Additional Problem for MA2C03 concerning Vector Algebra

6. Let A, B, C, D, E, F G and H be eight points in three-dimensional Euclidean space whose Cartesian coordinates are as as follows:—

$$A = (1, 3, 2), \quad B = (3, 7, 3), \quad C = (4, 5, 1), \quad D = (6, 9, 2),$$

$$E = (1, 4, 7), \quad F = (3, 8, 8), \quad G = (4, 6, 6), \quad H = (6, 10, 7).$$

Note that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = \mathbf{u},$$
$$\overrightarrow{AC} = \overrightarrow{BD} = \overrightarrow{EG} = \overrightarrow{FH} = \mathbf{v},$$
$$\overrightarrow{AE} = \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} = \mathbf{w},$$

where

$$\mathbf{u} = (2, 4, 1), \quad \mathbf{v} = (3, 2, -1), \quad \mathbf{w} = (0, 1, 5).$$

It follows that A, B, C, D, E, F, G and H are the vertices of a parallelepiped in three-dimensional Euclidean space.

(a) Calculate the length of the line segments BG BH, and the cosine of the angle between these two line segments at the point B.

(6 points)

(b) Calculate the equation of the plane passing through the points A, B and F, expressing the equation of the plane in the form $ax + by + cz = k$ for appropriate real constants a, b, c and k.

(8 points)

(c) Find the volume of the parallelepiped with vertices at A, B, C, D, E, F, G and H.

(6 points)