}\left( |\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q| \right).$$

But

$$\hat{\sigma} = \frac{1}{q+1}\mathbf{v}_i + \frac{q}{q+1}\mathbf{z}_i$$

for $i = 0, 1, \ldots, q$, where $\mathbf{z}_i$ is the barycentre of the $(q-1)$-face of $\sigma$ opposite to $\mathbf{v}_i$, given by

$$\mathbf{z}_i = \frac{1}{q}\sum_{j \neq i} \mathbf{v}_j.$$

Moreover $\mathbf{z}_i \in \sigma$. It follows that

$$|\hat{\sigma} - \mathbf{v}_i| = \frac{q}{q+1}|\mathbf{z}_i - \mathbf{v}_i| \leq \frac{qd}{q+1}$$

for $i = 1, 2, \ldots, q$, and thus

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum}\,(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q|) \leq \frac{qd}{q+1},$$

as required. ∎

The *mesh* $\mu(K)$ of a simplicial complex $K$ is the length of the longest edge of $K$.

**Lemma 3.10** *Let $K$ be a simplicial complex in $\mathbb{R}^k$ for some $k$, and let $n$ be the dimension of $K$. Let $K'$ be the first barycentric subdivision of $K$. Then*

$$\mu(K') \leq \frac{n}{n+1}\mu(K).$$

**Proof** A 1-simplex of $K'$ is of the form $(\hat{\tau}, \hat{\sigma})$, where $\sigma$ is a $q$-simplex of $K$ for some $q \leq n$ and $\tau$ is a proper face of $\sigma$. Then

$$|\hat{\tau} - \hat{\sigma}| \leq \frac{q}{q+1}\mu(K) \leq \frac{n}{n+1}\mu(K)$$

by Lemma 3.9, as required. ∎

It follows directly from the above lemma that $\lim_{j \to +\infty} \mu(K^{(j)}) = 0$, where $K^{(j)}$ is the $j$th barycentric subdivision of $K$.

## 3.7 The Barycentric Subdivision of a Simplex

**Proposition 3.11** *Let $\sigma$ be a simplex in $\mathbb{R}^N$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $m_0, m_1, \ldots, m_r$ be integers satisfying*

$$0 \leq m_0 < m_1 < \cdots < m_r \leq q.$$

*Let $\rho$ be the simplex in $\mathbb{R}^N$ with vertices $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r$, where $\hat{\tau}_k$ denotes the barycentre of the simplex $\tau_k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{m_k}$ for $k = 1, 2, \ldots, r$. Then the simplex $\rho$ is the set consisting of all points of $\mathbb{R}^N$ that can be represented in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are real numbers satisfying the following conditions:*

(i) $0 \leq t_j \leq 1$ *for* $j = 0, 1, \ldots, q$;

(ii) $\sum\limits_{j=0}^{q} t_j = 1$;

(iii) $t_0 \geq t_1 \geq \cdots \geq t_q$;

(iv) $t_j = t_{m_0}$ *for all integers $j$ satisfying $j \leq m_0$;*

(v) $t_j = t_{m_k}$ *for all integers $j$ and $k$ satisfying $0 < k \leq r$ and $m_{k-1} < j \leq m_k$;*

(vi) $t_j = 0$ *for all integers $j$ satisfying $j > m_r$.*

*Moreover the interior of the simplex $\rho$ is the set consisting of all points of $\mathbb{R}^N$ that can be represented in the form $\sum\limits_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are real numbers satisfying conditions (i)–(iv) above together with the following extra condition:*

(vii) $t_{m_{k-1}} > t_{m_k} > 0$ *for all integers $k$ satisfying $0 < k \leq r$.*

**Proof** Let $\mathbf{w}_k = \hat{\tau}_k$ for $k = 0, 1, \ldots, r$. Then

$$\mathbf{w}_k = \frac{1}{m_k + 1} \sum_{j=0}^{m_k} \mathbf{v}_j.$$

Let $\mathbf{x} \in \rho$, and let the real numbers $u_0, u_1, \ldots, u_r$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$ of $\rho$, so that $0 \leq u_k \leq 1$ for $k = 0, 1, \ldots, r$, $\sum\limits_{k=0}^{r} u_k \mathbf{w}_k = \mathbf{x}$, and $\sum\limits_{k=0}^{r} u_k = 1$. Also let

$$K(j) = \{k \in \mathbb{Z} : 0 \leq k \leq r \text{ and } m_k \geq j\}$$

for $j = 0, 1, \ldots, q$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where

$$t_j = \sum_{k \in K(j)} \frac{u_k}{m_k + 1}$$

when $0 \leq j \leq m_r$, and $t_j = 0$ when $m_r < j \leq q$. Moreover

$$\begin{aligned} \sum_{j=0}^{q} t_j &= \sum_{j=0}^{m_r} \sum_{k \in K(j)} \frac{u_k}{m_k + 1} = \sum_{(j,k) \in L} \frac{u_k}{m_k + 1} \\ &= \sum_{k=0}^{r} \sum_{j=0}^{m_k} \frac{u_k}{m_k + 1} = \sum_{k=0}^{r} u_k = 1, \end{aligned}$$

where

$$L = \{(j, k) \in \mathbb{Z}^2 : 0 \leq j \leq q,\ \ 0 \leq k \leq r \text{ and } j \leq m_k\}.$$

Now $t_j \geq 0$ for $j = 0, 1, \ldots, q$, because $u_k \geq 0$ for $k = 0, 1, \ldots, r$, and therefore

$$0 \leq t_j \leq \sum_{j=0}^{q} t_j = 1.$$

Also $t_{j'} \leq t_j$ for all integers $j$ and $j'$ satisfying $0 \leq j < j' \leq m_r$, because $K(j') \subset K(j)$. If $0 \leq j \leq m_0$ then $K(j) = K(m_0)$, and therefore $t_j = t_{m_0}$. Similarly if $0 < k \leq r$, and $m_{k-1} < j \leq m_k$ then $K(j) = K(m_k)$, and therefore $t_j = t_{m_k}$. Thus the real numbers $t_0, t_1, \ldots, t_k$ satisfy conditions (i)–(vi) above.

Now let $t_0, t_1, \ldots, t_q$ be real numbers satisfying conditions (i)-(vi), let

$$u_r = (m_r + 1) t_{m_r}$$

and

$$u_k = (m_k + 1)(t_{m_k} - t_{m_{k+1}})$$

for $k = 0, 1, \ldots, r - 1$. Then

$$t_{m_k} = \sum_{k'=k}^{r} \frac{u_{k'}}{m_{k'} + 1}$$

for $k = 0, 1, \ldots, r$. Also $u_k \geq 0$ for $k = 0, 1, \ldots, r$, and

$$\sum_{k=0}^{r} u_k = \sum_{k=0}^{r-1} (m_k + 1)(t_{m_k} - t_{m_{k+1}}) + (m_r + 1) t_{m_r}$$

$$
\begin{aligned}
&= \sum_{k=0}^{r-1}\Big((m_k+1)t_{m_k} - (m_{k+1}+1)t_{m_{k+1}}\Big) + (m_r+1)t_{m_r} \\
&\qquad + \sum_{k=0}^{r-1}(m_{k+1}-m_k)t_{m_{k+1}} \\
&= (m_0+1)t_{m_0} + \sum_{k=1}^{r}(m_k - m_{k-1})t_{m_k} \\
&= \sum_{j=0}^{m_0} t_j + \sum_{k=1}^{r}\sum_{j=m_{k-1}+1}^{m_k} t_j = \sum_{j=0}^{q} t_q \\
&= 1,
\end{aligned}
$$

because the real numbers $t_0, t_1, \ldots, t_q$ satisfy conditions (ii), (iv), (v) and (vi). It follows that $u_0, u_1, \ldots, u_r$ are the barycentric coordinates of a point of the simplex with vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$. Moreover

$$
t_j = \sum_{k \in K(j)} \frac{u_k}{m_k+1}
$$

for $j = 0, 1, \ldots, q$, and therefore

$$
\begin{aligned}
\sum_{k=0}^{r} u_k \mathbf{w_k} &= \sum_{k=0}^{r}\sum_{j=0}^{m_k} \frac{u_k}{m_k+1}\mathbf{v}_j \\
&= \sum_{(j,k)\in L} \frac{u_k}{m_k+1}\mathbf{v}_j \\
&= \sum_{j=0}^{q}\sum_{k\in K(j)} \frac{u_k}{m_k+1}\mathbf{v}_j \\
&= \sum_{j=0}^{q} t_j \mathbf{v}_j.
\end{aligned}
$$

We conclude the the simplex $\rho$ is the set of all points of $\mathbb{R}^N$ that are representable in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where the coefficients $t_0, t_1, \ldots, t_q$ are real numbers satisfying conditions (i)–(vi).

Now the point $\sum_{j=0}^{q} t_j \mathbf{v}_j$ belongs to the interior of the simplex $\rho$ if and only if $u_k > 0$ for $k = 0, 1, \ldots, r$, where $u_r = (m_r+1)t_{m_r}$ and $u_k = (m_k+1)(t_{m_k} - t_{m_{k+1}})$ for $k = 0, 1, \ldots, r-1$. This point therefore belongs to the interior of the simplex $\rho$ if and only if $t_{m_r} > 0$ and $t_{m_k} > t_{m_{k+1}}$ for $k = 0, 1, \ldots, r-1$.

Thus the interior of the simplex $\rho$ consists of those points $\sum_{j=0}^{q} t_j \mathbf{v}_j$ of $\sigma$ whose barycentric coordinates $t_0, t_1, \ldots, t_q$ with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of $\sigma$ satisfy conditions (i)–(vii), as required. ∎

**Corollary 3.12** *Let $\sigma$ be a simplex in some Euclidean space $\mathbb{R}^N$, and let $K_\sigma$ be the simplicial complex consisting of the simplex $\sigma$ together with all of its faces. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of $\sigma$, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of some point $\mathbf{x}$ of $\sigma$, so that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$. Then there exists a permutation $\pi$ of the set $\{0, 1, \ldots, q\}$ and integers $m_0, m_1, \ldots, m_r$ satisfying*

$$0 \leq m_0 < m_1 < \cdots < m_r \leq q.$$

*such the following conditions are satisfied:*

(iii) $t_{\pi(0)} \geq t_{\pi(1)} \geq \cdots \geq t_{\pi(q)}$;

(iv) $t_{\pi(j)} = t_{\pi(m_0)}$ *for all integers $j$ satisfying $j \leq m_0$;*

(v) $t_{\pi(j)} = t_{\pi(m_k)}$ *for all integers $j$ and $k$ satisfying $0 < k \leq r$ and $m_{k-1} < j \leq m_k$;*

(vi) $t_{\pi(j)} = 0$ *for all integers $j$ satisfying $j > m_r$.*

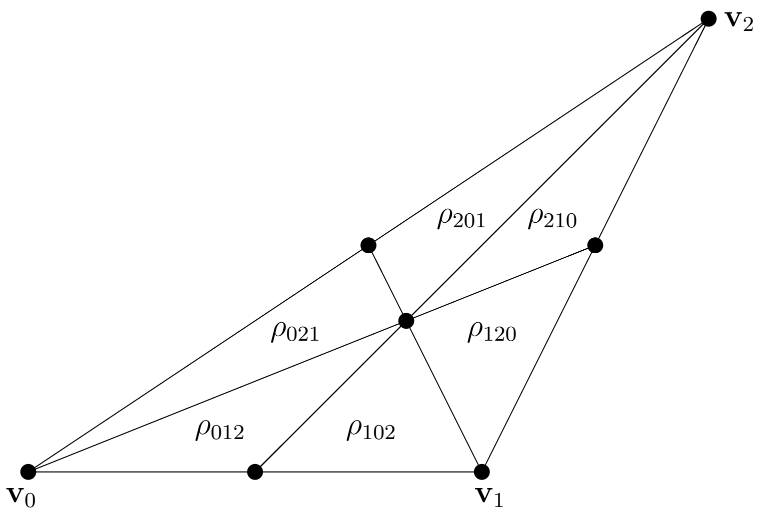
(vii) $t_{\pi(m_{k-1})} > t_{\pi(m_k)} > 0$ *for all integers $k$ satisfying $0 < k \leq r$.*

*Let $\rho$ be the simplex of the first barycentric subdivision $K'_\sigma$ of the simplical complex $K_\sigma$ with vertices $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r$, where $\hat{\tau}_k$ is the barycentre of the simplex $\tau_k$ with vertices $\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(m_k)}$ for $k = 0, 1, \ldots, r$. Then $\rho$ is the unique simplex of $K'_\sigma$ that contains the point $\mathbf{x}$ in its interior.*

**Proof** The required permutation $\pi$ can be any permutation that rearranges the barycentric coordinates in descending order, so that $1 \geq t_{\pi(0)} \geq t_{\pi(1)} \geq \ldots \geq t_{\pi(q)} \geq 0$. The required result then follows immediately on applying Proposition 3.11. ∎

Corollary 3.12 may be applied to determine the simplices of the first barycentric subdivision $K'_\sigma$ of the simplicial complex $K_\sigma$ that consists of some simplex $\sigma$ together with all of its faces.

**Example** Let $K$ be the simplicial complex consisting of a triangle with vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, together with all its edges and vertices, and let $K'$ be the first barycentric subdivision of the simplicial complex $K$. Then $K'$ consists of six triangles $\rho_{012}$, $\rho_{102}$, $\rho_{021}$, $\rho_{120}$, $\rho_{201}$ and $\rho_{210}$, together with all the edges and vertices of those triangles, where




$$\begin{aligned}
\rho_{012} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 \geq t_1 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{102} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_1 \geq t_0 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{021} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 \geq t_2 \geq t_1 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{120} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_1 \geq t_2 \geq t_0 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{201} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_2 \geq t_0 \geq t_1 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{210} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_2 \geq t_1 \geq t_0 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}.
\end{aligned}$$

The intersection of any two of those triangles is a common edge or vertex of those triangles. For example, the intersection of the triangles $\rho_{012}$ and $\rho_{102}$ is the edge $\rho_{012} \cap \rho_{102}$, where

$$\rho_{012} \cap \rho_{102} = \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 = t_1 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}.$$

And the intersection of the triangle $\rho_{012}$ and $\rho_{120}$ is the barycentre of the triangle $\mathbf{v}_0\,\mathbf{v}_1\,\mathbf{v}_2$, and is thus the point $\sum_{j=0}^{2} t_j \mathbf{v}_j$ whose barycentric coordinates $t_0, t_1, t_2$ satisfy $t_0 = t_1 = t_2 = \frac{1}{3}$.

Let $\sigma$ be a $q$-simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, let $K_\sigma$ be the simplicial complex consisting of the simplex $\sigma$, together with all its faces, and let $K'_\sigma$ be the first barycentric subdivision of the simplicial complex $K_\sigma$. Then the $q$-simplices of $K_\sigma$ are the simplices of the form $\rho_{m_0\, m_1 \ldots m_q}$, where the list $m_0, m_1, \ldots, m_q$ is a rearrangement of the list $0, 1, \ldots, q$ (so that each integer between 0 and $q$ occurs exactly one in the list $m_0, m_1, \ldots, m_q$), and where

$$\rho_{m_0\, m_1 \ldots m_q} = \left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 1 \geq t_{m_0} \geq t_{m_1} \geq \cdots \geq t_{m_q} \geq 0 \text{ and } \sum_{j=0}^{q} t_j = 1 \right\}.$$

A point of $\sigma$ belongs to the interior of one of the simplices of $K'_\sigma$ if and only if its barycentric coordinates $t_0, t_1, \ldots, t_q$ are all distinct and strictly positive. Moreover if a point $\sum_{j=0}^{q} t_j \mathbf{v}_j$ of $\sigma$ with barycentric coordinates $t_0, t_1, \ldots, t_q$ belongs to the interior of some $r$-simplex of $K'_\sigma$ then there are exactly $r+1$ distinct values amongst the real numbers $t_0, t_1, \ldots, t_q$ (i.e., $\{t_0, t_1, \ldots, t_q\}$ is a set with exactly $r+1$ elements).

## 3.8 The Simplicial Approximation Theorem

**Definition** Let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes $K$ and $L$. A simplicial map $s\colon K \to L$ is said to be a *simplicial approximation* to $f$ if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of $L$ which contains $f(\mathbf{x})$ in its interior.

**Definition** Let $X$ and $Y$ be topological spaces. Continuous maps $f\colon X \to Y$ and $g\colon X \to Y$ from $X$ to $Y$ are said to be *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

**Lemma 3.13** *Let $K$ and $L$ be simplicial complexes, let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of $K$ and $L$, and let $s\colon K \to L$ be a simplicial approximation to the map $f$. Then there is a well-defined homotopy $H\colon |K| \times [0,1] \to |L|$, between the maps $f$ and $s$, where*

$$H(\mathbf{x}, t) = (1-t)f(\mathbf{x}) + ts(\mathbf{x})$$

*for all $\mathbf{x} \in |K|$ and $t \in [0,1]$.*

**Proof** Let $\mathbf{x} \in |K|$. Then there is a unique simplex $\sigma$ of $L$ such that the point $f(\mathbf{x})$ belongs to the interior of $\sigma$. Then $s(\mathbf{x}) \in \sigma$. But, given any two points of a simplex embedded in some Euclidean space, the line segment joining those two points is contained within the simplex. It follows that $(1-t)f(\mathbf{x}) + ts(\mathbf{x}) \in |L|$ for all $\mathbf{x} \in K$ and $t \in [0,1]$. Thus the homotopy $H\colon |K| \times [0,1] \to |L|$ is a well-defined map from $|K| \times [0,1]$ to $|L|$. Moreover it follows directly from the definition of this map that $H(\mathbf{x}, 0) = f(\mathbf{x})$ and $H(\mathbf{x}, 1) = s(\mathbf{x})$ for all $\mathbf{x} \in |K|$ and $t \in [0,1]$. The map $H$ is thus a homotopy between the maps $f$ and $s$, as required. ∎

**Definition** Let $K$ be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in $K$ is the union of the interiors of all simplices of $K$ that contain the point $\mathbf{x}$.

**Lemma 3.14** *Let $K$ be a simplicial complex and let $\mathbf{x} \in |K|$. Then the star $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ is open in $|K|$, and $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$.*

**Proof** Every point of $|K|$ belongs to the interior of a unique simplex of $K$ (Lemma 3.6). It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of $K$ that do not contain the point $\mathbf{x}$. But if a simplex of $K$ does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of $K$ that do not contain the point $\mathbf{x}$. But each simplex of $K$ is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of $K$. ∎

**Proposition 3.15** *A function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ between the vertex sets of simplicial complexes $K$ and $L$ is a simplicial map, and a simplicial approximation to some continuous map $f\colon |K| \to |L|$, if and only if $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of $K$.*

**Proof** Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of $K$, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of $\sigma$, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to $\tau$ (since $s$ is a simplicial approximation to the map $f$), and therefore $s(\mathbf{x})$ must belong to the interior of some face of $\tau$. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, since $\mathbf{x}$ is in the interior of $\sigma$. It follows that $s(\sigma)$ must be a face of $\tau$, and therefore $s(\mathbf{v})$ must be a vertex of $\tau$. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a function with the property that $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of $K$. Let $\mathbf{x}$ be a point in the interior of some simplex of $K$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L\left(s(\mathbf{v}_j)\right)$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of $\tau$, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required. ∎

**Corollary 3.16** *If $s\colon K \to L$ and $t\colon L \to M$ are simplicial approximations to continuous maps $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$, where $K$, $L$ and $M$ are simplicial complexes, then $t \circ s\colon K \to M$ is a simplicial approximation to $g \circ f\colon |K| \to |M|$.*

**Theorem 3.17** (Simplicial Approximation Theorem) *Let $K$ and $L$ be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Then, for some sufficiently large integer $j$, there exists a simplicial approximation $s\colon K^{(j)} \to L$ to $f$ defined on the $j$th barycentric subdivision $K^{(j)}$ of $K$.*

**Proof** The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of $L$ is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ (Lemma 3.14) and the interior of any simplex of $L$ is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of $L$ is an open cover of $|K|$. It then follows from the Lebesgue Lemma that there exists some $\delta > 0$ with the property that every subset of $|K|$ whose diameter is less than $\delta$ is mapped by $f$ into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of $L$.

Now the mesh $\mu(K^{(j)})$ of the $j$th barycentric subdivision of $K$ tends to zero as $j \to +\infty$, since

$$\mu(K^{(j)}) \leq \left(\frac{\dim K}{\dim K + 1}\right)^j \mu(K)$$

for all $j$ (Lemma 3.10). Thus we can choose $j$ such that $\mu(K^{(j)}) < \frac{1}{2}\delta$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most $\delta$. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of $L$ such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of $L$. It follows directly from Proposition 3.15 that this is the desired simplicial approximation to $f$. ∎

# 4 Simplicial Homology Groups

## 4.1 Basic Properties of Permutations of a Finite Set

A *permutation* of a set $S$ is a bijection mapping $S$ onto itself. The set of all permutations of some set $S$ is a group; the group multiplication corresponds to composition of permutations. A *transposition* is a permutation of a set $S$ which interchanges two elements of $S$, leaving the remaining elements of the set fixed. If $S$ is finite and has more than one element then any permutation of $S$ can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and $j$ for some $j$.

Associated to any permutation $\pi$ of a finite set $S$ is a number $\epsilon_\pi$, known as the *parity* or *signature* of the permutation, which can take on the values $\pm 1$. If $\pi$ can be expressed as the product of an even number of transpositions, then $\epsilon_\pi = +1$; if $\pi$ can be expressed as the product of an odd number of transpositions then $\epsilon_\pi = -1$. The function $\pi \mapsto \epsilon_\pi$ is a homomorphism from the group of permutations of a finite set $S$ to the multiplicative group $\{+1, -1\}$ (i.e., $\epsilon_{\pi\rho} = \epsilon_\pi \epsilon_\rho$ for all permutations $\pi$ and $\rho$ of the set $S$). Note in particular that the parity of any transposition is $-1$.

## 4.2 The Chain Groups of a Simplicial Complex

Let $K$ be a simplicial complex, let $\mathrm{Vert}(K)$ denote the set of vertices of $K$, and let $R$ be a unital commutative ring. For each non-negative integer $q$, let $\Delta_q(K; R)$ denote the free $R$-module $F_R W_{q,K}$ on the set $W_{q,K}$, where $W_{q,K}$ denotes the subset of $\mathrm{Vert}(K)^{q+1}$ consisting of all $(q+1)$-tuples $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of vertices of $K$ with the property that $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$. (Thus if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are vertices of $K$ then $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in W_{q,K}$ if and only if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$.)

Now the set $W_{q,K}$ is a finite set for each non-negative integer $q$. It follows that elements of the free $R$-module $\Delta_q(K; R)$ can be represented as functions from the set $W_{q,K}$ to the ring $R$, where $(f + g)(w) = f(w) + g(w)$ and $(rf)(w) = rf(w)$ for all $f, g \in \Delta_q(K; R)$, $w \in W_{q,K}$ and $r \in R$. Each element $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of $W_{q,K}$ determines a corresponding element $\lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}$ of $\Delta_q(K; R)$, represented by the function from $W_{q,K}$ to the ring $R$ defined such that

$$\lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}(w) = \begin{cases} 1_R & \text{if } w = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q), \\ 0_R & \text{otherwise,} \end{cases}$$

where $0_R$ denotes the zero element of the ring $R$ and $1_R$ denotes the multiplicative identity element of the ring $R$. Any element of $\Delta_q(K; R)$ can then be represent by a (finite) sum of the form $\sum_{w \in W_{q,K}} r_w \lambda_w$, where $r_w \in R$ for all $w \in W_{q,K}$.

**Definition** Let $K$ be a simplicial complex, let $R$ be a unital commutative ring, and let $q$ be a non-negative integer Let $\Delta_q^0(K;R)$ be the submodule of $\Delta_q(K;R)$ generated by elements of the form $\lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$ where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct, and by elements of the form

$$\lambda_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)},$$

where $\pi$ is some permutation of $\{0, 1, \ldots, q\}$ with parity $\epsilon_\pi$. We define the $q$th *chain group* $C_q(K;R)$ of the simplicial complex $K$ with coefficients in the unital commutative ring $R$ to be the corresponding quotient module $\Delta_q(K;R)/\Delta_q^0(K;R)$.

An element of the chain group $C_q(K;R)$ is referred to as *$q$-chain* of the simplicial complex $K$ with coefficients in the ring $R$.

For convenience, we define both $\Delta_q(K;R)$ and $C_q(K;R)$ to be the zero module over the ring $R$ when $q < 0$.

**Remark** We have defined above the chain group $C_q(K;R)$ of a simplicial complex with coefficients in a unital commutative ring $R$. In topological applications this coefficient ring will often be the ring $\mathbb{Z}$ of integers, the field $\mathbb{R}$ of real numbers, or the finite field $\mathbb{Z}_p$ with $p$ elements, where $p$ is some prime number. In such cases we refer to chain groups (and associated homology and cohomology groups) with *integer coefficients*, *real coefficients*, or with *coefficients* in $\mathbb{Z}_p$. Chain groups, homology groups and cohomology groups with coefficients in the finite field $\mathbb{Z}_2$ are particularly important in studying the topology of manifolds. (A *manifold* of dimension $n$ is a topological space that locally resembles Euclidean space of dimension $n$.)

**Lemma 4.1** *Let $K$ be a simplicial complex, let $R$ be unital commutative ring, and, for each non-negative integer $q$, let $C_q(K;R)$ be the $q$th chain group of $K$ with coefficients in the ring $R$. Then $C_q(K;R) = 0$ for all integers $q$ satisfying $q > \dim K$.*

**Proof** The dimension $\dim K$ of the simplicial complex $K$ is by definition the maximum of the dimensions of the simplices of $K$. Thus if a finite list of distinct vertices spans a simplex of $K$ then the number of vertices in that list cannot exceed $\dim K + 1$. It follows that if $q > \dim K$ then $\Delta_q^0(K;R) = \Delta_q(K;R)$, because no $(q+1)$-tuple of vertices of $K$ can consist of distinct vertices of $K$. Therefore $C_q(K;R) = 0$ whenever $q > \dim K$, as required. ∎

Each $(q+1)$-tuple $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of vertices of $K$ that span a simplex of $K$ belongs to the set $W_{q,K}$, and therefore determines a corresponding element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of $C_q(K; R)$, where

$$\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = \Delta_q^0(K; R) + \lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}.$$

This element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of $C_q(K; R)$ is then the image of the generator $\lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}$ of the free $R$-module $\Delta_q(K; R)$ under the quotient homomorphism from $\Delta_q(K; R)$ to $C_q(K; R)$.

**Lemma 4.2** *Let $K$ be a simplicial complex, let $R$ be a unital commutative ring, let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices of $K$ that span a simplex of $K$. Then the following identities are satisfied within the $R$-module $C_q(K; R)$:—*

(i) $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = 0$ *if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct;*

(ii) $\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle = \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ *for any permutation $\pi$ of the set $\{0, 1, \ldots, q\}$.*

**Proof** If the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct then $\lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)} \in \Delta_0(K; R)$, and therefore the coset

$$\Delta_q^0(K; R) + \lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}$$

of $\Delta_q^0(K; R)$ in $\Delta_q(K; R)$ that contains the generator $\lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}$ of $\Delta_q(K; R)$ is the submodule $\Delta_q^0(K; R)$ itself. It follows that $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ is the zero element of the corresponding quotient module $\Delta(K; R)/\Delta_q^0(K; R)$. This proves (i).

Now suppose that the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of $K$ span a simplex of $K$ but are not necessarily distinct. Let $\pi$ be a permutation of the set $\{0, 1, \ldots, q\}$. Then the element

$$\lambda_{(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)})} - \epsilon_\pi \lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}$$

of $\Delta_q(K; R)$ is one of the generators of the submodule $\Delta_q^0(K; R)$ specified in the definition of this submodule. It follows that

$$\Delta_q^0(K; R) + \lambda_{(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)})} - \epsilon_\pi \lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)} = \Delta_q^0(K; R),$$

and therefore

$$\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle - \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = 0.$$

This proves (ii). ∎

**Example** Let $\mathbf{v}$ and $\mathbf{w}$ be the endpoints of a line segment in some Euclidean space, and let $K$ be the simplicial complex consisting of these two vertices together with the line segment joining them. Also let $R$ be a unital commutative ring. Then

$$\Delta_0(K; R) = \{r_{\mathbf{v}}\lambda_{\mathbf{v}} + r_{\mathbf{w}}\lambda_{\mathbf{w}} : r_{\mathbf{v}}, r_{\mathbf{w}} \in R\}$$

and

$$\begin{aligned}\Delta_1(K; R) &= \{r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{v},\mathbf{w})}\lambda_{(\mathbf{v},\mathbf{w})} + r_{(\mathbf{w},\mathbf{v})}\lambda_{(\mathbf{w},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})} : \\ &\qquad r_{(\mathbf{v},\mathbf{v})}, r_{(\mathbf{v},\mathbf{w})}, r_{(\mathbf{w},\mathbf{v})}, r_{(\mathbf{w},\mathbf{w})} \in R\}.\end{aligned}$$

Now the submodule $\Delta_0^0(K; R)$ of the $R$-module $\Delta_0(K; R)$ has no non-zero generators, and therefore $C_0(K; R) = \Delta_0(K; R)$. It follows that $C_0(K; R) \cong R^2$. Moreover

$$C_0(K; R) = \{r_{\mathbf{v}}\langle\mathbf{v}\rangle + r_{\mathbf{w}}\langle\mathbf{w}\rangle : r_{\mathbf{v}}, r_{\mathbf{w}} \in R\}.$$

The submodule $\Delta_1^0(K; R)$ of $\Delta_1(K; R)$ is generated by the elements $\lambda_{(\mathbf{v},\mathbf{v})}$, $\lambda_{(\mathbf{w},\mathbf{w})}$ and $\lambda_{(\mathbf{v},\mathbf{w})} + \lambda_{(\mathbf{w},\mathbf{v})}$ of $\Delta_1(K; R)$. It follows that

$$\langle\mathbf{v}, \mathbf{v}\rangle = 0, \quad \langle\mathbf{w}, \mathbf{w}\rangle = 0$$

and

$$\langle\mathbf{v}, \mathbf{w}\rangle + \langle\mathbf{w}, \mathbf{v}\rangle = 0.$$

Let $r_{(\mathbf{v},\mathbf{v})}, r_{(\mathbf{v},\mathbf{w})}, r_{(\mathbf{w},\mathbf{v})}, r_{(\mathbf{w},\mathbf{w})} \in R$. Then

$$\begin{aligned}&r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{v},\mathbf{w})}\lambda_{(\mathbf{v},\mathbf{w})} + r_{(\mathbf{w},\mathbf{v})}\lambda_{(\mathbf{w},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})} \\ &\qquad = r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})} + r_{(\mathbf{w},\mathbf{v})}(\lambda_{(\mathbf{v},\mathbf{w})} + \lambda_{(\mathbf{w},\mathbf{v})}) \\ &\qquad\qquad + (r_{(\mathbf{v},\mathbf{w})} - r_{(\mathbf{w},\mathbf{v})})\lambda_{(\mathbf{v},\mathbf{w})} \\ &\qquad \in \Delta_1^0(K; R) + (r_{(\mathbf{v},\mathbf{w})} - r_{(\mathbf{w},\mathbf{v})})\lambda_{(\mathbf{w},\mathbf{v})}\end{aligned}$$

Moreover if $\theta\colon \Delta_1(K; R) \to R$ is the $R$-module homomorphism defined such that

$$\theta\left(r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{v},\mathbf{w})}\lambda_{(\mathbf{v},\mathbf{w})} + r_{(\mathbf{w},\mathbf{v})}\lambda_{(\mathbf{w},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})}\right) = r_{(\mathbf{v},\mathbf{w})} - r_{(\mathbf{w},\mathbf{v})}$$

then $\lambda_{(\mathbf{v},\mathbf{v})} \in \ker\theta$, $\lambda_{(\mathbf{w},\mathbf{w})} \in \ker\theta$ and $\lambda_{(\mathbf{v},\mathbf{w})} + \lambda_{(\mathbf{w},\mathbf{v})} \in \ker\theta$, and therefore $\Delta_1^0(K; R) \subset \ker\theta$. But

$$r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{v},\mathbf{w})}\lambda_{(\mathbf{v},\mathbf{w})} + r_{(\mathbf{w},\mathbf{v})}\lambda_{(\mathbf{w},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})} \in \ker\theta$$

if and only if $r_{(\mathbf{w},\mathbf{v})} = r_{(\mathbf{v},\mathbf{w})}$, in which case

$$\begin{aligned} & r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{v},\mathbf{w})}\lambda_{(\mathbf{v},\mathbf{w})} + r_{(\mathbf{w},\mathbf{v})}\lambda_{(\mathbf{w},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})} \\ & \quad = r_{(\mathbf{v},\mathbf{v})}\lambda_{(\mathbf{v},\mathbf{v})} + r_{(\mathbf{w},\mathbf{w})}\lambda_{(\mathbf{w},\mathbf{w})} + r_{(\mathbf{v},\mathbf{w})}(\lambda_{(\mathbf{v},\mathbf{w})} + \lambda_{(\mathbf{w},\mathbf{v})}) \\ & \quad \in \ker\theta. \end{aligned}$$

Thus $\ker\theta = \Delta_1^0(K;R)$. Moreover the homomorphism $\theta\colon \Delta_1(K;R) \to R$ is surjective. It follows that the homomorphism $\theta$ induces an isomorphism $\tilde{\theta}\colon C_1(K;R) \to R$, where $C_1(K;R) = \Delta_1(K;R)/\Delta_1^0(K;R)$, and therefore $C_1(K;R) \cong R$ (see Corollary 1.8).

**Example** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the vertices of a triangle in some Euclidean space. Let $K$ be the simplicial complex consisting of this triangle, together with its edges and vertices, and let $R$ be a unital commutative ring. (The coefficient ring $R$ could for example be the ring $\mathbb{Z}$ of integers, or the field $\mathbb{R}$ of real numbers, or the finite field $\mathbb{Z}_2$ with exactly two elements.) Every element of the chain group $C_0(K;R)$ of $K$ in dimension zero can be expressed uniquely in the form

$$r_{\mathbf{u}}\langle\mathbf{u}\rangle + r_{\mathbf{v}}\langle\mathbf{v}\rangle + r_{\mathbf{w}}\langle\mathbf{w}\rangle$$

for some $r_{\mathbf{u}}, r_{\mathbf{v}}, r_{\mathbf{w}} \in R$. It follows that $C_0(K;R) \cong R^3$. Also

$$\langle\mathbf{w},\mathbf{v}\rangle = -\langle\mathbf{v},\mathbf{w}\rangle, \quad \langle\mathbf{u},\mathbf{w}\rangle = -\langle\mathbf{w},\mathbf{u}\rangle, \quad \langle\mathbf{v},\mathbf{u}\rangle = -\langle\mathbf{u},\mathbf{v}\rangle$$

and

$$\langle\mathbf{u},\mathbf{u}\rangle = \langle\mathbf{v},\mathbf{v}\rangle = \langle\mathbf{w},\mathbf{w}\rangle = 0$$

in the $R$-module $C_1(K;R)$. It follows that every element of $C_1(K;R)$ can be expressed uniquely in the form

$$r_0\langle\mathbf{v},\mathbf{w}\rangle + r_1\langle\mathbf{w},\mathbf{u}\rangle + r_2\langle\mathbf{u},\mathbf{v}\rangle$$

for some $r_0, r_1, r_2 \in R$. Indeed any element of $\Delta_1(K;R)$ may be represented as a linear combination of the form

$$\begin{aligned} & p_0\lambda_{(\mathbf{v},\mathbf{w})} + p_1\lambda_{(\mathbf{w},\mathbf{u})} + p_2\lambda_{(\mathbf{u},\mathbf{v})} \\ & \qquad + s_0\lambda_{(\mathbf{w},\mathbf{v})} + s_1\lambda_{(\mathbf{u},\mathbf{w})} + s_2\lambda_{(\mathbf{v},\mathbf{u})} \\ & \qquad + t_0\lambda_{(\mathbf{u},\mathbf{u})} + t_1\lambda_{(\mathbf{v},\mathbf{v})} + t_2\lambda_{(\mathbf{w},\mathbf{w})}, \end{aligned}$$

where $p_0, p_1, p_2, s_0, s_1, s_2, t_0, t_1, t_2 \in R$. The quotient homomorphism from $\Delta_1(K;R)$ to $C_1(K;R)$ maps this linear combination to the element $c_1$ of $C_1(K;R)$, where

$$\begin{aligned} c_1 &= p_0\langle\mathbf{v},\mathbf{w}\rangle + p_1\langle\mathbf{w},\mathbf{u}\rangle + p_2\langle\mathbf{u},\mathbf{v}\rangle \\ & \qquad + s_0\langle\mathbf{w},\mathbf{v}\rangle + s_1\langle\mathbf{u},\mathbf{w}\rangle + s_2\langle\mathbf{v},\mathbf{u}\rangle \\ & \qquad + t_0\langle\mathbf{u},\mathbf{u}\rangle + t_1\langle\mathbf{v},\mathbf{v}\rangle + t_2\langle\mathbf{w},\mathbf{w}\rangle. \end{aligned}$$

But it follows from Lemma 4.2 that

$$c_1 = (p_0 - s_0)\langle \mathbf{v}, \mathbf{w}\rangle + (p_1 - s_1)\langle \mathbf{w}, \mathbf{u}\rangle + (p_2 - s_2)\langle \mathbf{u}, \mathbf{v}\rangle.$$

Therefore

$$C_1(K; R) = \{r_0\langle \mathbf{v}, \mathbf{w}\rangle + r_1\langle \mathbf{w}, \mathbf{u}\rangle + r_2\langle \mathbf{u}, \mathbf{v}\rangle : r_0, r_1, r_2 \in R\},$$

and thus $C_1(K; R) \cong R^3$. Moreover

$$\begin{array}{rl} r_0\langle \mathbf{v}, \mathbf{w}\rangle + r_1\langle \mathbf{w}, \mathbf{u}\rangle + r_2\langle \mathbf{u}, \mathbf{v}\rangle & \\ = & -r_0\langle \mathbf{w}, \mathbf{v}\rangle + r_1\langle \mathbf{w}, \mathbf{u}\rangle + r_2\langle \mathbf{u}, \mathbf{v}\rangle \\ = & r_0\langle \mathbf{v}, \mathbf{w}\rangle - r_1\langle \mathbf{u}, \mathbf{w}\rangle + r_2\langle \mathbf{u}, \mathbf{v}\rangle \\ = & r_0\langle \mathbf{v}, \mathbf{w}\rangle + r_1\langle \mathbf{w}, \mathbf{u}\rangle - r_2\langle \mathbf{v}, \mathbf{u}\rangle \\ = & -r_0\langle \mathbf{w}, \mathbf{v}\rangle - r_1\langle \mathbf{u}, \mathbf{w}\rangle + r_2\langle \mathbf{u}, \mathbf{v}\rangle \\ & \text{etc.} \end{array}$$

Finally, we consider the structure of $C_2(K; R)$. Now

$$\begin{array}{rcl} \langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle & = & \langle \mathbf{v}, \mathbf{w}, \mathbf{u}\rangle = \langle \mathbf{w}, \mathbf{u}, \mathbf{v}\rangle = -\langle \mathbf{w}, \mathbf{v}, \mathbf{u}\rangle \\ & = & -\langle \mathbf{u}, \mathbf{w}, \mathbf{v}\rangle = -\langle \mathbf{v}, \mathbf{u}, \mathbf{w}\rangle. \end{array}$$

and

$$\langle \mathbf{u}, \mathbf{u}, \mathbf{w}\rangle = 0, \ \ \langle \mathbf{u}, \mathbf{v}, \mathbf{u}\rangle = 0, \ \ \langle \mathbf{u}, \mathbf{v}, \mathbf{v}\rangle = 0, \ \text{ etc.},$$

and therefore every element of $C_2(K; R)$ can be expressed uniquely in the form $r\langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle$ for some $r \in R$. It follows that $C_2(K; R) \cong R$.

Let $K$ be a simplicial complex, and let $q$ be a non-negative integer. Let some ordering be chosen on the vertices of each $q$-simplex of $K$. Let $W_{q,K}$ denote the subset of $\mathrm{Vert}(K)^{q+1}$ consisting of those $(q+1)$-tuples of vertices of $K$ that span simplices of $K$, and let $W^+_{q,K}$ be the subset of $W_{q,K}$ consisting of those $(q+1)$-tuples $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of vertices of $K$ satisfying the following conditions:—

(i) $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct;

(ii) $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span some simplex of $K$;

(iii) $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are listed in increasing order with respect to the chosen ordering of the vertices of the unique $q$-simplex of $K$ that is spanned by these vertices.

Let $R$ be a unital commutative ring, let $\Delta_q(K;R)$ denote the free $R$-module on the set $W_{q,K}$. Let us represent the elements of $\Delta_q(K;R)$ in the standard fashion, as functions from the finite set $W_{q,K}$ to the ring $R$. For each $w \in W_{q,K}$, let $\lambda_w$ denote the generator of $\Delta_q(K;R)$ that corresponds to $w$ and is represented as the function on $W_{q,K}$ that maps $w$ to the multiplicative identity element $1_R$ of $R$ and maps every other element of $W_{q,K}$ to the zero element $0_R$ of $R$.

Now the set $\{\lambda_w : w \in W_{q,K}^+\}$ generates a submodule $\Delta_q^+(K;R)$ of $\Delta_q(K;R)$. This submodule $\Delta_q^+(K;R)$ is itself a free module that is freely generated by $\{\lambda_w : w \in W_{q,K}^+\}$. We claim that the quotient homomorphism from $\Delta_q(K;R)$ to the chain group $C_q(K;R)$ with kernel $\Delta_q^0(K;R)$ maps $\Delta_q^+(K;R)$ isomorphically onto $C_q(K;R)$.

Let $w \in W_{q,K}$. Then $w = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are vertices of $K$ that span a simplex of $K$. Now if the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct then they span a $q$-simplex of $K$, and there is a unique permutation $\tau$ of the set $\{0, 1, \ldots, q\}$ such that the vertices

$$\mathbf{v}_{\tau(0)}, \mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(q)}$$

are listed in increasing order with respect to the chosen ordering of the vertices of this $q$-simplex of $K$. The permutatation $\tau$ is then the unique permutation of the set $\{0, 1, \ldots, q\}$ for which $(\mathbf{v}_{\tau(0)}, \mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(q)}) \in W_{q,K}^+$. There is thus a function $r: W_{q,K} \to \Delta_q^+(K;R)$ defined so as to satisfy the following two conditions:

(i) if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct then

$$r(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0;$$

(ii) if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct then

$$r(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \epsilon_\tau \lambda_{(\mathbf{v}_{\tau(0)}, \mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(q)})},$$

where $\tau$ is the unique permutation of the finite set $\{0, 1, \ldots, q\}$ for which $(\mathbf{v}_{\tau(0)}, \mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(q)}) \in W_{q,K}^+$.

This function $r: W_{q,K} \to \Delta_q^+(K;R)$ induces an $R$-module homomorphism $\rho_q: \Delta(K;R) \to \Delta_q^+(K;R)$, where $\rho_q(\lambda_w) = r(w)$ for all $w \in W_{q,K}$ (see Proposition 2.2).

We claim that $\lambda_w - \rho_q(\lambda_w) \in \Delta_q^0(K;R)$ for all $w \in W_{q,K}$, where $\Delta_q^0(K;R)$ is the kernel of the quotient homomorphism from $\Delta_q(K;R)$ to $C_q(K;R)$. Now $\Delta_q^0(K;R)$ is the submodule of $\Delta_q(K;R)$ generated by those elements of

$\Delta_q(K;R)$ that are of the form $\lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}$, where the vertices $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q$ of $K$ span a simplex of $K$ but are not all distinct, and by those elements of $\Delta_q(K;R)$ that are of the form

$$\lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)} - \epsilon_\tau \lambda_{(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)})},$$

where $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q$ are vertices of $K$ that span some simplex of $K$ and where $\tau$ is a permutation of the finite set $\{0,1,\ldots,q\}$.

Let $w \in W_{q,K}$, where $w = (\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)$. If the vertices $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q$ are not all distinct then $\lambda_w \in \Delta_q^0(K;R)$ and $\rho_q(\lambda_w) = 0$. It follows that $\lambda_w - \rho_q(\lambda_w) \in \Delta_q^0(K;R)$ if the vertices $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q$ are not all distinct. On the other hand, if these vertices are all distinct then

$$\rho_q(\lambda_w) = \rho_q(\lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}) = \epsilon_\tau \lambda_{(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)})},$$

where $\tau$ is the unique permutation of $\{0,1,\ldots,q\}$ for which

$$(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)}) \in W_{q,K}^+,$$

and therefore

$$\lambda_w - \rho_q(\lambda_w) = \lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)} - \epsilon_\tau \lambda_{(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)})} \in \Delta_q^0(K;R).$$

We have thus shown that $\lambda_w - \rho_q(\lambda_w) \in \Delta_q^0(K;R)$ for all $w \in W_{q,K}$. But the free $R$-module $\Delta_q(K;R)$ is generated by the set $\{\lambda_w : w \in W_{q,K}\}$. It follows that $x - \rho_q(x) \in \Delta_q^0(K;R)$ for all $x \in \Delta_q(K;R)$. Also $\rho_q(\lambda_w) = \lambda_w$ for all $w \in W_{q,K}^+$. and therefore $\rho_q(x) = x$ for all $x \in \Delta_q^+(K;R)$.

Let $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q$ be distinct vertices of $K$ that span a $q$-simplex of $K$, and let $\tau$ be the unique permutation of the set $\{0,1,\ldots,q\}$ for which $(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)}) \in W_{q,K}^+$. Also let $\pi$ be any permutation of $\{0,1,\ldots,q\}$. Then

$$\begin{aligned}
\rho_q(\lambda_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})}) &= \epsilon_{\tau\circ\pi^{-1}} \lambda_{(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)})} \\
&= \epsilon_\tau \epsilon_\pi \lambda_{(\mathbf{v}_{\tau(0)},\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(q)})} \\
&= \epsilon_\pi \rho_q(\lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}),
\end{aligned}$$

because the basic properties of the signature of parity of permutations of finite sets ensures that $\epsilon_{\tau\circ\pi^{-1}} = \epsilon_\tau \epsilon_{\pi^{-1}} = \epsilon_\tau \epsilon_\pi$. (We are here using the fact that the composition of two even permutations is an even permutation, as is the composition of two odd permutations, whereas the composition of an odd permutation and an even permutation is always an odd permutation, irrespective of the order in which those permutations are composed.) Thus

$$\rho_q\Bigl(\lambda_{(\mathbf{v}_{\pi(0)},\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(q)})} - \epsilon_\pi \lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}\Bigr) = 0$$

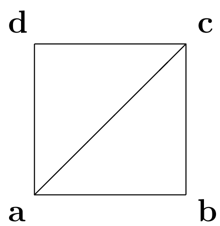
for all permutations $\pi$ of $\{0, 1, 2, \ldots, q\}$ whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct vertices of $K$ that span a simplex of $K$. Also $\rho_q(\lambda_{(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q)}) = 0$ when the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$ but are not distinct. It follows that $\rho_q(x) = 0$ for all $x \in \Delta_q^0(K; R)$ (because the generators of $\Delta_q^0(K; R)$ specified in the definition of this submodule of $\Delta_q(K; R)$ are all mapped by the homomorphism $\rho_q$ to the zero element of $\Delta_q^+(K; R)$), and therefore $\Delta_q^0(K; R) \subset \ker \rho_q$. But we have previously shown that $x - \rho_q(x) \in \Delta_q^0(K; R)$ for all $x \in \Delta_q(K; R)$. It follows that $\ker \rho_q \subset \Delta_q^0(K; R)$. Putting these results together, we see that $\ker \rho_q = \Delta_q^0(K; R)$.

Now the homomorphism $\rho_q \colon \Delta_q(K; R) \to \Delta_q^+(K; R)$ is surjective, because $\rho_q(x) = x$ for all $x \in \Delta_q^+(K; R)$. It follows that the $R$-module homomorphism $\rho_q$ induces an isomorphism from the chain group $C_q(K; R)$ to $\Delta_q^+(K; R)$, where

$$C_q(K; R) = \Delta_q(K; R)/\Delta_q^0(K; R) = \Delta_q(K; R)/\ker \rho_q$$

(see Corollary 1.8). Moreover this isomorphism is the inverse of the homomorphism from $\Delta_q^+(K; R)$ to $C_q(K; R)$ obtained on restricting to $\Delta_q^+(K; R)$ the quotient homorphism from $\Delta_q(K; R)$ to $C_q(K; R)$. We conclude therefore that $\Delta_q^+(K; R) \cong C_q(K; R)$. It follows that the chain group $C_q(K; R)$ is a free module over the ring $R$, since any $R$-module that is isomorphic to a free $R$-module must itself be a free $R$-module.

**Example** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the vertices of a square in the plane, listed in anticlockwise order round the square, and let $K$ be the 2-dimensional simplicial complex consisting of the following simplices: vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$; edges $\mathbf{a}\,\mathbf{b}$, $\mathbf{b}\,\mathbf{c}$, $\mathbf{c}\,\mathbf{d}$, $\mathbf{d}\,\mathbf{a}$ and $\mathbf{a}\,\mathbf{c}$; triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ and $\mathbf{a}\,\mathbf{c}\,\mathbf{d}$.




Let $R$ be an integral domain (which might for example be the ring of integers, the field of rational numbers, the field of real numbers, or some finite field). The $R$-module $\Delta_0(K; R)$ is then a free $R$-module of rank 4 over $R$ with free basis $\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}, \lambda_{\mathbf{d}}$. Moreover $\Delta_0^0(K; R)$ is the zero submodule of $\Delta_0(K; R)$, and therefore $C_0(K; R) \cong \Delta_0(K; R) \cong R^4$. Indeed

$$C_0(K; R) = \{r_{\mathbf{a}}\langle\mathbf{a}\rangle + r_{\mathbf{b}}\langle\mathbf{b}\rangle + r_{\mathbf{c}}\langle\mathbf{c}\rangle + r_{\mathbf{d}}\langle\mathbf{d}\rangle : r_{\mathbf{a}}, r_{\mathbf{b}}, r_{\mathbf{c}}, r_{\mathbf{d}} \in R\}.$$

We denote by $W_{1,K}$ the set of ordered pairs of vertices of $K$ that span a simplex of $K$. If the vertices in such an ordered pair coincide then the simplex spanned by them is is a vertex of $K$. On the other hand, if the vertices in the ordered pair are distinct then the simplex spanned by them is an edge of $K$. Moreover each edge of $K$ determines two elements of $W_{1,K}$, since there are two possible orderings of the endpoints of the edge. It follows that $W_{1,K}$ has 14 elements. Indeed

$$\begin{aligned} W_{1,K} &= \{(\mathbf{a},\mathbf{a}),\ (\mathbf{b},\mathbf{b}),\ (\mathbf{c},\mathbf{c}),\ (\mathbf{d},\mathbf{d}),\ (\mathbf{a},\mathbf{b}),\ (\mathbf{b},\mathbf{a}),\ (\mathbf{b},\mathbf{c}), \\ &\qquad (\mathbf{c},\mathbf{b}),\ (\mathbf{c},\mathbf{d}),\ (\mathbf{d},\mathbf{c}),\ (\mathbf{d},\mathbf{a}),\ (\mathbf{a},\mathbf{d}),\ (\mathbf{a},\mathbf{c}),\ (\mathbf{c},\mathbf{a})\}. \end{aligned}$$

It follows that $\Delta_1(K;R)$ is a free $R$-module of rank 14.

Let us order the vertices of each edges as indicated in the listing of edges in the specification of the simplicial complex $K$ given above. Then these orderings of the vertices of the edges of $K$ determines a subset $W_{1,K}^+$ of $W_{1,K}$, where

$$W_{1,K}^+ = \{(\mathbf{a},\mathbf{b}),\ (\mathbf{b},\mathbf{c}),\ (\mathbf{c},\mathbf{d}),\ (\mathbf{d},\mathbf{a}),\ (\mathbf{a},\mathbf{c})\}.$$

The corresponding elements $\lambda_{(\mathbf{a},\mathbf{b})}$, $\lambda_{(\mathbf{b},\mathbf{c})}$, $\lambda_{(\mathbf{c},\mathbf{d})}$, $\lambda_{(\mathbf{d},\mathbf{a})}$ and $\lambda_{(\mathbf{a},\mathbf{c})}$ of $\Delta_1(K;R)$ freely generate a submodule $\Delta_1^+(K;R)$ of $\Delta_1(K;R)$ which is itself a free $R$-module of rank 5.

Now $C_1(K;R) = \Delta_1(K;R)/\Delta_1^0(K;R)$, where $\Delta_1^0(K;R)$ is the submodule of $\Delta_1(K;R)$ generated by

$$\lambda_{(\mathbf{a},\mathbf{a})},\ \lambda_{(\mathbf{b},\mathbf{b})},\ \lambda_{(\mathbf{c},\mathbf{c})},\ \lambda_{(\mathbf{d},\mathbf{d})},\ \lambda_{(\mathbf{a},\mathbf{b})} + \lambda_{(\mathbf{b},\mathbf{a})},\ \lambda_{(\mathbf{b},\mathbf{c})} + \lambda_{(\mathbf{c},\mathbf{b})},$$

$$\lambda_{(\mathbf{c},\mathbf{d})} + \lambda_{(\mathbf{d},\mathbf{c})},\ \lambda_{(\mathbf{d},\mathbf{a})} + \lambda_{(\mathbf{a},\mathbf{d})} \text{ and } \lambda_{(\mathbf{a},\mathbf{c})} + \lambda_{(\mathbf{c},\mathbf{a})}.$$

Indeed let $\tau$ be the permutation of the set $\{\mathbf{a},\mathbf{b}\}$ that swaps the vertices $\mathbf{a}$ and $\mathbf{b}$. Then $\tau$ is an odd permutation, and therefore $\epsilon_\tau = -1$, where $\epsilon_\tau$ denotes the parity of the permutation $\tau$. The generator $g_\tau$ of $\Delta_1^0(K;R)$ determined by the permutation $\tau$ of $\{\mathbf{a},\mathbf{b}\}$ then satisfies

$$g_\tau = \lambda_{(\mathbf{a},\mathbf{b})} - \epsilon_\tau \lambda_{(\tau(\mathbf{a}),\tau(\mathbf{b}))} = \lambda_{(\mathbf{a},\mathbf{b})} + \lambda_{(\mathbf{b},\mathbf{a})}.$$

The generators corresponding to the other edges of $K$ are determined in an analogous fashion. And moreover each vertex of $K$ determines a corresponding generator of $\Delta_1^0(K;R)$.

Now the discussion preceding this example establishes the existence of a well-defined homomorphism $\rho_1\colon \Delta_1(K;R) \to \Delta_1^+(K;R)$ from $\Delta_1(K;R)$ to $\Delta_1^+(K;R)$ such that $\rho_1(x) = x$ for all $x \in \Delta_1^+(K;R)$ and $\rho_1(x) = 0$ for all $x \in \Delta_1^0(K;R)$. In the example under discussion this homomorphism is defined such that

$$\begin{aligned}
\rho_1(\lambda_{(\mathbf{a},\mathbf{a})}) &= 0, \quad \rho_1(\lambda_{(\mathbf{b},\mathbf{b})}) = 0, \quad \rho_1(\lambda_{(\mathbf{c},\mathbf{c})}) = 0, \quad \rho_1(\lambda_{(\mathbf{d},\mathbf{d})}) = 0,\\
\rho_1(\lambda_{(\mathbf{a},\mathbf{b})}) &= \lambda_{(\mathbf{a},\mathbf{b})}, \quad \rho_1(\lambda_{(\mathbf{b},\mathbf{a})}) = -\lambda_{(\mathbf{a},\mathbf{b})}, \quad \rho_1(\lambda_{(\mathbf{b},\mathbf{c})}) = \lambda_{(\mathbf{b},\mathbf{c})},\\
\rho_1(\lambda_{(\mathbf{c},\mathbf{b})}) &= -\lambda_{(\mathbf{b},\mathbf{c})}, \quad \rho_1(\lambda_{(\mathbf{c},\mathbf{d})}) = \lambda_{(\mathbf{c},\mathbf{d})}, \quad \rho_1(\lambda_{(\mathbf{d},\mathbf{c})}) = -\lambda_{(\mathbf{c},\mathbf{d})},\\
\rho_1(\lambda_{(\mathbf{d},\mathbf{a})}) &= \lambda_{(\mathbf{d},\mathbf{a})}, \quad \rho_1(\lambda_{(\mathbf{a},\mathbf{d})}) = -\lambda_{(\mathbf{d},\mathbf{a})}, \quad \rho_1(\lambda_{(\mathbf{a},\mathbf{c})}) = \lambda_{(\mathbf{a},\mathbf{c})} \quad \text{and}\\
\rho_1(\lambda_{(\mathbf{c},\mathbf{a})}) &= -\lambda_{(\mathbf{a},\mathbf{c})}.
\end{aligned}$$

Indeed the homomorphism $\rho_1$ defined on the generators of $\Delta_1(K;R)$ in this fashion maps each of the generators $\lambda_{(\mathbf{a},\mathbf{b})}$, $\lambda_{(\mathbf{b},\mathbf{c})}$, $\lambda_{(\mathbf{c},\mathbf{d})}$, $\lambda_{(\mathbf{d},\mathbf{a})}$ and $\lambda_{(\mathbf{a},\mathbf{c})}$ of $\Delta_1^+(K;R)$ to itself, whilst it maps each of the generators of $\Delta_1^0(K;R)$ to the zero element of $\Delta_1^+(K;R)$.

The homomorphism $\rho_1\colon \Delta_1(K;R) \to \Delta_1^+(K;R)$ is surjective, and it induces an isomorphism $\overline{\rho}_1\colon C_1(K;R) \to \Delta_1^+(K;R)$, where

$$C_1(K;R) = \Delta_1(K;R)/\Delta_1^0(K;R).$$

This induced isomorphism satisfies the following identities:

$$\begin{aligned}
\overline{\rho}_1(\langle\mathbf{a},\mathbf{a}\rangle) &= 0, \quad \overline{\rho}_1(\langle\mathbf{b},\mathbf{b}\rangle) = 0, \quad \overline{\rho}_1(\langle\mathbf{c},\mathbf{c}\rangle) = 0, \quad \overline{\rho}_1(\langle\mathbf{d},\mathbf{d}\rangle) = 0,\\
\overline{\rho}_1(\langle\mathbf{a},\mathbf{b}\rangle) &= \lambda_{(\mathbf{a},\mathbf{b})}, \quad \overline{\rho}_1(\langle\mathbf{b},\mathbf{a}\rangle) = -\lambda_{(\mathbf{a},\mathbf{b})}, \quad \overline{\rho}_1(\langle\mathbf{b},\mathbf{c}\rangle) = \lambda_{(\mathbf{b},\mathbf{c})},\\
\overline{\rho}_1(\langle\mathbf{c},\mathbf{b}\rangle) &= -\lambda_{(\mathbf{b},\mathbf{c})}, \quad \overline{\rho}_1(\langle\mathbf{c},\mathbf{d}\rangle) = \lambda_{(\mathbf{c},\mathbf{d})}, \quad \overline{\rho}_1(\langle\mathbf{d},\mathbf{c}\rangle) = -\lambda_{(\mathbf{c},\mathbf{d})},\\
\overline{\rho}_1(\langle\mathbf{d},\mathbf{a}\rangle) &= \lambda_{(\mathbf{d},\mathbf{a})}, \quad \overline{\rho}_1(\langle\mathbf{a},\mathbf{d}\rangle) = -\lambda_{(\mathbf{d},\mathbf{a})}, \quad \overline{\rho}_1(\langle\mathbf{a},\mathbf{c}\rangle) = \lambda_{(\mathbf{a},\mathbf{c})} \quad \text{and}\\
\overline{\rho}_1(\langle\mathbf{c},\mathbf{a}\rangle) &= -\lambda_{(\mathbf{a},\mathbf{c})}.
\end{aligned}$$

Note that these identities are consistent with the basic requirement that $\langle\mathbf{v},\mathbf{v}\rangle = 0$ and $\langle\mathbf{v},\mathbf{w}\rangle = -\langle\mathbf{w},\mathbf{v}\rangle$ for all vertices $\mathbf{v}$ and $\mathbf{w}$ of $K$.

The quotient homomorphism from $\Delta_1(K;R)$ to $C_1(K;R)$ maps $\lambda_{(\mathbf{v},\mathbf{w})}$ to $\langle\mathbf{v},\mathbf{w}\rangle$ for all vertices $\mathbf{v}$ and $\mathbf{w}$ of $K$. It therefore follows from the above identities characterizing that isomorphism $\overline{\rho}_1$ that this isomorphism $\overline{\rho}_1\colon C_1(K;R) \to \Delta_1^+(K;R)$ is the inverse of the restriction of the quotient homomorphism to the submodule $\Delta_1^+(K;R)$ of $\Delta_1(K;R)$. We note moreover that $C_1(K;R)$ is a free module of rank 5 over the integral domain $R$ which is freely generated by $\langle\mathbf{a},\mathbf{b}\rangle$, $\langle\mathbf{b},\mathbf{c}\rangle$, $\langle\mathbf{c},\mathbf{d}\rangle$, $\langle\mathbf{d},\mathbf{a}\rangle$ and $\langle\mathbf{a},\mathbf{c}\rangle$.

Let us now investigate the $R$-modules $\Delta_2(K;R)$ and $C_2(K;R)$. Now the $R$-module $\Delta_2(K;R)$ is by definition the free $R$-module on the set $W_{2,K}$ whose elements are those ordered triples of vertices of $K$ that span simplices of $K$. The vertices in such an ordered triple may be distinct, in which case they span a triangle of $K$. Otherwise there may be repeated vertices in the ordered triple, in which case the vertices comprising the ordered triple will span a vertex or edge of $K$.

Now there are 27 ordered triples composed of vertices that are vertices of the triangle $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$. There are also 27 ordered triples composed of vertices that are vertices of the triangle $\mathbf{a}\,\mathbf{c}\,\mathbf{d}$. Now there are 8 ordered triples that only involve the vertices $\mathbf{a}$ and $\mathbf{c}$. These 8 ordered triples are included in both the collection of 27 ordered triples involving vertices of the triangle $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$. and the collection of 27 orderet triples involving vertices of the triangle $\mathbf{a}\,\mathbf{c}\,\mathbf{d}$. It follows that the set $W_{2,K}$ consists of 46 ordered triples (since $27+27-8 = 46$), and therefore the $R$-module $\Delta_2(K;R)$ is a free $R$-module of rank 46.

The chain group $C_2(K;R)$ is a free $R$-module of rank 2, which is freely generated by $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ and $\langle \mathbf{a}, \mathbf{c}, \mathbf{d}\rangle$. Moreover the ordered triples $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(\mathbf{a}, \mathbf{c}, \mathbf{d})$ constitute a subset $W_{2,K}^+$ of $W_{2,K}$. The elements $\lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})}$ and $\lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})}$ of $\Delta_2(K;R)$ corresponding to the elements of $W_{2,K}^+$ freely generate a submodule $\Delta_2^+(K;R)$ of $\Delta_2(K;R)$ that is a free $R$-module of rank 2. Indeed

$$\Delta_2^+(K;R) = \{r\lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})} + s\lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})} : r, s \in R\}.$$

There is a homomorphism $\rho_2\colon \Delta_2(K;R) \to \Delta_2^+(K;R)$ defined as described in the discussion preceding this example. This homomorphism has the property that $\rho_2(x) = x$ for all $x \in \Delta^+_(K;R)$. Also $\ker \rho_2 = \Delta_2^0(K;R)$, where $\Delta_2^0(K;R)$ is the kernel of the quotient homomorphism from $\Delta_2(K;R)$ to $C_2(K;R)$. Moreover

$$\begin{array}{rcl}
\rho_2(\lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})}) & = & \rho_2(\lambda_{(\mathbf{b},\mathbf{c},\mathbf{a})}) = \rho_2(\lambda_{(\mathbf{c},\mathbf{a},\mathbf{b})}) = \lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})} \\
\rho_2(\lambda_{(\mathbf{a},\mathbf{c},\mathbf{b})}) & = & \rho_2(\lambda_{(\mathbf{b},\mathbf{a},\mathbf{c})}) = \rho_2(\lambda_{(\mathbf{c},\mathbf{b},\mathbf{a})}) = -\lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})} \\
\rho_2(\lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})}) & = & \rho_2(\lambda_{(\mathbf{c},\mathbf{d},\mathbf{a})}) = \rho_2(\lambda_{(\mathbf{d},\mathbf{a},\mathbf{c})}) = \lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})} \\
\rho_2(\lambda_{(\mathbf{a},\mathbf{d},\mathbf{c})}) & = & \rho_2(\lambda_{(\mathbf{c},\mathbf{a},\mathbf{d})}) = \rho_2(\lambda_{(\mathbf{d},\mathbf{c},\mathbf{a})}) = -\lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})}.
\end{array}$$

In addition $\rho_2(\lambda_{(\mathbf{u},\mathbf{v},\mathbf{w})}) = 0$ whenever $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vertices of $K$ that span a simplex of $K$ but are not all distinct. The homomorphism $\rho_2\colon \Delta_2(K;R) \to \Delta_2^+(K;R)$ induces an isomorphism $\overline{\rho}_2\colon C_2(K;R) \to \Delta_2^+(K;R)$. This isomorphism $\overline{\rho}_2$ satisfies the following identities:

$$\begin{array}{rcl}
\overline{\rho}_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle) & = & \overline{\rho}_2(\langle \mathbf{b}, \mathbf{c}, \mathbf{a}\rangle) = \overline{\rho}_2(\langle \mathbf{c}, \mathbf{a}, \mathbf{b}\rangle) = \lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})} \\
\overline{\rho}_2(\langle \mathbf{a}, \mathbf{c}, \mathbf{b}\rangle) & = & \overline{\rho}_2(\langle \mathbf{b}, \mathbf{a}, \mathbf{c}\rangle) = \overline{\rho}_2(\langle \mathbf{c}, \mathbf{b}, \mathbf{a}\rangle) = -\lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})} \\
\overline{\rho}_2(\langle \mathbf{a}, \mathbf{c}, \mathbf{d}\rangle) & = & \overline{\rho}_2(\langle \mathbf{c}, \mathbf{d}, \mathbf{a}\rangle) = \overline{\rho}_2(\langle \mathbf{d}, \mathbf{a}, \mathbf{c}\rangle) = \lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})} \\
\overline{\rho}_2(\langle \mathbf{a}, \mathbf{d}, \mathbf{c}\rangle) & = & \overline{\rho}_2(\langle \mathbf{c}, \mathbf{a}, \mathbf{d}\rangle) = \overline{\rho}_2(\langle \mathbf{d}, \mathbf{c}, \mathbf{a}\rangle) = -\lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})}.
\end{array}$$

The isomorphism $\overline{\rho}_2\colon C_2(K;R) \to \Delta_2^+(K;R)$ is the inverse of the isomorphism from $\Delta_2^+(K;R)$ to $C_2(K;R)$, sending $\lambda_{(\mathbf{a},\mathbf{b},\mathbf{c})}$ and $\lambda_{(\mathbf{a},\mathbf{c},\mathbf{d})}$ to $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$

and $\langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle$ respectively, that is obtained on restricting to $\Delta_2^+(K;R)$ the quotient homomorphism from $\Delta_2(K;R)$ to $C_2(K;R)$.

The following proposition and its proof summarize the essential ideas employed in the above example and in the discussion preceding it.

**Proposition 4.3** *Let $K$ be a simplicial complex, and let $R$ be an integral domain. Then the chain group $C_q(K;R)$ of $K$ in dimension $q$ with coefficients in the integral domain $R$ is a free module over $R$ whose rank is equal to the number of $q$-simplices of $K$. Moreover let an element $\gamma_\sigma$ of $C_q(K;R)$ be associated with each $q$-simplex $\sigma$ of $K$, where*

$$\gamma_\sigma = \langle \mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots \mathbf{v}_q^\sigma \rangle$$

*for some chosen ordering $\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots \mathbf{v}_q^\sigma$ of the vertices of $\sigma$. Then $C_q(K;R)$ is freely generated by the elements $\gamma_\sigma$ as $\sigma$ ranges over all the $q$-simplices of $\sigma$, and thus, given any element $c$ of $C_q(K;R)$, there exist uniquely-determined elements $r_\sigma$ of the coefficient ring $R$ such that*

$$x = \sum_{\substack{\sigma \in K \\ \dim \sigma = q}} r_\sigma \gamma_\sigma.$$

**Proof** Let $W_{q,K}$ be the set of all $(q+1)$-tuples of vertices of $K$ that span simplices of $K$, and, for each $w \in W_{q,K}$ let $\lambda_w$ be the corresponding generator of the free $R$-module $\Delta_q(K;R)$. This $R$-module $\Delta_q(K;R)$ is then freely generated by $\{\lambda_w : w \in W_{q,K}\}$.

Let

$$W_{q,K}^+ = \{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma) : \sigma \in K \text{ and } \dim \sigma = q\},$$

Then, for each $q$-simplex $\sigma$ of $K$, the vertices $\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots, \mathbf{v}_q^\sigma$ are distinct, and are the vertices of the simplex $\sigma$. Then $W_{q,K}^+$ is a subset of $W_{q,K}$, and, for each $q$-simplex of $K$, the set $W_{q,K}^+$ has exactly one element that is an ordered $(q+1)$-tuple whose components are distinct and are the vertices of the $q$-simplex $\sigma$. Let $\Delta_q^+(K;R)$ be the submodule of $\Delta_q(K;R)$ generated by $\{\lambda_w : w \in W_{q,K}^+\}$. (This submodule $\Delta_q^+(K;R)$ is determined by the chosen orderings of the vertices of the $q$-simplices of $K$, but is dependent on the choice of those orderings.)

Let $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q) \in W_{q,K}$. If $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ are not all distinct let $r(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q)$ be the zero element of $\Delta_q^+(K;R)$. If these vertices are distinct, let $r(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q) = \epsilon_\tau \lambda_{(\mathbf{w}_{\tau(0)}, \mathbf{w}_{\tau(1)}, \ldots, \mathbf{w}_{\tau(q)})}$, where $\tau$ is the unique permutation of the set $\{0, 1, \ldots, q\}$ for which $(\mathbf{w}_{\tau(0)}, \mathbf{w}_{\tau(1)}, \ldots, \mathbf{w}_{\tau(q)}) \in W_{q,K}^+$, and where $\epsilon_\tau$ denotes the parity of the permutation $\tau$. We then obtain a well-defined function $r\colon W_{q,K} \to \Delta_q^+(K;R)$ mapping $W_{q,K}$ into the $R$-module $\Delta_q^+(K;R)$. Moreover if the vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ are all distinct then

$$\begin{aligned} r(\mathbf{w}_{\pi(0)}, \mathbf{w}_{\pi(1)}, \ldots, \mathbf{w}_{\pi(q)}) &= \epsilon_{\tau\circ\pi^{-1}} \lambda_{(\mathbf{w}_{\tau(0)}, \mathbf{w}_{\tau(1)}, \ldots, \mathbf{w}_{\tau(q)})} \\ &= \epsilon_\tau \epsilon_\pi \lambda_{(\mathbf{w}_{\tau(0)}, \mathbf{w}_{\tau(1)}, \ldots, \mathbf{w}_{\tau(q)})} \\ &= \epsilon_\pi r(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q) \end{aligned}$$

for all permutations $\pi$ of $\{0, 1, \ldots, q\}$. Moreover

$$r(\mathbf{w}_{\pi(0)}, \mathbf{w}_{\pi(1)}, \ldots, \mathbf{w}_{\pi(q)}) = \epsilon_\pi r(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q)$$

for all $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q) \in W_{q,K}$ and for all permutations $\pi$ of $\{0, 1, \ldots, q\}$, irrespective of whether or not $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ are distinct vertices of $K$. (If these vertices are not distinct, then both sides of this identity are equal to the zero element of $\Delta_q^+(K;R)$.)

Now $\Delta_q(K;R)$ is freely generated by $\{\lambda_w : w \in W_{q,K}\}$, and therefore the function $r\colon W_{q,K} \to \Delta_q^+(K;R)$ induces an $R$-module homomorphism $\rho_q\colon \Delta_q(K;R) \to \Delta_q^+(K;R)$, where $\rho_q(\lambda_w) = r(w)$ for all $w \in W_{q,K}$ (see Proposition 2.2). Moreover

$$\rho_q(\lambda_{(\mathbf{w}_{\pi(0)}, \mathbf{w}_{\pi(1)}, \ldots, \mathbf{w}_{\pi(q)})}) = \epsilon_\pi \rho_q(\lambda_{(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q)})$$

for all $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q) \in W_{q,K}$ and for all permutations $\pi$ of $\{0, 1, \ldots, q\}$, where $\epsilon_\pi$ denotes the parity of the permutation $\pi$.

Now the kernel $\Delta_q^0(K;R)$ of the quotient homomorphism from $\Delta_q(K;R)$ to $C_q(K;R)$ is generated by elements of $\Delta_q(K;R)$ that are of the form $\lambda_{(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q)}$, where $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ span a simplex of $K$ but are not all distinct, and by elements of $\Delta_q(K;R)$ that are of the form

$$\lambda_{(\mathbf{w}_{\pi(0)}, \mathbf{w}_{\pi(1)}, \ldots, \mathbf{w}_{\pi(q)}))} - \epsilon_\pi \lambda_{(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q)},$$

where $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ are distinct and span a simplex of $K$ and $\pi$ is a permutation of the set $\{0, 1, \ldots, q\}$. All these generators of $\Delta_q^0(K;R)$ belong to the kernel of the homomorphism $\rho_q$, and therefore $\Delta_q^0(K;R) \subset \ker \rho_q$. Moreover the definition of the function $r$ ensures that $\rho_q(x) = x$ for all $x \in \Delta_q^+(K;R)$. The definitions of the function $r$ and the submodule $\Delta_q^0(K;R)$ also ensure that $\lambda_w - r(w) \in \Delta^0(K;R)$ for all $w \in W_{q,K}$, and thus $x - \rho_q(x) \in \Delta_q^0(K;R)$ for all $x \in \Delta_q(K;R)$. It follows that $\ker \rho_q \subset \Delta_q^0(K;R)$, and therefore $\ker \rho_q = \Delta_q^0(K;R)$. Moreover the homomorphism $\rho_q$ is surjective, because $\rho_q(x) = x$ for all $x \in \Delta_q^+(K;R)$. It follows that the homomorphism $\rho_q\colon \Delta_q(K;R) \to \Delta_q^+(K;R)$ induces an $R$-module isomorphism $\overline{\rho}_q\colon C_q(K;R) \to \Delta_q^+(K;R)$ (see Corollary 1.8).

Now

$$\overline{\rho}_q(\gamma_\sigma) = \overline{\rho}_q(\langle \mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots \mathbf{v}_q^\sigma \rangle) = \rho(\lambda_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots \mathbf{v}_q^\sigma)}) = \lambda_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots \mathbf{v}_q^\sigma)}$$

for each $q$-simplex $\sigma$ of $K$. But the free $R$-module $\Delta_q^+(K;R)$ is freely generated by the elements $\lambda_{(\mathbf{v}_0^\sigma, \mathbf{v}_1^\sigma, \ldots \mathbf{v}_q^\sigma)}$ as $\sigma$ ranges over all $q$-simplices of $K$. This free basis of $\Delta_q^+(K;R)$ corresponds under the isomorphism $\overline{\rho}$ to a free basis of $C_q(K;R)$. It follows that $C_q(K;R)$ is a free $R$-module which is freely generated by the elements $\gamma_\sigma$ as $\sigma$ ranges over all $q$-simplices of $K$. The result follows. ∎

**Example** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the vertices of a tetrahedron in some Euclidean space of dimension at least three, and let $K$ be the simplicial complex consisting of this tetrahedron, together with all its triangular faces, edges and vertices. Then the simplicial complex $K$ consists of the following simplices: the four vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$; the six edges $\mathbf{a}\,\mathbf{b}$, $\mathbf{a}\,\mathbf{c}$, $\mathbf{a}\,\mathbf{d}$, $\mathbf{b}\,\mathbf{c}$, $\mathbf{b}\,\mathbf{d}$ and $\mathbf{c}\,\mathbf{d}$; the four triangular faces $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$, $\mathbf{a}\,\mathbf{c}\,\mathbf{d}$, $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$; the tetrahedron $\mathbf{a}\,\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ itself. We shall investigate the structure of the chain groups $C_q(K,\mathbb{Z})$ with integer coefficients. (Analogous results apply to the chain groups $C_q(K,R)$ with coefficients in any integral domain $R$.)

Let us first consider the chain group $C_3(K;\mathbb{Z})$ in dimension three. It follows from Proposition 4.3 that this is a free Abelian group of rank 1, isomorphic to the group $\mathbb{Z}$ of integers itself. It is freely generated by the 3-chain $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle$. (We recall that, for each non-negative integer $q$, a *$q$-chain* of $K$ with coefficients in an integral domain $R$ is by definition an element of the group $C_q(K;R)$.) It follows that

$$C_3(K,\mathbb{Z}) = \{n\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle : n \in \mathbb{Z}\}.$$

We note also that, for each $n \in \mathbb{Z}$,

$$n\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle = -n\langle \mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle = n\langle \mathbf{b}, \mathbf{a}, \mathbf{d}, \mathbf{c} \rangle = \quad \text{etc.}$$

It also follows from Proposition 4.3 that the chain group $C_2(K;\mathbb{Z})$ in dimension two is a free Abelian group of rank 4, isomorphic to $\mathbb{Z}^4$, and moreover this group is freely generated by the elements

$$\langle \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle, \ \langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle, \ \langle \mathbf{a}, \mathbf{b}, \mathbf{d} \rangle \text{ and } \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle.$$

It follows that, given any 2-chain $c$ with integer coefficients, there exist uniquely determined integers $m_0$, $m_1$, $m_2$ and $m_3$ such that

$$c = m_0\langle \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle + m_1\langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle + m_2\langle \mathbf{a}, \mathbf{b}, \mathbf{d} \rangle + m_3\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle.$$

The group $C_2(K;\mathbb{Z})$ is also freely generated by the elements

$$\langle \mathbf{b}, \mathbf{c}, \mathbf{d}\rangle,\ \ \langle \mathbf{c}, \mathbf{a}, \mathbf{d}\rangle,\ \ \langle \mathbf{a}, \mathbf{b}, \mathbf{d}\rangle \text{ and } \langle \mathbf{a}, \mathbf{c}, \mathbf{b}\rangle,$$

for example, and

$$\begin{aligned} & m_0\langle \mathbf{b}, \mathbf{c}, \mathbf{d}\rangle + m_1\langle \mathbf{a}, \mathbf{c}, \mathbf{d}\rangle + m_2\langle \mathbf{a}, \mathbf{b}, \mathbf{d}\rangle + m_3\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle \\ & \quad = \ m_0\langle \mathbf{b}, \mathbf{c}, \mathbf{d}\rangle - m_1\langle \mathbf{c}, \mathbf{a}, \mathbf{d}\rangle + m_2\langle \mathbf{a}, \mathbf{b}, \mathbf{d}\rangle - m_3\langle \mathbf{a}, \mathbf{c}, \mathbf{b}\rangle \end{aligned}$$

for all integers $m_0$, $m_1$, $m_2$ and $m_3$.

The chain group $C_1(K;\mathbb{Z})$ in dimension one is a free Abelian of rank 6, isomorphic to $\mathbb{Z}^6$. It is freely generated by $\langle \mathbf{a}, \mathbf{b}\rangle$, $\langle \mathbf{a}, \mathbf{c}\rangle$, $\langle \mathbf{a}, \mathbf{d}\rangle$, $\langle \mathbf{b}, \mathbf{c}\rangle$, $\langle \mathbf{b}, \mathbf{d}\rangle$ and $\langle \mathbf{c}, \mathbf{d}\rangle$. Thus, given any 1-chain $e$ of $K$ with integer coefficients, there exist unique integers $k_{\mathbf{a},\mathbf{b}}$, $k_{\mathbf{a},\mathbf{c}}$, $k_{\mathbf{a},\mathbf{d}}$, $k_{\mathbf{b},\mathbf{c}}$, $k_{\mathbf{b},\mathbf{d}}$ and $k_{\mathbf{c},\mathbf{d}}$ such that

$$e = k_{\mathbf{a},\mathbf{b}}\langle \mathbf{a}, \mathbf{b}\rangle + k_{\mathbf{a},\mathbf{c}}\langle \mathbf{a}, \mathbf{c}\rangle + k_{\mathbf{a},\mathbf{d}}\langle \mathbf{a}, \mathbf{d}\rangle + k_{\mathbf{b},\mathbf{c}}\langle \mathbf{b}, \mathbf{c}\rangle + k_{\mathbf{b},\mathbf{d}}\langle \mathbf{b}, \mathbf{d}\rangle + k_{\mathbf{c},\mathbf{d}}\langle \mathbf{c}, \mathbf{d}\rangle.$$

Moreover $k_{\mathbf{a},\mathbf{b}}\langle \mathbf{a}, \mathbf{b}\rangle = -k_{\mathbf{a},\mathbf{b}}\langle \mathbf{b}, \mathbf{a}\rangle$ etc.

Finally we note that the chain group $C_0(K,\mathbb{Z})$ in dimension zero is a free Abelian group of rank 4, isomorphic to $\mathbb{Z}^4$. This group is freely generated by $\langle \mathbf{a}\rangle$, $\langle \mathbf{b}\rangle$, $\langle \mathbf{c}\rangle$ and $\langle \mathbf{d}\rangle$.

## 4.3 Homomorphisms defined on Chain Groups

**Lemma 4.4** *Let $K$ be a simplicial complex, and let $M$ be a module over a unital commutative ring $R$, let $W_{q,K}$ denote the set of $(q+1)$-tuples over vertices of $K$ that span simplices of $K$, and let $f\colon W_{q,K} \to M$ be a function from $W_{q,K}$ to $M$. Suppose that the function $f$ has the following properties:*

- $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ *unless* $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ *are all distinct,*
- $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ *changes sign on interchanging any two adjacent vertices* $\mathbf{v}_{j-1}$ *and* $\mathbf{v}_j$*.*

*Then there exists a unique $R$-module homomorphism $\varphi\colon C_q(K;R) \to M$ characterized by the property that*

$$\varphi(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) = f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

*whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$.*

**Proof** Let the elements of the free $R$-module $\Delta_q(K;R)$ be represented in the standard fashion as functions from the finite set $W_{q,K}$ to the coefficient ring $R$, where $W_{q,K}$ is the set of all $(q+1)$-tuples of vertices of $K$ whose components span some simplex of $K$. Then the free $R$-module $\Delta_q(K;R)$ is freely generated by the set

$$\{\lambda_w : w \in W_{q,K}\},$$

where, for each $w \in W_{q,K}$, the element $\lambda_w$ of $\Delta_q(K;R)$ is represented by the function from $W_{q,K}$ to $R$ that sends $w$ to the multiplicative identity element $1_R$ of $R$ and sends all other elements of $W_{q,K}$ to the zero element $0_R$ of $R$ (see Proposition 2.6). It follows that the $f: W_{q,K} \to M$ determines a homomorphism $\tilde{f}: \Delta_q(K;R) \to M$ characterized by the property that $\tilde{f}(\lambda_w) = f(w)$ for all $w \in W$ (see Proposition 2.2). Moreover

$$\tilde{f}\left(\sum_{w \in W_{q,K}} r_w \lambda_w\right) = \sum_{w \in W_{q,k}} r_w f(w)$$

for all collections $(r_w : w \in W_{q,K})$ of elements of the ring $R$ indexed by the finite set $W_{q,K}$. Now the conditions imposed on the function $f$ ensure that

$$\lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)} \in \ker \tilde{f}$$

unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. Also

$$\lambda_{(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)})} - \varepsilon_\pi \lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)} \in \ker \tilde{f}$$

for all permutations $\pi$ of $\{0, 1, \ldots, q\}$, since the permutation $\pi$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ with $j$ for some $j$ and leave the rest of the set fixed, and the parity $\varepsilon_\pi$ of $\pi$ is given by $\varepsilon_\pi = +1$ when the number of such transpositions is even, and by $\varepsilon_\pi = -1$ when the number of such transpositions is odd. Thus all the generators of $\Delta_q^0(K;R)$ specified in the definition of this submodule are contained in $\ker \tilde{f}$, and therefore $\Delta_q^0(K;R) \subset \ker \tilde{f}$. It follows that $\tilde{f}: \Delta_q(K;R) \to M$ induces a well-defined $R$-module homomorphism $\varphi: C_q(K;R) \to M$, where

$$\varphi\left(\Delta_q^0(K;R) + \sum_{w \in W_{q,K}} r_w \lambda_w\right) = \tilde{f}\left(\sum_{w \in W_{q,K}} r_w \lambda_w\right) = \sum_{w \in W_{q,K}} r_w f(w)$$

for all collections $(r_w : w \in W_{q,K})$ of elements of the ring $R$ indexed by the finite set $W_{q,K}$. Then

$$\varphi(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \tilde{f}(\Delta_q^0(K;R) + \lambda_{(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)}) = f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

whenever the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of $K$ span a simplex of $K$. This homomorphism $\varphi: C_q(K;R) \to M$ is uniquely determined by the function $f: W_{q,K} \to M$, as required. ∎

## 4.4 Orientations on Simplices

Let $V$ be a finite-dimensional real vector space. Then each ordered basis of $V$ determines one of two possible orientations on this vector space. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ and $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_q$ be two ordered bases of a vector space $V$ of dimension $q$. Then there exists a non-singular $q \times q$ matrix $(A_k^j)$ such that $\mathbf{f}_k = \sum\limits_{j=1}^{q} A_k^j \mathbf{e}_j$ for $k = 1, 2, \ldots, q$. If this matrix $(A_k^j)$ has positive determinant then the two bases determine the same orientation on the vector space $V$. On the other hand, if the matrix $(A_k^j)$ has negative determinant then the two bases determine the opposite orientation on the vector space $V$. In particular if any two elements of an ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of the vector space $V$ are interchanged with one another, then this reverses the orientation of the vector space.

Let $\pi$ be a permutation of the set $\{0, 1, \ldots, q\}$, and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ be an ordered basis of the vector space $V$, determining a particular orientation of this vector space. If the permutation $\pi$ is even then the basis $\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(q)}$ of $V$ obtained on reordering the elements of the given basis by means of the permutation $\pi$ determines the same orientation on the vector space $V$ as the original basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$. On the other hand, if the permutation $\pi$ is odd then the basis $\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(q)}$ determines the opposite orientation on $V$ to that determined by the original basis.

Let $\sigma$ be a $q$-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$, and let $V$ be the unique $q$-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between any two points of $\sigma$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ an ordered list of the vertices of $\sigma$. Then these vertices are geometrically independent and determine an ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of the vector space $V$, where $\mathbf{e}_j = \mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$. This ordered basis then determines an orientation on the vector space $V$. We see therefore that each ordering of the vertices of the $q$-simplex $\sigma$ determines a corresponding orientation on the $q$-dimensional vector space $V$ determined by the $q$-simplex $\sigma$.

**Proposition 4.5** *Let $\sigma$ be a $q$-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$, and let $V$ be the unique $q$-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between any two points of $\sigma$ (so that $V$ is parallel to the tangent space to $\sigma$ at each point in the interior of $\sigma$). Given any ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the vertices of $\sigma$, let the corresponding orientation on the vector space $V$ be the orientation determined by the ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of $V$, where $\mathbf{e}_j = \mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$. Then any even permutation of the order of the vertices in the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$*

*preserves the orientation on the vector space $V$, whereas any odd permutation of the order of these vertices reverses the orientation on $V$.*

**Proof** Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the ordered list of vertices determining the orientation on the vector space $V$. If the vertex $\mathbf{v}_j$ is transposed with $\mathbf{v}_k$, where $j > 0$ and $k > 0$, then the corresponding basis elements $\mathbf{e}_j$ and $\mathbf{e}_k$ in the ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ of $V$ are also transposed, and this reverses the orientation on $V$ determined by that ordered basis.

If the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ are interchanged, then this has the effect of replacing the ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ corresponding to the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ by the ordered basis $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_q$, where

$$\mathbf{f}_1 = \mathbf{v}_0 - \mathbf{v}_1 = -\mathbf{e}_1$$

and

$$\mathbf{f}_j = \mathbf{v}_j - \mathbf{v}_1 = \mathbf{e}_j - \mathbf{e}_1 \quad \text{for } j = 2, 3, \ldots, q.$$

The non-singular $q \times q$ matrix that implements this change of basis is the upper triangular matrix $A$, where

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The determinant of an upper triangular matrix is the product of the matrix elements along the leading diagonal, and therefore $\det A = -1$. It follows that transposing the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ occurring in the first two positions in the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of vertices of $\sigma$ reverses the orientation on the vector space $V$ determined by the ordering of the vertices of $\sigma$.

It now follows from standard properties of permutations of finite sets that interchanging any two of the vertices in any ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the vertices of the $q$-simplex $\sigma$ reverses the orientation on the $q$-dimensional real vector space $V$ that is determined by the ordering of these vertices. Indeed if the positions in the list are numbered from 0 to $q$ then the vertex in position 0 can be transposed with the vertex in position $q$ by first transposing the vertices in positions 1 and $q$, then transposing the vertices in positions 0 and 1, and then again transposing the vertices in positions 1 and $q$. This involves three transpositions of vertices in the list, and each of these transpositions reverses the orientation on the vector space $V$. It follows that any even permutation of the ordering of the vertices in the ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ preserves the corresponding orientation on the vector space $V$, whereas any odd permutation of the ordering of these vertices reverses the orientation on this vector space, as required. ∎

We can regard the orientation on the vector space $V$ as an orientation of the simplex $\sigma$ itself. Indeed this orientation may be viewed as an orientation on the $q$-dimensional tangent space to the simplex $\sigma$ at any interior point of $\sigma$. In this fashion any ordering of the vertices of a simplex $\sigma$ determines a corresponding orientation on that simplex. If the ordering of the vertices is permuted by means of an even permutation then the orientation of the simplex is preserved. But if the ordering of the vertices is permuted by means of an odd permutation then the orientation of the simplex is reversed.

**Example** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the vertices of a triangle in a Euclidean space $\mathbb{R}^k$ of dimension at least two. Then this triangle determines a 2-dimensional vector subspace $V$ of $\mathbb{R}^k$. This 2-dimensional subspace $V$ is spanned by the displacement vectors $\mathbf{v} - \mathbf{u}$ and $\mathbf{w} - \mathbf{u}$, and is parallel to the tangent plane to the triangle at any interior point of the triangle.

Now it follows from Proposition 4.5 that the orientation of the triangle should be preserved under cyclic permutations of its vertices. Now the ordering $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of these vertices determines an ordered basis $\mathbf{b}_1, \mathbf{b}_2$ of the vector space $V$, where $\mathbf{b}_1 = \mathbf{v} - \mathbf{u}$ and $\mathbf{b}_2 = \mathbf{w} - \mathbf{u}$. The ordering $\mathbf{v}, \mathbf{w}, \mathbf{u}$ of the vertices of the triangle corresponds to the orientation on the vector space $V$ determined by the ordered basis $\mathbf{w} - \mathbf{v}$, $\mathbf{u} - \mathbf{v}$. Now $\mathbf{w} - \mathbf{v} = \mathbf{b}_2 - \mathbf{b}_1$ and $\mathbf{u} - \mathbf{v} = -\mathbf{b}_1$. Moreover the $2 \times 2$ matrix implementing the change of basis from the ordered basis $\mathbf{b}_1, \mathbf{b}_2$ to the ordered basis $\mathbf{b}_2 - \mathbf{b}_1, -\mathbf{b}_1$ is the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

and this matrix has determinant 1. Similarly the ordering $\mathbf{w}, \mathbf{u}, \mathbf{v}$ of the vertices of the triangle determines a corresponding ordered basis $\mathbf{u}-\mathbf{w}$, $\mathbf{v}-\mathbf{w}$ of the vector space $V$. Moreover $\mathbf{u} - \mathbf{w} = -\mathbf{b}_2$ and $\mathbf{v} - \mathbf{w} = \mathbf{b}_1 - \mathbf{b}_2$, and the $2 \times 2$ matrix implementing the change of basis from the ordered basis $\mathbf{b}_1, \mathbf{b}_2$ to the ordered basis $-\mathbf{b}_2$, $\mathbf{b}_1 - \mathbf{b}_2$ is the $2 \times 2$ matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

and this matrix also has determinant 1. It follows that an even permutation of the ordering of the vertices of the triangle (resulting from a cyclic permutation of those vertices) preserves the orientation on the vector space $V$ determined by the ordering of the vertices.

On the other hand the $2\times 2$ matrices that implement the change of ordered basis of the vector space $V$ resulting from odd permutations of the order of the vertices $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

and these three matrices all have determinant $-1$. It follows that any odd permutation of the vertices (resulting from a transposition of two of those vertices that fixes the remaining vertex) results in a reversal of the orientation on the vector space $V$.

Thus even permutations of the ordering of the vertices of the triangle preserve the orientation of the triangle determined by the ordering of its vertices, whereas odd permutations of the ordering reverse the orientation determined by the ordering.

Let $K$ be a simplicial complex, and let $\sigma$ be a $q$-simplex of $K$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\sigma$, with the chosen ordering of its vertices, determines a corresponding element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_q(K; R)$. This element is in fact determined by the orientation on the simplex $\sigma$. If the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the simplex are reordered by means of an even permutation of the vertices in the list then both the orientation on the simplex determined by the ordering of its vertices remains unchanged and the corresponding element of $C_q(K; R)$ determined by the ordered list of the vertices of the simplex also remains unchanged. On the other hand, if the vertices are reordered through an odd permutation of the vertices in the list then both the orientation of the simplex determined by the ordering of its vertices is reversed, and the corresponding element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of $C_q(K; R)$ determined by the ordered list of those vertices is replaced by the element $-\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$.

It follows from Proposition 4.3 that if we choose an orientation on each $q$-simplex of the simplicial complex $K$, then the $q$-simplices of $K$, together with the chosen orientations, determine corresponding generators of $C_q(K; R)$ that constitute a free basis of this $R$-module.

Now $r \neq -r$ for all non-zero elements $r$ of the integral domain $R$, provided that $\operatorname{char} R \neq 2$, where $\operatorname{char} R$ denotes the characteristic of $R$ (see Lemma 1.2 and Lemma 1.3). It then follows from a direct application of Proposition 4.3 that

$$\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \neq -\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle,$$

provided that $\operatorname{char} R \neq 2$. Thus, provided that charR $\neq 2$, the element $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_q(K; R)$ determined by an ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of the vertices of a simplex represents one of two possible orientations on that simplex. The negative of this element of $C_q(K; R)$ represents the other orientation on the simplex.

**Example** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the vertices of a triangle in a Euclidean space $\mathbb{R}^k$ of dimension at least two, and let $K$ be the simplicial complex consisting of this triangle $\mathbf{u}\,\mathbf{v}\,\mathbf{w}$ together with its edges and vertices. Then the element $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ of the chain group $C_2(K, \mathbb{Z})$ represents the triangle $\mathbf{u}\,\mathbf{v}\,\mathbf{w}$, provided with the orientation determined by the ordered list $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of the vertices of this triangle. Moreover this orientation on the triangle corresponds to the orientation on the vector space $V$ determined by the ordered basis $\mathbf{v} - \mathbf{u},\ \mathbf{w} - \mathbf{u}$ of $V$. The algebraic properties of the chain group $C_2(K; \mathbb{Z})$ then ensure that

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle &= \langle \mathbf{v}, \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{u}, \mathbf{v} \rangle = -\langle \mathbf{u}, \mathbf{w}, \mathbf{v} \rangle \\ &= -\langle \mathbf{v}, \mathbf{u}, \mathbf{w} \rangle = -\langle \mathbf{w}, \mathbf{v}, \mathbf{u} \rangle \end{aligned}$$

Thus a cyclic permutation of the vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ preserves both the orientation of the triangle determined by the ordering of its vertices and also the value of the corresponding 2-chain $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ in the chain group $C_2(K; \mathbb{Z})$. On the other hand, if two of the vertices are transposed, whilst the remaining vertex remains unchanged, then this reverses the orientation on the triangle determined by the ordering of its vertices and also changes the sign of the 2-chain $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ in $C_2(K; \mathbb{Z})$. We see therefore that the orientation on the triangle determined by each ordering of the vertices of the triangle is correctly encoded in the corresponding element of the chain group $C_2(K; \mathbb{Z})$ determined by the ordered list of vertices of the triangle.

A 0-simplex is a single vertex. Let $\mathbf{v}$ be a vertex of a simplicial complex $K$. Then $\mathbf{v}$ is considered to admit a positive orientation, represented by the element $\langle \mathbf{v} \rangle$ of $C_0(K; \mathbb{Z})$, and a negative orientation, represented by the element $-\langle \mathbf{v} \rangle$ of $C_0(K; \mathbb{Z})$.

## 4.5 Boundary Homomorphisms

Let $K$ be a simplicial complex, and let $R$ be an integral domain. We introduce below *boundary homomorphisms* $\partial_q \colon C_q(K; R) \to C_{q-1}(K; R)$ between the chain groups of $K$ with coefficients in $R$.

In order to define and investigate the properties of this boundary homomorphism, we introduce a notation that is frequently used to indicate that some particular vertex is to be omitted from a ordered list of vertices of a simplex. Let $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ be the element of the chain group $C_q(K; R)$ determined by some ordered list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of vertices of $K$ that span a simplex of $K$. We denote by $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ the element

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q \rangle$$

of $C_{q-1}(K; R)$ obtained on omitting the vertex $\mathbf{v}_j$ from the list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of vertices of $K$. Thus

$$\begin{aligned}
\langle \hat{\mathbf{v}}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle &= \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle, \\
\langle \mathbf{v}_0, \hat{\mathbf{v}}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle &= \langle \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle, \\
\langle \mathbf{v}_0, \mathbf{v}_1, \hat{\mathbf{v}}_2, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle &= \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_q \rangle, \\
&\vdots \\
\langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \hat{\mathbf{v}}_q \rangle &= \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{q-1} \rangle.
\end{aligned}$$

We may employ analogous notation when omitting two or more vertices from an ordered list of vertices. Thus if $j$ and $k$ are integers between 0 and $q$, where $j < k$, we denote by

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots \mathbf{v}_q \rangle$$

the element $\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_q \rangle$ of $C_{q-2}(K; R)$ determined by the ordered list of vertices that results on omitting both vertices $\mathbf{v}_j$ and $\mathbf{v}_k$ from the list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.

If the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct then they are the vertices of a $q$-simplex $\sigma$ of $K$, and this simplex is represented by the corresponding generators $\pm\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_q(K; R)$. Moreover if $\operatorname{char} R \neq 2$ then there are exactly two such generators in $C_q(K; R)$ corresponding to the simplex $\sigma$, and these two generators represent the two possible orientations on the simplex. The elements $\pm\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ of the chain group $C_{q-1}(K; R)$ obtained by omitting the vertex $\mathbf{v}_j$ from the list of vertices then represent the unique $(q-1)$-dimensional face of the simplex $\sigma$ that does not contain the vertex $\mathbf{v}_j$.

**Proposition 4.6** *Let $K$ be a simplicial complex, and let $R$ be an integral domain. Then there exist well-defined homomorphisms*

$$\partial_q \colon C_q(K; R) \to C_{q-1}(K; R)$$

*for all integers $q$ characterized by the requirement that*

$$\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

*whenever the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of $K$ span a simplex of $K$.*

**Proof** If $q \leq 0$, or if $q > \dim K$, then at least one of the $R$-modules $C_q(K; R)$ and $C_{q-1}(K; R)$ is the zero module: in those case we define $\partial_q : C_q(K; R) \to C_{q-1}(K; R)$ to be the zero homomorphism.

Suppose then that $0 < q \leq \dim K$. We prove the existence of the required homomorphism $\partial_q$ by means of Lemma 4.4.

Given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ spanning a simplex of $K$, let

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Let $i$ be an integer between 1 and $q$. If $0 \leq j < i - 1$ then

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \ldots, \mathbf{v}_q \rangle$$

changes sign (i.e., it is replaced by the negative of itself) when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed. Similarly if $i < j \leq q$ then

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

changes sign when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed. Also

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_{i-1}, \ldots, \mathbf{v}_q \rangle \quad \text{and} \quad \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle$$

are transposed when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed. It follows that the $(q-1)$-chain $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign when the vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are transposed for some integer $i$ satisfying $1 \leq i \leq q$.

Next suppose that $\mathbf{v}_i = \mathbf{v}_k$ for some $i$ and $k$ satisfying $i < k$. Then

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = (-1)^i \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle + (-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle,$$

since the remaining terms in the expression defining $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ contain both $\mathbf{v}_i$ and $\mathbf{v}_k$ and are therefore equal to the zero element of $C_{q-1}(K; R)$ when $\mathbf{v}_i = \mathbf{v}_k$. Also

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{k-i-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle.$$

Indeed this identity is immediate when $k = i+1$. Suppose that $k > i+1$. Let $\mathbf{w} = \mathbf{v}_i = \mathbf{v}_k$. Then the vertex $\mathbf{w}$ occurs in the ordered list $\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q$ before $\mathbf{v}_{i+1}$ but is omitted after $\mathbf{v}_{k-1}$, whereas the vertex $\mathbf{w}$ occurs in the ordered list $\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q$ after $\mathbf{v}_{k-1}$ but is omitted before $\mathbf{v}_{i+1}$. Thus, in order to convert the first ordered list to the second by successively transposing vertices, it suffices to transpose the vertex $\mathbf{w}$ occurring before $\mathbf{v}_{i+1}$ in the first list successively with the vertices $\mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_{k-1}$, shuffling it along the list until it occurs after $\mathbf{v}_{k-1}$. This process requires $k - i - 1$ successive transpositions and is thus results in a permutation of the vertices in the list which is of parity $(-1)^{k-i-1}$. It follows that

$$(-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{i-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle$$

and thus

$$f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$$

whenever $\mathbf{v}_i = \mathbf{v}_k$, where $0 \leq i < k \leq q$. We conclude therefore that $f(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ unless the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct.

It now follows directly from Lemma 4.4 that there is a well-defined homomorphism $\partial_q \colon C_q(K; R) \to C_{q-1}(K; R)$, characterized by the property that

$$\partial_q \left( \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$. ∎

Let $K$ be a simplicial complex, and let $R$ be an integral domain. The $R$-module homomorphism $\partial_q \colon C_q(K; R) \to C_{q-1}(K; R)$ between the chain groups of $K$ in dimensions $q$ and $q-1$ is referred to as the *boundary homomorphism* between these chain groups.

**Example** Let $K$ be a simplicial complex consisting of a triangle with vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, together with all the vertices and edges of this triangle, and let $R$ be an integral domain. Then

$$C_2(K; R) = \{ r \, \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle : r \in R \}.$$

Now

$$\partial_2 \Big( r \, \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle \Big) = r \, \partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle),$$

because $\partial_2 \colon C_3(K; R) \to C_2(K; R)$ is a homomorphism of $R$-modules. It follows that this boundary homomorphism is determined by the value of $\partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle)$. Moreover

$$\partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle) = \langle \mathbf{b}, \mathbf{c} \rangle - \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle,$$

and

$$\begin{aligned}
\partial_1(\langle \mathbf{b}, \mathbf{c}, \rangle) &= \langle \mathbf{c} \rangle - \langle \mathbf{b} \rangle, \\
\partial_1(\langle \mathbf{a}, \mathbf{c}, \rangle) &= \langle \mathbf{c} \rangle - \langle \mathbf{a} \rangle, \\
\partial_1(\langle \mathbf{a}, \mathbf{b}, \rangle) &= \langle \mathbf{b} \rangle - \langle \mathbf{a} \rangle.
\end{aligned}$$

Therefore

$$\partial_1\Bigl(\partial_2\Bigl(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle\Bigr)\Bigr) = \langle \mathbf{c} \rangle - \langle \mathbf{b} \rangle - \langle \mathbf{c} \rangle + \langle \mathbf{a} \rangle + \langle \mathbf{b} \rangle - \langle \mathbf{a} \rangle = 0.$$

It follows that $\partial_1(\partial_2(x)) = 0$ for all $x \in C_2(K; R)$.

**Example** Let $K$ be a simplicial complex consisting of a tetrahedron with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, together with all the vertices, edges and triangular faces of this tetrahedron, and let $R$ be an integral domain. Then

$$C_3(K; R) = \{r \langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle : r \in R\}.$$

Now

$$\partial_3\Bigl(r \langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle\Bigr) = r\, \partial_3(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle),$$

because $\partial_3 \colon C_3(K; R) \to C_2(K; R)$ is a homomorphism of $R$-modules. It follows that this boundary homomorphism is determined by the value of $\partial_3(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle)$. Moreover

$$\partial_3\Bigl(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle\Bigr) = \langle \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle + \langle \mathbf{a}, \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle,$$

and

$$\begin{aligned}
\partial_2(\langle \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle) &= \langle \mathbf{c}, \mathbf{d} \rangle - \langle \mathbf{b}, \mathbf{d} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle, \\
\partial_2(\langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle) &= \langle \mathbf{c}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{d} \rangle + \langle \mathbf{a}, \mathbf{c} \rangle, \\
\partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{d} \rangle) &= \langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{d} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle, \\
\partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle) &= \langle \mathbf{b}, \mathbf{c} \rangle - \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\partial_2\Bigl(\partial_3\Bigl(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle\Bigr)\Bigr) &= \partial_2(\langle \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle) - \partial_2(\langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle) + \partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{d} \rangle) \\
&\qquad - \partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle) \\
&= \langle \mathbf{c}, \mathbf{d} \rangle - \langle \mathbf{b}, \mathbf{d} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle \\
&\qquad - \langle \mathbf{c}, \mathbf{d} \rangle + \langle \mathbf{a}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{c} \rangle \\
&\qquad + \langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{d} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle \\
&\qquad - \langle \mathbf{b}, \mathbf{c} \rangle + \langle \mathbf{a}, \mathbf{c} \rangle - \langle \mathbf{a}, \mathbf{b} \rangle \\
&= 0.
\end{aligned}$$

It follows that $\partial_2(\partial_3(x)) = 0$ for all $x \in C_3(K;R)$. Also the boundary homomorphism $\partial_2\colon C_2(K;R) \to C_1(K;R)$ is determined by the values of

$$\partial_2(\langle \mathbf{b}, \mathbf{c}, \mathbf{d}\rangle), \quad \partial_2(\langle \mathbf{a}, \mathbf{c}, \mathbf{d}\rangle), \quad \partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{d}\rangle) \quad \text{and} \quad \partial_2(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle).$$

It follows from the calculation in the preceding example that $\partial_1(\partial_2(x)) = 0$ for all $x \in C_2(K;R)$.

**Lemma 4.7** *Let $K$ be a simplicial complex, let $R$ be an integral domain, and, for each integer $q$, let $\partial_q\colon C_q(K;R) \to C_{q-1}(K;R)$ be the boundary homomorphism between the chain groups $C_q(K;R)$ and $C_{q-1}(K;R)$. Then $\partial_{q-1}\circ\partial_q = 0$ for all integers $q$.*

**Proof** The result is trivial if $q < 2$, since in this case $\partial_{q-1} = 0$. Suppose that $q \geq 2$. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of $K$. Then

$$\begin{aligned}
\partial_{q-1}\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1}\left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle\right) \\
&= \sum_{j=1}^{q}\sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\
&\quad + \sum_{j=0}^{q-1}\sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q\rangle \\
&= 0
\end{aligned}$$

(since each term in this summation over $j$ and $k$ cancels with the corresponding term with $j$ and $k$ interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1}\circ\partial_q$ is determined by its values on the elements of any free basis of $C_q(K;R)$. ∎

## 4.6 The Homology Groups of a Simplicial Complex

Let $K$ be a simplicial complex, and let $R$ be an integral domain, and, for each non-negative integer $q$, let $C_q(K;R)$ denote the $R$-module whose elements are $q$-chains of $K$ with coefficients in the coefficient ring $R$. A $q$-chain $z$ is said to be a *$q$-cycle* if $\partial_q z = 0$. A $q$-chain $b$ is said to be a *$q$-boundary* if $b = \partial_{q+1}c'$ for some $(q+1)$-chain $c'$. The $R$-module consisting of the $q$-cycles of $K$ with coefficients in the integral domain $R$ is denoted by $Z_q(K;R)$, and the $R$-module consisising of the $q$-boundaries of $K$ with coefficients in $R$ is denoted by $B_q(K;R)$. Thus $Z_q(K;R)$ is the kernel of the boundary homomorphism $\partial_q\colon C_q(K;R) \to C_{q-1}(K;R)$, and $B_q(K;R)$ is the image of the boundary homomorphism $\partial_{q+1}: C_{q+1}(K; R) \to C_q(K; R)$. However $\partial_q \circ \partial_{q+1} = 0$ (see Lemma 4.7). It follows that $B_q(K; R) \subset Z_q(K; R)$. But these $R$-modules are submodules of the $R$-module $C_q(K; R)$. We can therefore form the quotient module $H_q(K; R)$, where $H_q(K; R) = Z_q(K; R)/B_q(K; R)$. The $R$-module $H_q(K; R)$ is referred to as the *qth homology group* of the simplicial complex $K$ with coefficients in the integral domain $R$. Note that $H_q(K; R) = 0$ if $q < 0$ or $q > \dim K$ (since $Z_q(K; R) = 0$ and $B_q(K; R) = 0$ in these cases).

The element $[z] \in H_q(K; R)$ of the homology group $H_q(K; R)$ determined by an element $z$ of $Z_q(K; R)$ is referred to as the *homology class* of the $q$-cycle $z$. Note that $[z_1 + z_2] = [z_1] + [z_2]$ for all $z_1, z_2 \in Z_q(K; R)$, and $[z_1] = [z_2]$ if and only if $z_1 - z_2 = \partial_{q+1} c$ for some $(q + 1)$-chain $c$ with coefficients in the coefficient ring $R$.

An important special case of the above definitions is that in which the coefficient ring $R$ is the ring $\mathbb{Z}$ of integers. The resultant Abelian groups $C_q(K; \mathbb{Z})$, $Z_q(K; \mathbb{Z})$, $B_q(K; \mathbb{Z})$ and $H_q(K; \mathbb{Z})$ defined as described above are often denoted simply by $C_q(K)$, $Z_q(K)$, $B_q(K)$ and $H_q(K)$ respectively. Thus if a group of $q$-dimensional chains, cycles, boundaries or homology classes is specified, but the ring of coefficients is not specified, then the coefficient ring is by default taken to be the ring of integers.

**Remark** It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex. This fact is far from obvious, and a lot of basic theory must be developed in order to establish the tools to prove this result.

**Proposition 4.8** *Let $K$ be a simplicial complex, and let $R$ be an integral domain. Suppose that there exists a vertex $\mathbf{w}$ of $K$ with the following property:*

- *if vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$ then so do $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.*

*Then $H_0(K; R) \cong R$, and $H_q(K; R)$ is the zero module for all $q > 0$.*

**Proof** Using Lemma 4.4, we see that there is a well-defined $R$-module homomorphism $D_q: C_q(K; R) \to C_{q+1}(K; R)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$. Now $\partial_1(D_0(\langle \mathbf{v} \rangle)) = \langle \mathbf{v} \rangle - \langle \mathbf{w} \rangle$ for all vertices $\mathbf{v}$ of $K$. It follows that

$$\sum_{k=1}^{s} r_k \langle \mathbf{v}_k \rangle - \left( \sum_{k=1}^{s} r_k \right) \langle \mathbf{w} \rangle = \sum_{k=1}^{s} r_k (\langle \mathbf{v}_k \rangle - \langle \mathbf{w} \rangle) \in B_0(K; R)$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$. It follows that

$$z - \varepsilon(z)\langle w\rangle \in B_0(K;R)$$

for all $z \in C_0(K;R)$, where $\varepsilon\colon C_0(K;R) \to R$ is the $R$-module homomorphism from $C_0(K;R)$ to $R$ defined such that

$$\varepsilon\left(\sum_{k=1}^{s} r_k\langle\mathbf{v}_k\rangle\right) = \sum_{k=1}^{s} r_k$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$. It follows that $\ker\varepsilon \subset B_0(K;R)$. But

$$\varepsilon(\partial_1(\langle\mathbf{u},\mathbf{v}\rangle)) = \varepsilon(\langle\mathbf{v}\rangle - \langle\mathbf{u}\rangle) = 0$$

for all edges $\mathbf{u}\,\mathbf{v}$ of $K$, and therefore $B_0(K;R) \subset \ker\varepsilon$. We conclude therefore that $B_0(K;R) = \ker\varepsilon$.

Now $Z_0(K;R) = C_0(K;R)$ (because $\partial_0\colon C_0(K;R) \to C_{-1}(K;R)$ is defined to be the zero homomorphism from $C_0(K;R)$ to the zero module $C_{-1}(K;R)$), and therefore

$$H_0(K;R) = C_0(K;R)/B_0(K;R),$$

where $B_0(K;R) = \ker\varepsilon$. It follows that the $R$-module homomorphism $\varepsilon\colon C_0(K;R) \to R$ induces a well-defined isomorphism from $H_0(K;R)$ to the coefficient ring $R$ that sends the homology class of $\sum_{k=1}^{s} r_k\langle\mathbf{v}_k\rangle$ to $\sum_{k=1}^{s} r_k$ for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$ (see Corollary 1.8).

Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle)) \\
&\qquad = \partial_{q+1}(\langle\mathbf{w},\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle) \\
&\qquad = \langle\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle + \sum_{j=0}^{q}(-1)^{j+1}\langle\mathbf{w},\mathbf{v}_0,\ldots,\hat{\mathbf{v}}_j,\ldots,\mathbf{v}_q\rangle \\
&\qquad = \langle\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle - D_{q-1}(\partial_q(\langle\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_q\rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K;R)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K;R)$, and hence $Z_q(K;R) = B_q(K;R)$. It follows that $H_q(K;R)$ is the zero group for all $q > 0$, as required. ∎

**Example** The hypotheses of the proposition are satisfied for the complex $K_\sigma$ consisting of a simplex $\sigma$ together with all of its faces: we can choose $\mathbf{w}$ to be any vertex of the simplex $\sigma$.

## 4.7 Simplicial Maps and Induced Homomorphisms

Let $K$ and $L$ be simplicial complexes, and let $R$ be an integral domain. Any simplicial map $\varphi\colon K \to L$ between the simplicial complexes $K$ and $L$ induces well-defined homomorphisms $\varphi_q\colon C_q(K;R) \to C_q(L;R)$ of chain groups, where

$$\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$. (The existence of these induced homomorphisms follows from a straightforward application of Lemma 4.4.) Note that $\varphi_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) = 0$ unless $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ are all distinct.

Now $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$ for each integer $q$. Therefore

$$\varphi_q(Z_q(K;R)) \subset Z_q(L;R) \quad \text{and} \quad \varphi_q(B_q(K;R)) \subset B_q(L;R)$$

for all integers $q$. It follows that any simplicial map $\varphi\colon K \to L$ induces well-defined homomorphisms

$$\varphi_*\colon H_q(K;R) \to H_q(L;R)$$

of homology groups, where $\varphi_*([z]) = [\varphi_q(z)]$ for all $q$-cycles $z \in Z_q(K;R)$. It is a trivial exercise to verify that if $K$, $L$ and $M$ are simplicial complexes and if $\varphi\colon K \to L$ and $\psi\colon L \to M$ are simplicial maps then the induced homomorphisms of homology groups satisfy $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

## 4.8 Connectedness and $H_0(K;R)$

**Lemma 4.9** *Let $K$ be a simplicial complex. Then $K$ can be partitioned into pairwise disjoint subcomplexes $K_1, K_2, \ldots, K_s$ whose polyhedra are the connected components of the polyhedron $|K|$ of $K$.*

**Proof** Let $X_1, X_2, \ldots, X_s$ be the connected components of the polyhedron of $K$, and, for each $j$, let $K_j$ be the collection of all simplices $\sigma$ of $K$ for which $\sigma \subset X_j$. If a simplex belongs to $K_j$ for all $j$ then so do all its faces. Therefore $K_1, K_2, \ldots, K_s$ are subcomplexes of $K$. These subcomplexes are pairwise disjoint since the connected components $X_1, X_2, \ldots, X_s$ of $|K|$ are pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some $j$, since $\sigma$ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup \cdots \cup K_s$ and $|K| = |K_1| \cup |K_2| \cup \cdots \cup |K_s|$, as required. ∎

Let $R$ be an integral domain. The *direct sum* $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ of $R$-modules $M_1, M_2, \ldots, M_k$ is defined to be the $R$-module consisting of all $k$-tuples $(x_1, x_2, \ldots, x_k)$ with $x_i \in M_i$ for $i = 1, 2, \ldots, k$, where

$$(x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$$

and

$$r(x_1, x_2, \ldots, x_k) = (rx_1, rx_2, \ldots, rx_k)$$

for all elements $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of $M_1 \oplus M_2 \oplus \cdots \oplus M_k$, and for all $r \in R$.

**Lemma 4.10** *Let $K$ be a simplicial complex, and let $R$ be an integral domain. Suppose that $K = K_1 \cup K_2 \cup \cdots \cup K_s$, where $K_1, K_2, \ldots K_s$ are pairwise disjoint. Then*

$$H_q(K; R) \cong H_q(K_1; R) \oplus H_q(K_2; R) \oplus \cdots \oplus H_q(K_s; R)$$

*for all integers $q$.*

**Proof** We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K; R) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any $q$-chain $c$ of $K$ with coefficients in the integral domain $R$ can be expressed uniquely as a sum of the form $c = c_1 + c_2 + \cdots + c_s$, where $c_j$ is a $q$-chain of $K_j$ for $j = 1, 2, \ldots, s$. It follows that

$$C_q(K; R) \cong C_q(K_1; R) \oplus C_q(K_2; R) \oplus \cdots \oplus C_q(K_s; R).$$

Now let $z \in Z_q(K; R)$. We can express $z$ uniquely in the form $z = z_1 + z_2 + \cdots + z_s$, where $z_j \in C_q(K_j; R)$ for $j = 1, 2, \ldots, s$. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + \cdots + \partial_q(z_s),$$

and $\partial_q(z_j)$ is a $(q-1)$-chain of $K_j$ for $j = 1, 2, \ldots, s$. It follows that $\partial_q(z_j) = 0$ for $j = 1, 2, \ldots, s$. Hence each $z_j$ is a $q$-cycle of $K_j$, and thus

$$Z_q(K; R) \cong Z_q(K_1; R) \oplus Z_q(K_2; R) \oplus \cdots \oplus Z_q(K_s; R).$$

Now let $b \in B_q(K; R)$. Then $b = \partial_{q+1}(c)$ for some $c \in C_{q+1}(K; R)$. Moreover $c = c_1 + c_2 + \cdots c_s$, where $c_j \in C_{q+1}(K_j; R)$ for $j = 1, 2, \ldots, s$. Thus $b = b_1 + b_2 + \cdots b_s$, where $b_j = \partial_{q+1} c_j$ for $j = 1, 2, \ldots, s$. Moreover $b_j \in B_q(K_j; R)$ for $j = 1, 2, \ldots, s$. We deduce that

$$B_q(K; R) \cong B_q(K_1; R) \oplus B_q(K_2; R) \oplus \cdots \oplus B_q(K_s; R).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu \colon H_q(K_1; R) \oplus H_q(K_2; R) \oplus \cdots \oplus H_q(K_s; R) \to H_q(K; R)$$

which maps $([z_1], [z_2], \ldots, [z_s])$ to $[z_1 + z_2 + \cdots + z_s]$, where $[z_j]$ denotes the homology class of a $q$-cycle $z_j$ of $K_j$ for $j = 1, 2, \ldots, s$. ∎

Let $K$ be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of $K$. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of $K$ with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to $K$ for $j = 1, 2, \ldots, m$.

**Lemma 4.11** *The polyhedron $|K|$ of a simplicial complex $K$ is a connected topological space if and only if any two vertices of $K$ can be joined by an edge path.*

**Proof** It is easy to verify that if any two vertices of $K$ can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of $K$ can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of $K$. It suffices to verify that every vertex of $K$ can be joined to $\mathbf{v}_0$ by an edge path.

Let $K_0$ be the collection of all of the simplices of $K$ having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If $\sigma$ is a simplex belonging to $K_0$ then every vertex of $\sigma$ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of $\sigma$ belongs to $K_0$. Thus $K_0$ is a subcomplex of $K$. Clearly the collection $K_1$ of all simplices of $K$ which do not belong to $K_0$ is also a subcomplex of $K$. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of $K_0$ and $K_1$ are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of $K$ can be joined to $\mathbf{v}_0$ by an edge path, as required. ∎

**Theorem 4.12** *Let $K$ be a simplicial complex and let $R$ be an integral domain. Suppose that the polyhedron $|K|$ of $K$ is connected. Then $H_0(K; R) \cong R$.*

**Proof** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ be the vertices of the simplicial complex $K$. Every 0-chain of $K$ with coefficients in $R$ can be expressed uniquely as a formal sum of the form

$$r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + \cdots + r_s\langle \mathbf{v}_s \rangle$$

for some $r_1, r_2, \ldots, r_s \in R$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K;R) \to R$ defined such that

$$\varepsilon\left(r_1\langle \mathbf{v}_1\rangle + r_2\langle \mathbf{v}_2\rangle + \cdots + r_s\langle \mathbf{v}_s\rangle\right) = r_1 + r_2 + \cdots + r_s.$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$.

Now $\varepsilon(\partial_1(\langle \mathbf{u}, \mathbf{v}\rangle)) = \varepsilon(\langle \mathbf{v}\rangle - \langle \mathbf{u}\rangle) = 0$ whenever $\mathbf{u}$ and $\mathbf{v}$ are endpoints of an edge of $K$. It follows that $\varepsilon \circ \partial_1 = 0$, and therefore $B_0(K;R) \subset \ker \varepsilon$.

Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ be vertices of $K$ determining an edge path. Then $\mathbf{w}_{j-1}\,\mathbf{w}_j$ is an edge of $K$ for $j = 1, 2, \ldots, m$, and

$$\langle \mathbf{w}_m\rangle - \langle \mathbf{w}_0\rangle = \sum_{j=1}^{m}\Big(\langle \mathbf{w}_j\rangle - \langle \mathbf{w}_{j-1}\rangle\Big) = \partial_1\left(\sum_{j=1}^{m}\langle \mathbf{w}_{j-1}, \mathbf{w}_j\rangle\right) \in B_0(K;R).$$

Now $|K|$ is connected, and therefore any pair of vertices of $K$ can be joined by an edge path (Lemma 4.11). We deduce that $\langle \mathbf{v}\rangle - \langle \mathbf{u}\rangle \in B_0(K;R)$ for all vertices $\mathbf{u}$ and $\mathbf{v}$ of $K$.

Choose a vertex $\mathbf{u} \in K$. Then

$$\sum_{j=1}^{s} r_j\langle \mathbf{v}_j\rangle = \sum_{j=1}^{s} r_j(\langle \mathbf{v}_j\rangle - \langle \mathbf{u}\rangle) + \left(\sum_{j=1}^{s} r_j\right)\langle \mathbf{u}\rangle \in B_0(K;R) + \left(\sum_{j=1}^{s} r_j\right)\langle \mathbf{u}\rangle$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of $K$, and therefore

$$z - \varepsilon(z)\langle \mathbf{u}\rangle \in B_0(K;R)$$

for all $z \in C_0(K;R)$. It follows that $\ker \varepsilon \in B_0(K;R)$. But we have already shown that $B_0(K;R) \subset \ker \varepsilon$. It follows that $\ker \varepsilon = B_0(K;R)$.

Now the homomorphism $\varepsilon\colon C_0(K;R) \to R$ is surjective and its kernel is $B_0(K;R)$. Moreover $Z_0(K;R) = C_0(K;R)$ (because $\partial_0\colon C_0(K;R) \to C_{-1}(K;R)$ is defined to be the zero homomorphism from $C_0(K;R)$ to the zero module $C_{-1}(K;R)$), and therefore

$$H_0(K;R) = Z_0(K;R)/B_0(K;R) = C_0(K;R)/B_0(K;R).$$

It follows that the homomorphism $\varepsilon$ induces an isomorphism from $H_0(K;R)$ to $R$ (see Corollary 1.8), and therefore $H_0(K;R) \cong R$, as required. ∎

On combining Theorem 4.12 with Lemmas 4.9 and 4.10 we obtain immediately the following result.

**Corollary 4.13** *Let $K$ be a simplicial complex, and let $R$ be an integral domain. Then $H_0(K;R) \cong R^s$, where $s$ is the number of connected components of $|K|$.*

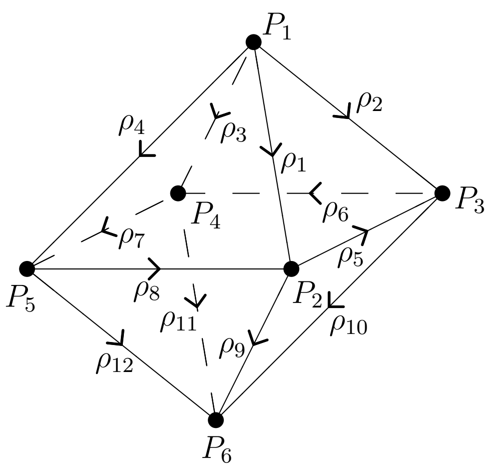
# 5 Homology Calculations

## 5.1 The Homology Groups of an Octahedron

Let $K$ be the simplicial complex consisting of the triangular faces, edges and vertices of an octahedron in $\mathbb{R}^3$ with vertices $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ and $P_6$, where

$$P_1 = (0,0,1), \quad P_2 = (1,0,0), \quad P_3 = (0,1,0),$$

$$P_4 = (-1,0,0), \quad P_5 = (0,-1,0), \quad P_6 = (0,0,-1)$$




This octahedron consists of the four triangular faces $P_1P_2P_3$, $P_1P_3P_4$, $P_1P_4P_5$ and $P_1P_5P_2$ of the pyramid whose base is the square $P_2P_3P_4P_5$ and whose apex is $P_1$, together with the four triangular faces $P_6P_2P_3$, $P_6P_3P_4$, $P_6P_4P_5$ and $P_6P_5P_2$ of the pyramid whose base is $P_2P_3P_4P_5$ and whose apex is $P_6$.

A typical 2-chain $c_2$ of $K$ is a linear combination, with integer coefficients, of eight oriented 2-simplices that represent the triangular faces of the octahedron. Thus we can write

$$c_2 = \sum_{i=1}^{8} n_i \sigma_i,$$

where $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, 8$ and

$$\sigma_1 = \langle P_1, P_2, P_3 \rangle, \quad \sigma_2 = \langle P_1, P_3, P_4 \rangle, \quad \sigma_3 = \langle P_1, P_4, P_5 \rangle,$$

$$\sigma_4 = \langle P_1, P_5, P_2 \rangle, \quad \sigma_5 = \langle P_6, P_3, P_2 \rangle, \quad \sigma_6 = \langle P_6, P_4, P_3 \rangle,$$

$$\sigma_7 = \langle P_6, P_5, P_4 \rangle, \quad \sigma_8 = \langle P_6, P_2, P_5 \rangle.$$

(The orientation on each of these triangles has been chosen such that the vertices of the triangle are listed in anticlockwise order when viewed from a point close to the centre of triangle that lies outside the octahedron.)

Similarly a typical 1-chain $c_1$ of $K$ is a linear combination, with integer coefficients, of twelve 1-simplices that represent the edges of the octahedron. Thus we can write

$$c_1 = \sum_{j=1}^{12} m_j \rho_j,$$

where $m_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, 12$ and

$$\rho_1 = \langle P_1, P_2 \rangle, \quad \rho_2 = \langle P_1, P_3 \rangle, \quad \rho_3 = \langle P_1, P_4 \rangle, \quad \rho_4 = \langle P_1, P_5 \rangle,$$

$$\rho_5 = \langle P_2, P_3 \rangle, \quad \rho_6 = \langle P_3, P_4 \rangle, \quad \rho_7 = \langle P_4, P_5 \rangle, \quad \rho_8 = \langle P_5, P_2 \rangle,$$

$$\rho_9 = \langle P_2, P_6 \rangle, \quad \rho_{10} = \langle P_3, P_6 \rangle, \quad \rho_{11} = \langle P_4, P_6 \rangle, \quad \rho_{12} = \langle P_5, P_6 \rangle,$$

A typical 0-chain $c_0$ takes the form

$$c_0 = \sum_{k=1}^{6} r_k \langle P_k \rangle,$$

where $r_k \in \mathbb{Z}$ for $k = 1, 2, \ldots, 6$.

We now calculate the boundary of a 2-chain. It follows from the definition of the boundary homomorphism $\partial_2$ that

$$\partial_2 \sigma_1 = \partial_2 \langle P_1, P_2, P_3 \rangle = \langle P_2 P_3 \rangle - \langle P_1 P_3 \rangle + \langle P_1 P_2 \rangle = \rho_5 - \rho_2 + \rho_1.$$

Similarly

$$\begin{array}{rcl}
\partial_2 \sigma_2 & = & \partial_2 \langle P_1, P_3, P_4 \rangle = \rho_6 - \rho_3 + \rho_2, \\
\partial_2 \sigma_3 & = & \partial_2 \langle P_1, P_4, P_5 \rangle = \rho_7 - \rho_4 + \rho_3, \\
\partial_2 \sigma_4 & = & \partial_2 \langle P_1, P_5, P_2 \rangle = \rho_8 - \rho_1 + \rho_4, \\
\partial_2 \sigma_5 & = & \partial_2 \langle P_6, P_3, P_2 \rangle = -\rho_5 + \rho_9 - \rho_{10}, \\
\partial_2 \sigma_6 & = & \partial_2 \langle P_6, P_4, P_3 \rangle = -\rho_6 + \rho_{10} - \rho_{11}, \\
\partial_2 \sigma_7 & = & \partial_2 \langle P_6, P_5, P_4 \rangle = -\rho_7 + \rho_{11} - \rho_{12}, \\
\partial_2 \sigma_8 & = & \partial_2 \langle P_6, P_2, P_5 \rangle = -\rho_8 + \rho_{12} - \rho_9.
\end{array}$$

Thus

$$\partial_2 c_2 \quad = \quad \partial_2 \left( n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3 + n_4 \sigma_4 + n_5 \sigma_5 + n_6 \sigma_6 + n_7 \sigma_7 + n_8 \sigma_8 \right)$$

$$
\begin{aligned}
&= n_1\partial_2\sigma_1 + n_2\partial_2\sigma_2 + n_3\partial_2\sigma_3 + n_4\partial_2\sigma_4 \\
&\qquad + n_5\partial_2\sigma_5 + n_6\partial_2\sigma_6 + n_7\partial_2\sigma_7 + n_8\partial_2\sigma_8 \\
&= (n_1 - n_4)\rho_1 + (n_2 - n_1)\rho_2 + (n_3 - n_2)\rho_3 + (n_4 - n_3)\rho_4 \\
&\qquad + (n_1 - n_5)\rho_5 + (n_2 - n_6)\rho_6 + (n_3 - n_7)\rho_7 + (n_4 - n_8)\rho_8 \\
&\qquad + (n_5 - n_8)\rho_9 + (n_6 - n_5)\rho_{10} + (n_7 - n_6)\rho_{11} + (n_8 - n_7)\rho_{12}
\end{aligned}
$$

It follows that $\partial_2 c_2 = 0$ if and only if

$$n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8.$$

Therefore

$$Z_2(K;\mathbb{Z}) = \ker\partial_2 = \{n\mu : n \in \mathbb{Z}\},\ \text{ where } \mu = \sum_{i=1}^{8} \sigma_i.$$

Now $C_3(K;\mathbb{Z}) = 0$, and thus $B_2(K;\mathbb{Z}) = 0$ (where 0 here denotes the zero group), since the complex $K$ has no 3-simplices. Therefore

$$H_2(K;\mathbb{Z}) \cong Z_2(K;\mathbb{Z}) \cong \mathbb{Z}.$$

Next we calculate the boundary of a 1-chain. It follows from the definition of the boundary homomorphism $\partial_1$ that

$$
\begin{aligned}
\partial_1 c_1 &= \partial_1\left(\sum_{j=1}^{12} m_j\rho_j\right) \\
&= m_1(\langle P_2\rangle - \langle P_1\rangle) + m_2(\langle P_3\rangle - \langle P_1\rangle) \\
&\qquad + m_3(\langle P_4\rangle - \langle P_1\rangle) + m_4(\langle P_5\rangle - \langle P_1\rangle) \\
&\qquad + m_5(\langle P_3\rangle - \langle P_2\rangle) + m_6(\langle P_4\rangle - \langle P_3\rangle) \\
&\qquad + m_7(\langle P_5\rangle - \langle P_4\rangle) + m_8(\langle P_2\rangle - \langle P_5\rangle) \\
&\qquad + m_9(\langle P_6\rangle - \langle P_2\rangle) + m_{10}(\langle P_6\rangle - \langle P_3\rangle) \\
&\qquad + m_{11}(\langle P_6\rangle - \langle P_4\rangle) + m_{12}(\langle P_6\rangle - \langle P_5\rangle) \\
&= -(m_1 + m_2 + m_3 + m_4)\langle P_1\rangle + (m_1 - m_5 + m_8 - m_9)\langle P_2\rangle \\
&\qquad + (m_2 + m_5 - m_6 - m_{10})\langle P_3\rangle + (m_3 + m_6 - m_7 - m_{11})\langle P_4\rangle \\
&\qquad + (m_4 + m_7 - m_8 - m_{12})\langle P_5\rangle + (m_9 + m_{10} + m_{11} + m_{12})\langle P_6\rangle
\end{aligned}
$$

It follows that the 1-chain $c_1$ is a 1-cycle if and only if

$$m_1 + m_2 + m_3 + m_4 = 0, \quad m_1 - m_5 + m_8 - m_9 = 0,$$

$$m_2 + m_5 - m_6 - m_{10} = 0, \quad m_3 + m_6 - m_7 - m_{11} = 0,$$

$$m_4 + m_7 - m_8 - m_{12} = 0 \quad \text{and} \quad m_9 + m_{10} + m_{11} + m_{12} = 0.$$

On examining the structure of these equations, we see that, when $c_1$ is a 1-cycle, it is possible to eliminate five of the integer quantities $m_j$, expressing them in terms of the remaining quantities. For example, we can eliminate $m_4$, $m_6$, $m_7$, $m_8$ and $m_{12}$, expressing these quantities in terms of $m_1$, $m_2$, $m_3$, $m_5$, $m_9$ $m_{10}$ and $m_{11}$ by means of the equations

$$\begin{aligned} m_4 &= -m_1 - m_2 - m_3, \\ m_6 &= m_2 - m_{10} + m_5, \\ m_7 &= m_2 + m_3 - m_{10} - m_{11} + m_5, \\ m_8 &= -m_1 + m_9 + m_5, \\ m_{12} &= -m_9 - m_{10} - m_{11} \end{aligned}$$

It follows that

$$Z_2(K;\mathbb{Z}) = \{m_1 z_1 + m_2 z_2 + m_3 z_3 + m_5 z_5 + m_9 z_9 + m_{10} z_{10} + m_{11} z_{11}\},$$

where

$$\begin{aligned} z_1 &= \rho_1 - \rho_4 - \rho_8 = -\partial_2 \sigma_4, \\ z_2 &= \rho_2 - \rho_4 + \rho_6 + \rho_7 = \partial_2(\sigma_2 + \sigma_3), \\ z_3 &= \rho_3 - \rho_4 + \rho_7 = \partial_2 \sigma_3, \\ z_5 &= \rho_5 + \rho_6 + \rho_7 + \rho_8 = \partial_2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4), \\ z_9 &= \rho_8 + \rho_9 - \rho_{12} = -\partial_2 \sigma_8, \\ z_{10} &= -\rho_6 - \rho_7 + \rho_{10} - \rho_{12} = \partial_2(\sigma_6 + \sigma_7), \\ z_{11} &= \rho_{11} - \rho_7 - \rho_{12} = \partial_2 \sigma_7. \end{aligned}$$

From these equations, we see that the generators $z_1$, $z_2$, $z_3$, $z_5$, $z_9$, $z_{10}$ and $z_{11}$ of the group $Z_1(K;\mathbb{Z})$ of 1-cycles all belong to the group $B_1(K;\mathbb{Z})$ of 1-boundaries. It follows that $Z_1(K;\mathbb{Z}) = B_1(K;\mathbb{Z})$, and therefore $H_1(K;\mathbb{Z}) = 0$.

In order to determine $H_0(K;\mathbb{Z})$ it suffices to note that the 0-chains

$$\langle P_2\rangle - \langle P_1\rangle, \quad \langle P_3\rangle - \langle P_1\rangle, \quad \langle P_4\rangle - \langle P_1\rangle, \quad \langle P_5\rangle - \langle P_1\rangle \quad \text{and} \quad \langle P_6\rangle - \langle P_1\rangle$$

are 0-boundaries. Indeed

$$\langle P_2\rangle - \langle P_1\rangle = \partial_1 \rho_1, \quad \langle P_3\rangle - \langle P_1\rangle = \partial_1 \rho_2, \quad \langle P_4\rangle - \langle P_1\rangle = \partial_1 \rho_3,$$

$$\langle P_5\rangle - \langle P_1\rangle = \partial_1 \rho_4 \quad \text{and} \quad \langle P_6\rangle - \langle P_1\rangle = \partial_1(\rho_1 + \rho_9).$$

Therefore

$$\sum_{k=1}^{6} r_k \langle P_k \rangle - \left( \sum_{k=1}^{6} r_k \right) \langle P_1 \rangle \in B_0(K;\mathbb{Z})$$

for all integers $r_1$, $r_2$, $r_3$, $r_4$, $r_5$ and $r_6$. It follows that $B_0(K;\mathbb{Z}) = \ker \varepsilon$, where $\varepsilon\colon C_0(K;\mathbb{Z}) \to \mathbb{Z}$ is the homomorphism defined such that

$$\varepsilon \left( \sum_{k=1}^{6} r_k \langle P_k \rangle \right) = \sum_{k=1}^{6} r_k$$

for all integers $r_k$ $(k = 1, 2, \ldots, 6)$. Now $Z_0(K;\mathbb{Z}) = C_0(K;\mathbb{Z})$ since the homomorphism $\partial_0\colon C_0(K;\mathbb{Z}) \to C_{-1}(K;\mathbb{Z})$ is the zero homomorphism mapping $C_0(K;\mathbb{Z})$ to the zero group. It follows that

$$H_0(K;\mathbb{Z}) = C_0(K;\mathbb{Z})/B_0(K;\mathbb{Z}) = C_0(K;\mathbb{Z})/\ker\varepsilon \cong \mathbb{Z}.$$

(Here we are using the result that the image of a homomorphism is isomorphic to the quotient of the domain of the homomorphism by the kernel of the homomorphism.)

We have thus shown that

$$H_2(K;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K;\mathbb{Z}) = 0, \quad H_0(K;\mathbb{Z}) \cong \mathbb{Z}.$$

One can show that $Z_1(K;\mathbb{Z}) = B_1(K;\mathbb{Z})$ by employing an alternative approach to that used above. An element $z$ of $Z_1(K;\mathbb{Z})$ is of the form $z = \sum\limits_{j=1}^{12} m_j \rho_j$, where

$$m_1 + m_2 + m_3 + m_4 = 0, \quad m_1 - m_5 + m_8 - m_9 = 0,$$

$$m_2 + m_5 - m_6 - m_{10} = 0, \quad m_3 + m_6 - m_7 - m_{11} = 0,$$

$$m_4 + m_7 - m_8 - m_{12} = 0 \quad \text{and} \quad m_9 + m_{10} + m_{11} + m_{12} = 0.$$

The 1-cycle $z$ belongs to the group $B_1(K;\mathbb{Z})$ if and only if there exists some 2-chain $c_2$ such that $z = \partial_2 c_2$. It follows that $z \in B_1(K;\mathbb{Z})$ if and only if there exist integers $n_1, n_2, \ldots, n_8$ such that

$$m_1 = n_1 - n_4, \quad m_2 = n_2 - n_1, \quad m_3 = n_3 - n_2, \quad m_4 = n_4 - n_3,$$

$$m_5 = n_1 - n_5, \quad m_6 = n_2 - n_6, \quad m_7 = n_3 - n_7, \quad m_8 = n_4 - n_8,$$

$$m_9 = n_5 - n_8, \quad m_{10} = n_6 - n_5, \quad m_{11} = n_7 - n_6, \quad m_{12} = n_8 - n_7.$$

The integers $n_1, n_2, \ldots, n_8$ solving the above equations are not uniquely determined, since, given one collection of integers $n_1, n_2, \ldots, n_8$ satisfying these equations, another solution can be obtained by adding some fixed integer to each of $n_1, n_2, \ldots, n_8$. It follows from this that if there exists some collection $n_1, n_2, \ldots, n_8$ of integers that solves the above equations, then there exists a solution which satisfies the extra condition $n_1 = 0$. We then find that

$$n_1 = 0, \quad n_2 = m_2, \quad n_3 = m_2 + m_3, \quad n_4 = -m_1,$$

$$n_5 = -m_5, \quad n_6 = m_2 - m_6, \quad n_7 = m_2 + m_3 - m_7, \quad n_8 = -m_1 - m_8.$$

On substituting $n_1, n_2, \ldots, n_8$ into the relevant equations, and making use of the constraints on the values of $m_1, m_2, \ldots, m_{12}$, we find that we do indeed have a solution to the equations that express the integers $m_j$ in terms of the integers $n_i$. It follows that every 1-cycle of $K$ is a 1-boundary. Thus $Z_1(K;\mathbb{Z}) = B_1(K;\mathbb{Z})$, and therefore $H_1(K;\mathbb{Z}) = 0$.

Note that the results of many of the calculations of boundaries of chains can be verified by consulting the diagram representing the vertices and edges of the octahedron with their labels and orientations. For example, direct calculation using the definition of the boundary homomorphism $\delta_2 \colon C_2(K;\mathbb{Z}) \to C_1(K;\mathbb{Z})$ shows that

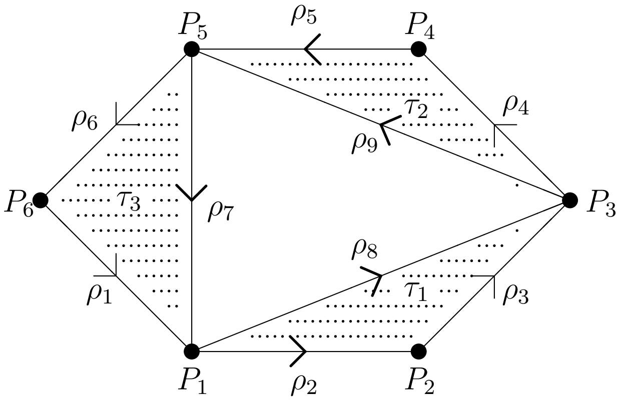
$$\partial_2\sigma_1 = \partial_2\langle P_1, P_2, P_3\rangle = \langle P_2 P_3\rangle - \langle P_1 P_3\rangle + \langle P_1 P_2\rangle = \rho_5 - \rho_2 + \rho_1.$$

Now if we follow round the edges of the triangle $P_1\,P_2\,P_3$ represented by $\sigma$, starting at $P_1$ and proceeding to $P_2$, then $P_3$ then back to $P_1$ we traverse the edge $\rho_1$ in the direction of the arrow, then the edge $\rho_5$ in the direction of the arrow, and finally the edge $\rho_2$ in the reverse direction to the arrow. In consequence, both $\rho_1$ and $\rho_5$ occur in the 1-boundary $\partial_2\sigma_1$ with coefficient $+1$, whereas $\rho_2$ occurs in this 1-boundary with coefficient $-1$.

Consider also the coefficient corresponding to the vertex $P_2$ in the 0-boundary $\partial_1 c_1$, where $c_1 = \sum\limits_{j=1}^{12} m_j \rho_j$. The vertex $P_2$ is an endpoint of four edges. The arrows indicating the orientation on the edges $\rho_1$ and $\rho_8$ are directed towards the vertex $P_2$, whereas the arrows indicating the orientation on the edges $\rho_5$ and $\rho_9$ are directed away from the vertex $P_2$. In consequence, the coefficient of $\langle P_2\rangle$ in $\partial_1 c_1$ is $m_1 - m_5 + m_8 - m_9$.

## 5.2 Another Homology Example

Let $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ and $P_6$ be the vertices of a hexagon in the plane, listed in cyclic order, and let $K$ be simplicial complex consisting of the triangles $P_1P_2P_3$, $P_3P_4P_5$ and $P_5P_6P_1$, together with all the edges and vertices of these triangles.




Then

$$C_2(K;\mathbb{Z}) = \{n_1\tau_1 + n_2\tau_2 + n_3\tau_3 : n_1, n_2, n_3 \in \mathbb{Z}\},$$

where

$$\tau_1 = \langle P_1P_2P_3\rangle, \quad \tau_2 = \langle P_3P_4P_5\rangle \quad \text{and} \quad \tau_3 = \langle P_5P_6P_1\rangle.$$

(Note $\tau_1$, $\tau_2$ and $\tau_3$ represent the three triangles of the simplicial complex with the orientations that results from an anticyclic ordering of the vertices in the diagram above.) Also

$$C_1(K;\mathbb{Z}) = \left\{\sum_{j=1}^{9} m_j\rho_j : m_j \in \mathbb{Z} \text{ for } j = 1, 2, \ldots, 9\right\},$$

where

$$\rho_1 = \langle P_6P_1\rangle, \quad \rho_2 = \langle P_1P_2\rangle, \quad \rho_3 = \langle P_2P_3\rangle, \quad \rho_4 = \langle P_3P_4\rangle, \quad \rho_5 = \langle P_4P_5\rangle,$$

$$\rho_6 = \langle P_5P_6\rangle, \quad \rho_7 = \langle P_5P_1\rangle, \quad \rho_8 = \langle P_1P_3\rangle \quad \text{and} \quad \rho_9 = \langle P_3P_5\rangle,$$

and

$$C_0(K;\mathbb{Z}) = \left\{\sum_{k=1}^{6} r_k\langle P_k\rangle : r_k \in \mathbb{Z} \text{ for } k = 1, 2, \ldots, 6\right\}.$$

(Note that the 1-chains $\rho_1, \rho_2, \ldots, \rho_9$ represent the 9 edges of the simplicial complex with the orientations indicated by the arrows on the above diagram.)

We now calculate the images of the 2-chains $\tau_1$, $\tau_2$ and $\tau_3$ under the boundary homomorphism $\partial_2\colon C_2(K;\mathbb{Z}) \to C_1(K;\mathbb{Z})$. We find that

$$\partial_2\tau_1 = \rho_3 - \rho_8 + \rho_2, \quad \partial_2\tau_2 = \rho_5 - \rho_9 + \rho_4, \quad \partial_2\tau_3 = \rho_1 - \rho_7 + \rho_6,$$

Now

$$\begin{aligned}
&\partial_2(n_1\tau_1 + n_2\tau_2 + n_3\tau_3) \\
&\quad = \quad n_3\rho_1 + n_1\rho_2 + n_1\rho_3 + n_2\rho_4 + n_2\rho_5 + n_3\rho_6 - n_3\rho_7 - n_1\rho_8 - n_2\rho_9.
\end{aligned}$$

The simplicial complex $K$ has no non-zero 2-cycles, and therefore $Z_2(K;\mathbb{Z}) = 0$. It follows that $H_2(K;\mathbb{Z}) = 0$.

Let

$$c_1 = \sum_{j=1}^{9} m_j\rho_j.$$

Then

$$\begin{aligned}
\partial_1 c_1 \quad = \quad & (m_1 - m_2 + m_7 - m_8)\langle P_1\rangle + (m_2 - m_3)\langle P_2\rangle \\
& + (m_3 - m_4 + m_8 - m_9)\langle P_3\rangle + (m_4 - m_5)\langle P_4\rangle \\
& + (m_5 - m_6 + m_9 - m_7)\langle P_5\rangle + (m_6 - m_1)\langle P_6\rangle
\end{aligned}$$

It follows that $c_1$ is a 1-cycle of $K$ if and only if

$$m_2 = m_3, \quad m_4 = m_5, \quad m_6 = m_1$$

and

$$m_1 + m_7 = m_3 + m_8 = m_5 + m_9.$$

Moreover $c_1$ is a 1-boundary of $K$ if and only if

$$m_2 = m_3 = -m_8, \quad m_4 = m_5 = -m_9, \quad m_6 = m_1 = -m_7.$$

We see from this that not every 1-cycle of $K$ is a 1-boundary of $K$. Indeed

$$Z_1(K;\mathbb{Z}) = \{n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz : n_1, n_2, n_3, n \in \mathbb{Z}\},$$

where $z = \rho_7 + \rho_8 + \rho_9$. Let $\theta\colon Z_1(K;\mathbb{Z}) \to \mathbb{Z}$ be the homomorphism defined such that

$$\theta\left(n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz\right) = n$$

for all $n_1, n_2, n_3, n \in \mathbb{Z}$. Now

$$n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz \in B_1(K;\mathbb{Z}) \quad \text{if and only if} \quad n = 0.$$

It follows that $B_1(K;\mathbb{Z}) = \ker\theta$. Therefore the homomorphism $\theta$ induces an isomorphism from $H_1(K;\mathbb{Z})$ to $\mathbb{Z}$, where $H_1(K;\mathbb{Z}) = Z_1(K;\mathbb{Z})/B_1(K;\mathbb{Z})$. Indeed $H_1(K;\mathbb{Z}) = \{n[z] : n \in \mathbb{Z}\}$, where $z = \rho_7 + \rho_8 + \rho_9$ and $[z]$ denotes the homology class of the 1-cycle $z$.

It is a straightforward exercise to verify that

$$B_0(K;\mathbb{Z}) = \left\{ \sum_{k=1}^{6} r_k\langle P_k\rangle : r_k \in \mathbb{Z} \text{ for } k = 1, 2, \ldots, 6 \text{ and } \sum_{k=1}^{6} r_k = 0 \right\}.$$

It follows from this that $H_0(K;\mathbb{Z}) \cong \mathbb{Z}$. Indeed this result is a consquence of the fact that the polyhedron $|K|$ of the simplicial complex $K$ is connected.

## 5.3 The Homology Groups of the Boundary of a Simplex

**Proposition 5.1** *Let $K$ be the simplicial complex consisting of all the proper faces of an $(n+1)$-dimensional simplex $\sigma$, where $n > 0$. Then*

$$H_0(K;\mathbb{Z}) \cong \mathbb{Z}, \quad H_n(K;\mathbb{Z}) \cong Z, \quad H_q(K;\mathbb{Z}) = 0 \text{ when } q \neq 0, n.$$

**Proof** Let $M$ be the simplicial complex consisting of the $(n+1)$-dimensional simplex $\sigma$, together with all its faces. Then $K$ is a subcomplex of $M$, and $C_q(K;\mathbb{Z}) = C_q(M;\mathbb{Z})$ when $q \leq n$.

It follows from Proposition 4.8 that $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$ and $H_q(M;\mathbb{Z}) = 0$ when $q > 0$. (Here 0 denotes the zero group.) Now $Z_q(K;\mathbb{Z}) = Z_q(M;\mathbb{Z})$ when $q \leq n$, and $B_q(K;\mathbb{Z}) = B_q(M;\mathbb{Z})$ when $q < n$. It follows that $H_q(K;\mathbb{Z}) = H_q(M;\mathbb{Z})$ when $q < n$. Thus $H_0(K;\mathbb{Z}) \cong \mathbb{Z}$ and $H_q(K;\mathbb{Z}) = 0$ when $0 < q < n$. Also $H_q(K;\mathbb{Z}) = 0$ when $q > n$, since the simplicial complex $K$ is of dimension $n$. Thus, to determine the homology of the complex $K$, it only remains to find $H_n(K;\mathbb{Z})$.

Let the $(n+1)$-dimensional simplex $\sigma$ have vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$. Then

$$C_{n+1}(M;\mathbb{Z}) = \{n\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\rangle : n \in \mathbb{Z}\}.$$

and therefore $B_n(M;\mathbb{Z}) = \{nz : n \in \mathbb{Z}\}$, where

$$z = \partial_{n+1}\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\rangle\right).$$

Now $H_n(M;\mathbb{Z}) = 0$ (Proposition 4.8). It follows that $Z_n(M;\mathbb{Z}) = B_n(M;\mathbb{Z})$. But $Z_n(K;\mathbb{Z}) = Z_n(M;\mathbb{Z})$, since $C_n(K;\mathbb{Z}) = C_n(M;\mathbb{Z})$ and the definition of the boundary homomorphism on $C_n(K;\mathbb{Z})$ is consistent with the definition of the boundary homomorphism on $C_n(M;\mathbb{Z})$. Also $B_n(K;\mathbb{Z}) = 0$, because the simplicial complex $K$ is of dimension $n$, and therefore has no non-zero $n$-boundaries. It follows that

$$H_n(K;\mathbb{Z}) \cong Z_n(K;\mathbb{Z}) = Z_n(M;\mathbb{Z}) = B_n(M;\mathbb{Z}) \cong \mathbb{Z}.$$

Indeed $H_n(K;\mathbb{Z}) = \{n[z] : n \in \mathbb{Z}\}$, where $[z]$ denotes the homology class of the $n$-cycle $z$ of $K$ defined above. ∎

**Remark** Note that the $n$-cycle $z$ is an $n$-cycle of the simplicial complex $K$, since it is a linear combination, with integer coefficients, of oriented $n$-simplices of $K$. The $n$-cycle $z$ is an $n$-boundary of the large simplicial complex $M$. However it is not an $n$-boundary of $K$. Indeed the $n$-dimensional simplicial complex $K$ has no non-zero $(n+1)$-chains, therefore has no non-zero $n$-boundaries. Therefore $z$ represents a non-zero homology class $[z]$ of $H_n(K;\mathbb{Z})$. This homology class generates the homology group $H_n(K;\mathbb{Z})$.

**Remark** The boundary of a 1-simplex consists of two points. Thus if $K$ is the simplicial complex representing the boundary of a 1-simplex then $H_0(K;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ (Corollary 4.13), and $H_q(K;\mathbb{Z}) = 0$ when $q > 0$.