Sample Questions

5. In this question, all graphs are undirected graphs.

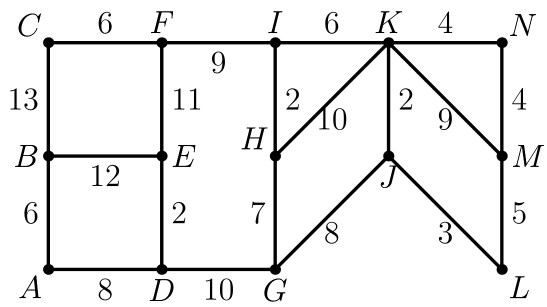
   (a) Let (V, E) be a connected graph, and let v and w be vertices of this graph that determine an edge $v\,w$ of the graph. Let $E' = E \setminus \{v\,w\}$. Suppose that the edge $v\,w$ forms part of a circuit in the graph (V, E). Prove that (V, E') is a connected graph.

   (8 points)

   (b) Consider the connected graph with vertices A, B, C, D, E, F, G, H, I, J, K, L, M and N and with edges, listed with associated costs, in the following table:

DE	HI	JK	JL	KN	MN	LM	AB	CF	IK
2	2	2	3	4	4	5	6	6	6
GH	AD	GJ	FI	KM	DG	HK	EF	BE	BC
7	8	8	9	9	10	10	11	12	13

   This graph is depicted below.



   Determine the minimum spanning tree generated by Kruskal's Algorithm, starting from the vertex A, where that algorithm is applied with the edges ordered as specified above. In particular, list the edges in the order in which they are added to the Kruskal subtree generated by the application algorithm, so that edges AB and AD as the first and second edges added to the Kruskal subtree.

   (12 points)
   (End of Question)

6. Let A, B, C, D, E, F G and H be eight points in three-dimensional Euclidean space whose Cartesian coordinates are as as follows:—

$$A = (1,2,2), \quad B = (3,3,3), \quad C = (4,5,0), \quad D = (6,6,1),$$

$$E = (5,5,7), \quad F = (7,6,8), \quad G = (8,8,5), \quad H = (10,9,6).$$

Note that

$$\begin{aligned} \overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} &= \mathbf{u}, \\ \overrightarrow{AC} = \overrightarrow{BD} = \overrightarrow{EG} = \overrightarrow{FH} &= \mathbf{v}, \\ \overrightarrow{AE} = \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} &= \mathbf{w}, \end{aligned}$$

where

$$\mathbf{u} = (2,1,1), \quad \mathbf{v} = (3,3,-2), \quad \mathbf{w} = (4,3,5).$$

It follows that A, B, C, D, E, F, G and H are the vertices of a parallelepiped in three-dimensional Euclidean space.

(a) Calculate the length of the line segments BD, BG, and the cosine of the angle between these two line segments at the point B.

(6 points)

(b) Calculate the equation of the plane passing through the points A, B and C, expressing the equation of the plane in the form $ax + by + cz = k$ for appropriate real constants a, b, c and k.

(8 points)

(c) Find the volume of the parallelepiped with vertices at A, B, C, D, E, F, G and H.

(6 points)
(End of Question)

7. In this question we consider differential equations of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\cos kx + h\sin kx$$

where the real numbers b, c, g, h and k are constants. We suppose also that the auxiliary polynomial $z^2 + bz + c$ of this differential equation has a repeated root r.

(a) Prove that if $y_C = (A+Bx)e^{rx}$, where A and B are real constants, then y_C satisfies the associated homogeneous linear differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

(6 points)

(b) Let y_P be a function of the form

$$y_P = u\cos kx + v\sin kx,$$

where u and v are constants to be determined. Calculate algebraic expressions that represent u and v in terms of b, c, g, h and k so as to ensure that y_P is a solution of the differential equation specified at the beginning of the question. [Your work should be fully justified.]

(10 points)

(c) Determine the form of the general solution of differential equations that have the form specified at the beginning of the question.

(4 points)
(End of Question)

8. (a) Find an integer x such that $x \equiv 2$ (mod. 5), $x \equiv 3$ (mod. 7) and $x \equiv 8$ (mod. 11).

(12 points)

(b) Find the value of the unique integer x satisfying $0 \leq x < 13$ for which $8^{200000023} \equiv x$ (mod. 13).

(8 points)
(End of Question)