Course MA3427, Michaelmas Term 2018
Problems

1. Determine which of the following maps are covering maps:—

   (i) the map from $\mathbb{R}$ to $[-1, 1]$ sending θ to $\sin\theta$,

   (ii) the map from S^1 to S^1 sending $(\cos\theta, \sin\theta)$ to $(\cos n\theta, \sin n\theta)$, where n is some non-zero integer,

   (iii) the map from $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$ sending z to $\exp(z)$,

   (iv) the map from $\{z \in \mathbb{C} : -4\pi < \operatorname{Im} z < 4\pi\}$ to $\{z \in \mathbb{C} : |z| > 0\}$ sending z to $\exp(z)$.

   [Briefly justify your answers.]

2. A continuous function $f\colon X \to Y$ between topological spaces X and Y is said to be a *local homeomorphism* if, given any point x of X there exists an open set V in X containing the point x and an open set W in Y containing the point $f(x)$ such that the function f maps V homeomorphically onto W. Explain why any covering map is a local homeomorphism.

3. Determine which of the maps described in question 1 are local homeomorphisms.

4. (a) Let X and Y be topological spaces, let $f\colon X \to Y$ and $h\colon Y \to X$ continuous maps, and let x be a point of X. Suppose that $h(f(x)) = x$ and that $h \circ f \simeq 1_X \operatorname{rel}\{x\}$ and $f \circ h \simeq 1_Y \operatorname{rel}\{f(x)\}$, where 1_X and 1_Y denote the identity maps of the spaces X and Y. Explain why the fundamental groups $\pi_1(X, x)$ and $\pi_1(Y, f(x))$ are isomorphic.

   (b) Using (a), explain why the fundamental groups $\pi_1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{p})$ and $\pi^1(S^{n-1}, \mathbf{p})$ of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and the $(n-1)$-dimensional sphere S^{n-1} are isomorphic for all $n > 1$, where $\mathbf{p} \in S^{n-1}$.

5. Let X be a topological space, and let $\alpha\colon [0, 1] \to X$ and $\beta\colon [0, 1] \to X$ be paths in X. We say that the path β is a *reparameterization* of the path α if there exists a strictly increasing continuous function $\sigma\colon [0, 1] \to [0, 1]$ such that $\sigma(0) = 0$, $\sigma(1) = 1$ and $\beta = \alpha \circ \sigma$. (Note that if β is a reparameterization of α then $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and the paths α and β have the same image in X.)

(a) Show that there is a well-defined equivalence relation on the set of all paths in X, where a path α is related to a path β if and only if β is a reparameterization of a path α. [Hint: use the basic result of analysis which states that a strictly increasing continuous function mapping one interval onto another has a continuous inverse.]

(b) Show that if the path β is a reparameterization of the path α, then $\beta \simeq \alpha \operatorname{rel}\{0, 1\}$.

Given paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ in a topological space X, where $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, \ldots, n-1$, we define the *concatenation* $\gamma_1.\gamma_2. \cdots .\gamma_n$ of the paths by the formula $(\gamma_1.\gamma_2. \ldots .\gamma_n)(t) = \gamma_i(nt - i + 1)$ for all t satisfying $(i-1)/n \leq t \leq i/n$.

(c) Show that the path $(\gamma_1. \ldots .\gamma_r).(\gamma_{r+1}. \ldots .\gamma_n)$ is a reparameterization of $\gamma_1.\gamma_2. \cdots .\gamma_n$ for any r between 1 and $n-1$.

(d) By making repeated applications of (c), or otherwise, show that $(\gamma_1.\gamma_2).\gamma_3.(\gamma_4.\gamma_5)$ is a reparameterization of $\gamma_1.(\gamma_2.\gamma_3.\gamma_4).\gamma_5$ for all paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ in X satisfying $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, 3, 4$.

6. Let X be a topological space.

(a) Show that, given any path $\alpha\colon [0, 1] \to X$ in X, there is a well-defined homomorphism $\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ of fundamental groups which sends the homotopy class $[\gamma]$ of any loop γ based at $\alpha(1)$ to the homotopy class $[\alpha.\gamma.\alpha^{-1}]$ of the loop $\alpha.\gamma.\alpha^{-1}$, where

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \gamma(3t - 1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \alpha(3 - 3t) & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}$$

(i.e., $\alpha.\gamma.\alpha^{-1}$ represents 'α followed by γ followed by α reversed').

(b) Show that $\Theta_{\alpha.\beta} = \Theta_\alpha \circ \Theta_\beta$ for all paths α and β in X satisfying $\beta(0) = \alpha(1)$.

(c) Show that Θ_α is the identity homomorphism whenever α is a constant path.

(d) Let α and $\hat{\alpha}$ be paths in X satisfying $\alpha(0) = \hat{\alpha}(0)$ and $\alpha(1) = \hat{\alpha}(1)$. Suppose that $\alpha(0) \simeq \hat{\alpha}(0) \operatorname{rel}\{0, 1\}$. Show that $\Theta_\alpha = \Theta_{\hat{\alpha}}$.

(e) Explain why the homomorphism $\Theta_\alpha \colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ is an isomorphism for all paths α in X. (This shows that, up to isomorphism, the fundamental group of a path-connected topological space does not depend on the choice of basepoint.)

7. Let X and Y be topological spaces, and let x_0 and y_0 be points of X and Y. Prove that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. [Hint: you should make use of the result that a function mapping a topological space into $X \times Y$ is continuous if and only if its components are continuous.]