Module MA3427: Michaelmas Semester Examination 2018 Worked solutions

David R. Wilkins

October 19, 2018

Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3427/`

1. (a) [*Standard definition.*] Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let w be a complex number that does not lie on γ. The *winding number* of γ about w is defined to be the unique integer $n(\gamma, w)$ with the property that $\varphi(b) - \varphi(a) = 2\pi i n(\gamma, w)$ for all paths $\varphi\colon [a,b] \to \mathbb{C}$ in the complex plane that satisfy $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a,b]$.

(b) [*Problem, not currently included in the course notes, but which may well be discussed in the final weeks of teaching.*]

There exist lifts $\tilde{\alpha}\colon [0,1] \to \mathbb{C}$ and $\tilde{\beta}\colon [0,1] \to \mathbb{C}$ of α and β respectively such that $\exp(\tilde{\alpha}(t)) = \alpha(t)$ and $\exp(\tilde{\beta}(t)) = \beta(t)$ for all $t \in [0,1]$. Let $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ be defined so that $\tilde{\gamma}(t) = \tilde{\alpha}(t) + \tilde{\beta}(t)$ for all $t \in [0,1]$. Then $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [0,1]$, and therefore

$$\begin{aligned} n(\gamma, 0) &= \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = \frac{\tilde{\alpha}(1) - \tilde{\alpha}(0)}{2\pi i} + \frac{\tilde{\beta}(1) - \tilde{\beta}(0)}{2\pi i} \\ &= n(\alpha, 0) + n(\beta, 0), \end{aligned}$$

as required.

(c) [*Bookwork, a theorem included in the course notes, but not recently examined.*]

We shall prove that any polynomial that is everywhere non-zero must be a constant polynomial.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_m$ are complex numbers and $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and $Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$. Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then $|z| \geq 1$, and therefore

$$\begin{aligned} \left| \frac{Q(z)}{P_m(z)} \right| &= \frac{1}{|a_m z|} \left| \frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1} \right| \\ &\leq \frac{1}{|a_m|\,|z|} \left(\left| \frac{a_0}{z^{m-1}} \right| + \left| \frac{a_1}{z^{m-2}} \right| + \cdots + |a_{m-1}| \right) \\ &\leq \frac{1}{|a_m|\,|z|} (|a_0| + |a_1| + \cdots + |a_{m-1}|) \leq \frac{R}{|z|} < 1. \end{aligned}$$

It follows that $|P(z) - P_m(z)| < |P_m(z)|$ for all complex numbers z satisfying $|z| > R$.

For each non-negative real number r, let

$$\gamma_r\colon [0,1] \to \mathbb{C} \quad \text{and} \quad \varphi_r\colon [0,1] \to \mathbb{C}$$

be the closed paths defined such that $\gamma_r(t) = P(r\exp(2\pi i t))$ and

$$\varphi_r(t) = P_m(r\exp(2\pi i t)) = a_m r^m \exp(2\pi i m t)$$

for all $t \in [0,1]$. If $r > R$ then $|\gamma_r(t) - \varphi_r(t)| < |\varphi_r(t)|$ for all $t \in [0,1]$. It then follows from the Dog-Walking Lemma that $n(\gamma_r, 0) = n(\varphi_r, 0) = m$ whenever $r > R$.

Now if the polynomial P is everywhere non-zero then the function sending each non-negative real number r to the winding number $n(\gamma_r, 0)$ of the closed path γ_r about zero is a continuous function on the set of non-negative real numbers. But any continuous integer-valued function on an interval is necessarily constant. It follows that $n(\gamma_r, 0) = n(\gamma_0, 0)$ for all positive real-numbers r. But γ_0 is the constant path defined by $\gamma_0(t) = P(0)$ for all $t \in [0,1]$, and therefore $n(\gamma_0, 0) = 0$. It follows that is the polynomial P is everywhere non-zero then $n(\gamma_r, 0) = 0$ for all non-negative real numbers r. But we have shown that $n(\gamma_r, 0) = m$ for sufficiently large values of r, where m is the degree of the polynomial P. It follows that if the polynomial P is everywhere non-zero, then it must be a constant polynomial. The result follows.

2. (a) [*Standard definition, but not stated exactly as below in course notes.*]

Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0, where two loops γ_1 and γ_2 are in the same based homotopy class if and only if $\gamma_1 \simeq \gamma_2$ rel $\{0, 1\}$. Then $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0, where $\gamma_1.\gamma_2$ denotes the concatenation of the loops γ_1 and γ_2. This group is the *fundamental group* of X based at the point x_0. The identity element of the fundamental loop is represented by the constant loop at the basepoint x_0. The inverse of a loop $\gamma: [0, 1] \to X$ is represented by the loop $\gamma^{-1}: [0, 1] \to X$, where $\gamma^{-1}(t) = \gamma(1 - t)$ for all $t \in [0, 1]$.

(b) [*Standard definition.*]

A topological space X is said to be *simply-connected* if it is path-connected, and any continuous map $f: \partial D \to X$ mapping the boundary circle ∂D of a closed disc D into X can be extended continuously over the whole of the disk.

(c) [*Bookwork, the proof of a theorem included in the course notes, which on the annual examination paper for module MA421 in 2009 but has not appeared on examination papers subsequently.*]

We must show that any continuous function $f: \partial D \to X$ defined on the unit circle ∂D can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(U)$ and $f^{-1}(V)$ of U and V are open in ∂D (since f is continuous), and $\partial D = f^{-1}(U) \cup f^{-1}(V)$. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that any arc in ∂D whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(U)$ and $f^{-1}(V)$. Choose points $z_1, z_2, \ldots, z_n$ around ∂D such that the distance from z_i to z_{i+1} is less than δ for $i = 1, 2, \ldots, n - 1$ and the distance from z_n to z_1 is also less than δ. Then, for each i, the short arc joining z_{i-1} to z_i is mapped by f into one or other of the open sets U and V.

Let x_0 be some point of $U \cap V$. Now the sets U, V and $U \cap V$ are all path-connected. Therefore we can choose paths $\alpha_i: [0, 1] \to X$ for $i = 1, 2, \ldots, n$ such that $\alpha_i(0) = x_0$, $\alpha_i(1) = f(z_i)$, $\alpha_i([0, 1]) \subset U$ whenever $f(z_i) \in U$, and $\alpha_i([0, 1]) \subset V$ whenever $f(z_i) \in V$. For convenience let $\alpha_0 = \alpha_n$.

Now, for each i, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points z_{i-1} and z_i and by the short arc joining z_{i-1} to z_i. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary ∂T_i of T_i into a simply-connected space can be extended continuously over the whole of T_i. In particular, let F_i be the function on ∂T_i defined by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in T_i \cap \partial D, \\ \alpha_{i-1}(t) & \text{if } z = tz_{i-1} \text{ for any } t \in [0,1], \\ \alpha_i(t) & \text{if } z = tz_i \text{ for any } t \in [0,1], \end{cases}$$

Note that $F_i(\partial T_i) \subset U$ whenever the short arc joining z_{i-1} to z_i is mapped by f into U, and $F_i(\partial T_i) \subset V$ whenever this short arc is mapped into V. But U and V are both simply-connected. It follows that each of the functions F_i can be extended continuously over the whole of the sector T_i. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect, and can therefore be pieced together to yield a continuous map defined over the the whole of the closed disk D which extends the map f, as required.

3. (a) [*Standard Definitions.*] Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) [*Not bookwork, though similar but not identical scenarios are likely to be discussed in the final weeks of teaching.*]

The map $f: H \to \mathbf{C} \setminus \{0\}$ is not a covering map. There is no evenly covered open neighbourhood of i, where $i = \sqrt{-1}$.

One way of verifying this without engaging with the definition of evenly covered open sets is to consider the path $\gamma: [0,6] \to \mathbb{C}$ defined such that $\gamma(t) = e^{\pi i t/2}$ for all $t \in [0,1]$. Consider the path $\sigma: [0,6] \to \mathbf{C}$ defined such that $\sigma(t) = e^{\pi i t/10}$ for all $t \in [0,6]$. Then $f(\sigma(t)) = \gamma(t)$ for all $t \in [0,6]$. If the map $f: H \to \mathbb{C} \setminus \{0\}$ were a covering map then the path γ would have a uniquely determined lift $\tilde{\gamma}$ starting at 0 for which $\tilde{\gamma}([0,6]) \subset H$. The uniqueness of lifts determined according to the Path-Lifting Theorem would ensure that $\tilde{\gamma}(t) = \sigma(t)$ for all $t \in [0,6]$, and therefore $\sigma(t) \in H$ for all $t \in [0,6]$. But $\sigma(t) \notin H$ when $t > 5$. It follows that the conclusions for the Path-Lifting Theorem are not valid for the map f, and therefore the map $f: H \to \mathbb{C} \to \{0\}$ cannot be a covering map.

(c) [*Bookwork, a theorem included in the course notes, often examined, and examined in MA342R in 2017.*]

Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z

into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}: \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required.

4. (a) [*Bookwork, but explained in the discussion following the definition of free and properly discontinuous group actions, without being isolated as a separate numbered lemma in its own right.*]

Let $g \in G$, and let e denote the identity element of G. Then $\theta_{g^{-1}} \circ \theta_g = \theta_{g^{-1}g} = \theta_e$ and $\theta_g \circ \theta_{g^{-1}} = \theta_{gg^{-1}} = \theta_e$, and thus $\theta_{g^{-1}} \circ \theta_g$ and $\theta_g \circ \theta_{g^{-1}}$ are both equal to the identity map of G. It follows that $\theta_{g^{-1}} = \theta_g^{-1}$. Thus the continuous map θ_g has a continuous inverse $\theta_{g^{-1}}$, and is thus a homeomorphism.

(b) [*The proof is essentially bookwork but only occurs in distributed notes in the particular case where the group acts freely and properly discontinuously, as the first paragraph of the proof of the result that the quotient map onto the orbit space in that case is a covering map.*]

The quotient map $q: X \to X/G$ is surjective. Let V be an open set in X. Then $q^{-1}(q(V))$ is the union $\bigcup_{g \in G} \theta_g(V)$ of the open sets $\theta_g(V)$ as g ranges over the group G, since $q^{-1}(q(V))$ is the subset of X consisting of all elements of X that belong to the orbit of some element of V. But any union of open sets in a topological space is an open set. We conclude therefore that if V is an open set in X then $q(V)$ is an open set in X/G.

(c) [*Not bookwork.*]

For all $z \in S$ and ψ satisfying $0 < \varphi < 2\pi$, let $A_{z,\varphi} = \{ze^{i\psi} : -\frac{1}{2}\varphi \leq \psi \leq \frac{1}{2}\varphi\}$. The subset $A_{z,\varphi}$ of S is then a closed arc of the circle S centred on z and subtending an angle of φ at 0.

If the group $\mathbb{Z}$ of integers under addition were to act freely and properly discontinuously on S in the manner specified, then, given any point z of S, there would exist an open set U in S such that $z \in U$ and $\theta_n(U) \cap U = \emptyset$ for all non-zero integers n. A real number φ satisfing $0 < \varphi < 2\pi$ could then be chosen small enough to ensure that $A_{z,\varphi} \subset U$. The closed arcs $\theta_n(A_{z,\varphi})$ as n ranges over the set of integers would then be disjoint arcs of the circle, each subtending an angle φ. Let m is an integer chosen such that $2\pi/m < \varphi$. Then the circle S cannot include m or more such arcs. This contradiction demonstrates that the group $\mathbb{Z}$ does not act freely and properly discontinuously on the circle S in the manner specified.

(d) [*Not bookwork.*]

Let γ be a loop in S based at 1. Then $\gamma: [0,1] \to S$ is continuous, and $\gamma(0) = \gamma(1) = 1$. There exists a continuous path $\tilde{\gamma}: [0,1] \to S$

for which $\tilde{\gamma}(0) = 1$ and $p \circ \tilde{\gamma} = \gamma$. Also the map from $\mathbb{R}$ to S mapping each real number t to $e^{2\pi i t}$ is also a covering map (where $i = \sqrt{-1}$), and therefore there exists a continuous function $\sigma\colon [0,1] \to \mathbb{R}$ with the properties that $\sigma(0) = 0$ and $\exp(2\pi i \sigma(t)) = \gamma(t)$ for all $t \in [0,1]$. Then $p(\exp(2\pi i \sigma(t)/m)) = \gamma(t)$ for all $t \in [0,1]$ and the lift $\tilde{\gamma}$ is uniquely determined by its starting point. It follows that $\tilde{\gamma}(t) = \exp(2\pi i \sigma(t)/m)$ for all $t \in [0,1]$, It follows that the path $\tilde{\gamma}$ is a loop based at 1 if and only if $\sigma(1)/m$ is an integer.

Now the standard isomorphism from $\pi_1(S,1)$ to $\mathbb{Z}$ sends the based homotopy class $[\gamma]$ of the loop γ to its winding number $\sigma(1)$ about zero. Moroever $[\gamma] \in p_\#(\pi_1(S,1))$ if and only if the lift $\tilde{\gamma}$ of γ is itself a loop in S. It follows that the standard isomorphism from $\pi_1(S,1)$ to $\mathbb{Z}$ maps the subgroup $p_\#(\pi_1(S,1))$ of $\pi_1(S,1)$ onto the subgroup $m\mathbb{Z}$. of integer multiples of the positive integer m. It follows that $\pi_1(S,1)/p_\#(\pi_1(S,1)) \cong \mathbb{Z}/m\mathbb{Z}$, as required.