Module MA2C03: Discrete Mathematics
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Section 36 to 38: Vectors

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Contents

36 Vectors in Three-Dimensional Space

36.1 Vector Quantities

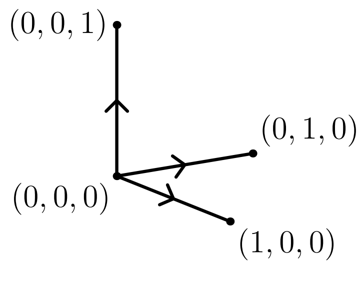
Vector quantities are objects that have attributes of magnitude and direction. Many physical quantities, such as velocity, acceleration, force, electric field and magnetic field are examples of vector quantities. Displacements between points of space may also be represented using vectors.

Quantities that do not have a sense of direction associated with them are known as *scalar quantities*. Such physical quantities as temperature and energy are scalar quantities. Scalar quantities are usually represented by real numbers.

36.2 Displacement Vectors

Points of three-dimensional space may be represented, in a Cartesian coordinate system, by ordered triples (x, y, z) of real numbers. Two ordered triples (x_1, y_1, z_1) and (x_2, y_2, z_2) of real numbers represent the same point of three-dimensional space if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$. The point whose Cartesian coordinates are given by the ordered triple $(0, 0, 0)$ is referred to as the *origin* of the Cartesian coordinate system.

It is usual to employ a Coordinate system such that the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are situated at a unit distance from the origin $(0, 0, 0)$, and so that the three lines that join the origin to these points are mutually perpendicular. Moreover it is customary to require that if the thumb of your right hand points in the direction from the origin to the point $(1, 0, 0)$, and if the first finger of that hand points in the direction from the origin to the point $(0, 1, 0)$, and if the second finger of that hand points in a direction perpendicular to the directions of the thumb and first finger, then that second finger points in the direction from the origin to the point $(0, 0, 1)$. (Thus if, at a point on the surface of the earth, away from the north and south pole, the point $(1, 0, 0)$ is located to the east of the origin, and the point $(0, 1, 0)$ is located to the north of the origin, then the point $(0, 0, 1)$ will be located above the origin.



Let P_1, P_2, P_3 and P_4 denote four points of three-dimensional space, represented in a Cartesian coordinate system by ordered triples as follows:

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2),$$

$$P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

The *displacement vector* $\overrightarrow{P_1, P_2}$ from the point P_1 to the point P_2 measures the distance and the direction in which one would have to travel in order to get from P_1 to P_2. This displacement vector may be represented by an ordered triple as follows:

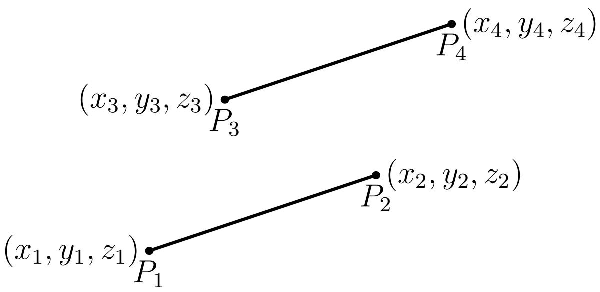
$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

The displacement vector $\overrightarrow{P_3P_4}$ is *equal* to the displacement vector $\overrightarrow{P_1P_2}$ if and only if

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3, \\ y_2 - y_1 &= y_4 - y_3, \\ z_2 - z_1 &= z_4 - z_3, \end{aligned}$$

in which case we represent the fact that these two displacement vectors are equal by writing

$$\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}.$$



Note: $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$ and therefore

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3, \\ y_2 - y_1 &= y_4 - y_3, \\ z_2 - z_1 &= z_4 - z_3, \end{aligned}$$

Now

$$\begin{aligned} & x_2 - x_1 = x_4 - x_3 \\ \iff & x_2 + x_3 = x_1 + x_4 \\ \iff & x_3 - x_1 = x_4 - x_2 \end{aligned}$$

Thus

$$x_2 - x_1 = x_4 - x_3 \quad \text{if and only if} \quad x_3 - x_1 = x_4 - x_2.$$

Similarly

$$y_2 - y_1 = y_4 - y_3 \quad \text{if and only if} \quad y_3 - y_1 = y_4 - y_2,$$

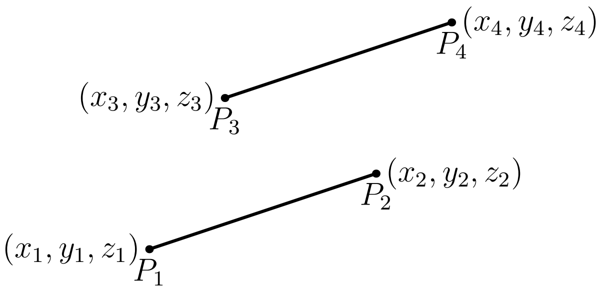
$$z_2 - z_1 = z_4 - z_3 \quad \text{if and only if} \quad z_3 - z_1 = z_4 - z_2.$$

Geometrically, these two displacement vectors are equal if and only if P_1, P_2, P_4 and P_3 are the vertices of a parallelogram in three-dimensional space, in which case

$$\begin{aligned} x_3 - x_1 &= x_4 - x_2, \\ y_3 - y_1 &= y_4 - y_2, \\ z_3 - z_1 &= z_4 - z_2, \end{aligned}$$

and thus

$$\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$



Note:

$$\left\{ \begin{array}{rcl} x_2 - x_1 &=& x_4 - x_3, \\ y_2 - y_1 &=& y_4 - y_3, \\ z_2 - z_1 &=& z_4 - z_3, \end{array} \right\} \iff \left\{ \begin{array}{rcl} x_3 - x_1 &=& x_4 - x_2, \\ y_3 - y_1 &=& y_4 - y_2, \\ z_3 - z_1 &=& z_4 - z_2, \end{array} \right\}.$$

These displacement vectors may be regarded as objects in their own right, and denoted by symbols of their own: we use a symbol such as $\mathbf{u}$ to denote the displacement vector $\overrightarrow{P_1P_2}$ from the point P_1 to the point P_2, and we write $\mathbf{u} = (u_x, u_y, u_z)$ where $u_x = x_2 - x_1$, $u_y = y_2 - y_1$ and $u_z = z_2 - z_1$.

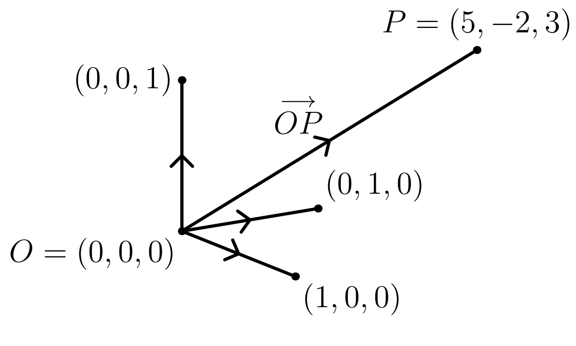
Remark It is traditional in mathematics texts to denote vectors with boldface letters (e.g., $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$). The traditional way of writing the equivalent on paper or blackboards is to put a tilde underneath the letter (e.g., $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{w}$). When vectors are taught at second level, they are often written with an arrow on top (e.g., $\vec{u}$, $\vec{v}$, $\vec{w}$).

Vectors are used to record displacements and positions. Let P_1 and P_2 be points of three-dimensional Euclidean space with Cartesian coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. The *displacement vector* $\overrightarrow{P_1P_2}$ from P_1 to P_2 is the vector with components

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

that contains the information necessary to determine the distance of P_2 from P_1 and also the direction of P_2 in relation to P_1.

A Cartesian coordinate system in three-dimensional space determines an *origin* O that is the point whose Cartesian coordinates are $(0, 0, 0)$. The position of a point P of the plane with respect to the origin is specified by a vector $\mathbf{r}$, where $\mathbf{r} = \overrightarrow{OP}$. This vector $\mathbf{r}$ is the *position vector* of the point P. It represents the displacement of the point P from the origin of the Cartesian coordinate system.



Note: The position vector $\overrightarrow{OP}$ of the point P, where $P = (5, -2, 3)$.

36.3 The Parallelogram Law of Vector Addition

Let P_1, P_2, P_3 and P_4 denote four points of three-dimensional space, located such that $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$. Then (as we have seen) $\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}$ and the geometrical figure $P_1P_2P_4P_3$ is a parallelogram. Let

$$\mathbf{u} = \overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}, \quad \mathbf{v} = \overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$

Let

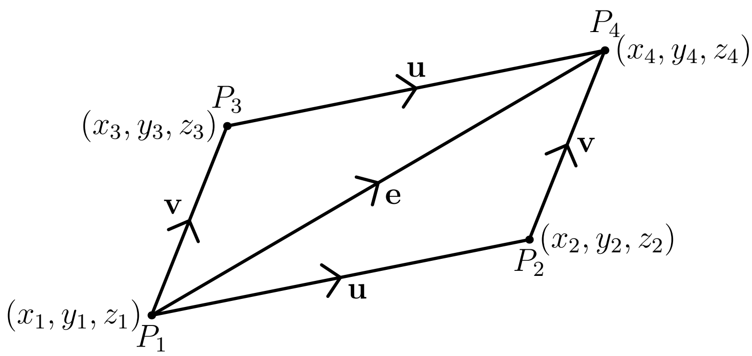
$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2),$$
$$P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

Then $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$, where

$$\begin{aligned} u_x &= x_2 - x_1 = x_4 - x_3, \\ u_y &= y_2 - y_1 = y_4 - y_3, \\ u_z &= z_2 - z_1 = z_4 - z_3, \\ v_x &= x_3 - x_1 = x_4 - x_2, \\ v_y &= y_3 - y_1 = y_4 - y_2, \\ v_z &= z_3 - z_1 = z_4 - z_2, \end{aligned}$$

Let $\mathbf{e} = \overrightarrow{P_1P_4}$. Then $\mathbf{e} = (e_x, e_y, e_z)$, where

$$\begin{aligned} e_x &= x_4 - x_1 = u_x + v_x, \\ e_y &= y_4 - y_1 = u_y + v_y, \\ e_z &= z_4 - z_1 = u_z + v_z, \end{aligned}$$



Note: $\mathbf{u} = \overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$ and $\mathbf{v} = \overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}$, and

$$\begin{aligned} u_x &= x_2 - x_1 = x_4 - x_3 \quad \&\text{c.}, \\ v_x &= x_3 - x_1 = x_4 - x_2 \quad \&\text{c.}, \\ e_x &= x_4 - x_1 = u_x + v_x \quad \&\text{c.}. \end{aligned}$$

We say that the vector $\mathbf{e}$ is the *sum* of the vectors $\mathbf{u}$ and $\mathbf{v}$, and denote this fact by writing

$$\mathbf{e} = \mathbf{u} + \mathbf{v}.$$

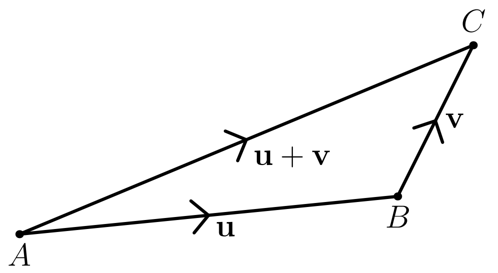
This rule for addition of vectors is known as the *parallelogram rule*, due to its association with the geometry of parallelograms. Note that vectors are added, by adding together the corresponding components of the two vectors. For example,

$$(0, 3, 2) + (4, 8, -5) = (4, 11, -3).$$

Note that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

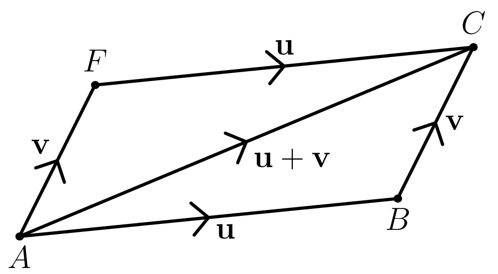
for all points A, B and C of space.



The identity

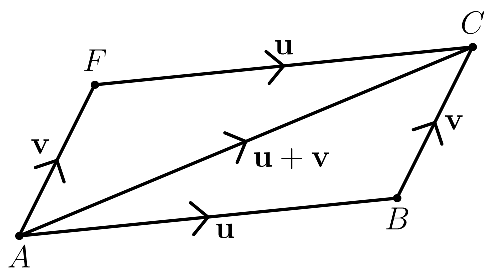
$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

holds for all vectors $\mathbf{u}$ and $\mathbf{v}$ in three-dimensional space.



The identity $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ may be interpreted geometrically as follows. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$, where A, B and C are points of three-dimensional space. Then there exists a point F in three-dimensional space such that $\overrightarrow{AF} = \overrightarrow{BC}$. Then $ABCF$ is a parallelogram, and $\overrightarrow{FC} = \overrightarrow{AB}$. It follows that

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{u} + \mathbf{v}, \\
\overrightarrow{AC} &= \overrightarrow{AF} + \overrightarrow{FC} = \mathbf{v} + \mathbf{u}.
\end{aligned}$$



In Cartesian coordinates

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = (u_x + v_x, u_y + v_y, u_z + v_z),$$

where

$$\mathbf{u} = (u_x, u_y, u_z) \quad \text{and} \quad \mathbf{v} = (v_x, v_y, v_z).$$

Let **u**, **v** and **w** be vectors in three-dimensional space. Then

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

This identity may be verified algebraically as follows. Let

$$\mathbf{u} = (u_x, u_y, u_z), \quad \mathbf{v} = (v_x, v_y, v_z), \quad \mathbf{w} = (w_x, w_y, w_z).$$

Then

$$\mathbf{u} + \mathbf{v} = (u_x + v_x, u_y + v_y, u_z + v_z), \quad \mathbf{v} + \mathbf{w} = (v_x + w_x, v_y + w_y, v_z + w_z),$$

and therefore

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (u_x + v_x + w_x, u_y + v_y + w_y, u_z + v_z + w_z) \\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w}).\end{aligned}$$

This identity can be interpreted geometrically as follows. Let A be a point of three-dimensional space. Then there exist points B, C and D of three-dimensional space such that

$$\mathbf{u} = \overrightarrow{AB}, \quad \mathbf{v} = \overrightarrow{BC}, \quad \mathbf{w} = \overrightarrow{CD}.$$

Then

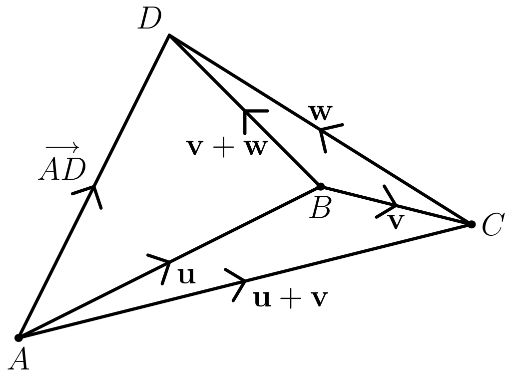
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{u} + \mathbf{v} \quad \text{and} \quad \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \mathbf{v} + \mathbf{w},$$

and hence

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AC} + \overrightarrow{CD} = (\mathbf{u} + \mathbf{v}) + \mathbf{w}, \\ \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).\end{aligned}$$

and thus

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \overrightarrow{AD} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$



Note:

$$
\begin{aligned}
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD} \\
&= \mathbf{u}+(\mathbf{v}+\mathbf{w}).
\end{aligned}
$$

The *zero vector* $\mathbf{0}$ is the vector $(0,0,0)$ that represents the displacement from any point in space to itself. The zero vector $\mathbf{0}$ has the property that

$$\mathbf{u}+\mathbf{0}=\mathbf{u}$$

for all vectors $\mathbf{u}$.

Given any vector $\mathbf{u}$, there exists a vector, denoted by $-\mathbf{u}$, characterized by the property that

$$\mathbf{u}+(-\mathbf{u})=\mathbf{0}.$$

If $\mathbf{u}=(u_x,u_y,u_z)$, then $-\mathbf{u}=(-u_x,-u_y,-u_z)$.

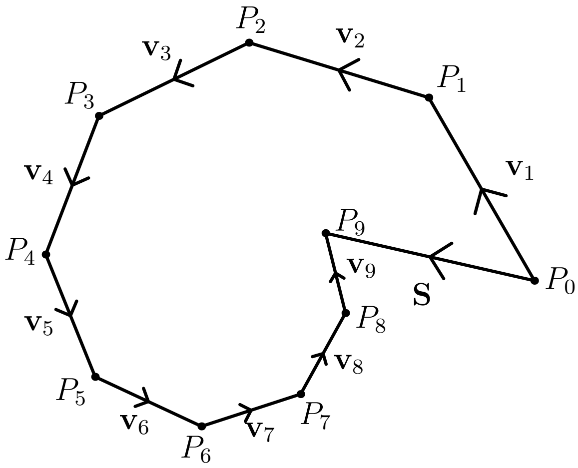
We have shown that addition of vectors satisfies the Commutative Law and the Associative Law.

Given three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we define their sum $\mathbf{u}+\mathbf{v}+\mathbf{w}$ so that

$$\mathbf{u}+\mathbf{v}+\mathbf{w}=(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

More generally, let $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ be vectors in three-dimensional space, and let P_0 be a point of three-dimensional space. Then there exist points $P_1,P_2,\ldots,P_k$ such that $\mathbf{v}_j=\overrightarrow{P_{j-1},P_j}$ for $j=1,2,\ldots,n$. We define the sum of the vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ such that

$$\mathbf{v}_1+\mathbf{v}_2+\ldots+\mathbf{v}_k=\overrightarrow{P_0P_k}.$$



Note: case $k = 9$, with

$$\mathbf{S} = \overrightarrow{P_0P_9} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_9,$$

where $\mathbf{v}_j = \overrightarrow{P_{j-1}P_j}$ for $j = 1, 2, \ldots, 9$.

Lemma 36.1 *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in three-dimensional space, where $\mathbf{v}_j = (v_x^{(j)}, v_y^{(j)}, v_z^{(j)})$ for $j = 1, 2, \ldots, n$, and let*

$$\mathbf{S} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k.$$

Then $\mathbf{S} = (S_x, S_y, S_z)$, where

$$S_x = \sum_{j-1}^{n} v_x^{(j)}, \quad S_y = \sum_{j-1}^{n} v_y^{(j)}, \quad S_z = \sum_{j-1}^{n} v_z^{(j)}.$$

Proof Let points P_0 be a point in three-dimensional space, and let points $P_1, P_2, \ldots, P_k$ be successively constructed such that $\mathbf{v}_j = \overrightarrow{P_{j-1}P_j}$ for $j = 1, 2, \ldots, k$. Let $P_j = (x_j, y_j, z_j)$ for $j = 0, 1, 2, \ldots, k$. Then

$$\mathbf{v}_j = (x_j - x_{j-1}, y_j - y_{j-1}, z_j - z_{j-1})$$

for $j = 1, 2, \ldots, k$. Thus if

$$\mathbf{S} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k,$$

then

$$\mathbf{S} = \overrightarrow{P_0P_k} = (x_k - x_0, y_k - y_0, z_k - z_0) = (S_x, S_y, S_z)$$

where

$$
\begin{aligned}
S_x &= x_k - x_0 = \sum_{j=1}^{k} (x_j - x_{j-1}) = \sum_{j-1}^{n} v_x^{(j)}, \\
S_y &= y_k - y_0 = \sum_{j=1}^{k} (y_j - y_{j-1}) = \sum_{j-1}^{n} v_y^{(j)}, \\
S_z &= z_k - z_0 = \sum_{j=1}^{k} (z_j - z_{j-1}) = \sum_{j-1}^{n} v_z^{(j)},
\end{aligned}
$$

as required. ∎

36.4 Scalar Multiples of Vectors

Let $P_0, P_1, P_2, P_3, \ldots$ be an infinite sequence of points in three-dimensional space, where

$$\overrightarrow{P_0P_1} = \overrightarrow{P_1P_2} = \overrightarrow{P_2P_3} = \cdots .$$

Let $\mathbf{v} = \overrightarrow{P_0P_1}$, and let $\mathbf{v} = (v_x, v_y, v_z)$. Then $\overrightarrow{P_jP_{j+1}} = \mathbf{v}$ for all positive integers j. It then follows immediately from Lemma 36.1 that

$$
\begin{aligned}
\mathbf{v} &= \overrightarrow{P_0P_1} = (v_x, v_y, v_z) \\
\mathbf{v} + \mathbf{v} &= \overrightarrow{P_0P_2} = (2v_x, 2v_y, 2v_z) \\
\mathbf{v} + \mathbf{v} + \mathbf{v} &= \overrightarrow{P_0P_3} = (3v_x, 3v_y, 3v_z) \\
\mathbf{v} + \mathbf{v} + \mathbf{v} + \mathbf{v} &= \overrightarrow{P_0P_4} = (4v_x, 4v_y, 4v_z) \\
&\vdots
\end{aligned}
$$

It follows that

$$\mathbf{v} + \mathbf{v} = 2\mathbf{v}, \quad \mathbf{v} + \mathbf{v} + \mathbf{v} = 3\mathbf{v}, \quad \mathbf{v} + \mathbf{v} + \mathbf{v} + \mathbf{v} = 4\mathbf{v}, \quad \text{\&c.},$$

where

$$k\mathbf{v} = \overrightarrow{P_0P_k} = (kv_x, kv_y, kv_z)$$

for all non-negative integers k.

More generally, let $\mathbf{v}$ be a vector, represented by the ordered triple (v_x, v_y, v_z), and let t be a real number. We define $t\mathbf{v}$ to be the vector represented by the ordered triple (tv_x, tv_y, tv_z). Thus $t\mathbf{v}$ is the vector obtained on multiplying each of the components of $\mathbf{v}$ by the real number t. The vector $t\mathbf{v}$ is said to be a *scalar multiple* of the vector $\mathbf{v}$, obtained by multiplying the vector $\mathbf{v}$ by the *scalar* t.

It follows from this definition of scalar multiples of vectors that

$$(s+t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}, \quad t(\mathbf{u}+\mathbf{v}) = t\mathbf{u} + t\mathbf{v}, \text{ and } s(t\mathbf{u}) = (st)\mathbf{u},$$

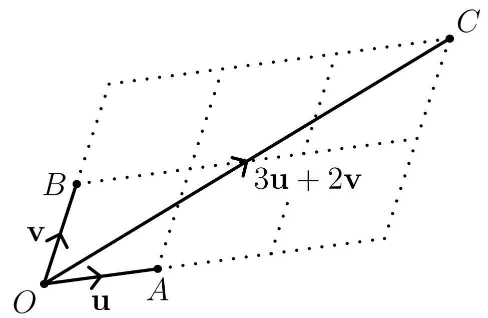
for all vectors $\mathbf{u}$ and $\mathbf{v}$ and real numbers s and t. Also $1\mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$.

36.5 Linear Combinations of Vectors

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in three-dimensional space. A vector $\mathbf{v}$ is said to be a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there exist real numbers $t_1, t_2, \ldots, t_k$ such that

$$\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k.$$

Let O, A and B be distinct points of three-dimensional space that are not collinear (i.e., that do not all lie on any one line in that space). The displacement vector $\overrightarrow{OP}$ of a point P in three-dimensional space is a linear combination of the displacement vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ if and only if the point P lies in the unique plane that contains the points O, A and B.



Note: $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OB} = \mathbf{v}$, $\overrightarrow{OC} = 3\mathbf{u} + 2\mathbf{v}$.

36.6 Linear Dependence and Independence

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are said to be be *linearly dependent* if there exist real numbers $t_1, t_2, \ldots, t_k$, not all zero, such that

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are not linearly dependent, then they are said to be *linearly independent*.

Note that if any of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ is the zero vector, then those vectors are linearly dependent. Indeed if $\mathbf{v}_i = \mathbf{0}$ then these vectors satisfy a relation of the form

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

where $t_j = 0$ if $j \neq i$ and $t_i \neq 0$. We conclude that, in any list of linearly independent vectors, the vectors are all non-zero.

Also if any vector in the list $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ is a scalar multiple of some other vector in the list then these vectors are linearly dependent. Indeed suppose that $\mathbf{v}_k = t\mathbf{v}_j$, where $j \neq k$. Then $t\mathbf{v}_j - \mathbf{v}_k = \mathbf{0}$, and thus

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0},$$

where $t_j = t$, $t_k = -1$ and $t_i = 0$ whenever i is distinct from both j and k.

If a vector $\mathbf{v}$ is expressible as a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$ are linearly dependent. For there exist real numbers $s_1, \ldots, s_k$ such that

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k.$$

But then

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k - \mathbf{v} = \mathbf{0}.$$

Theorem 36.2 *Let* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be three vectors in three-dimensional space which are linearly independent. Then, given any vector* $\mathbf{s}$, *there exist unique real numbers* p, q *and* r *such that*

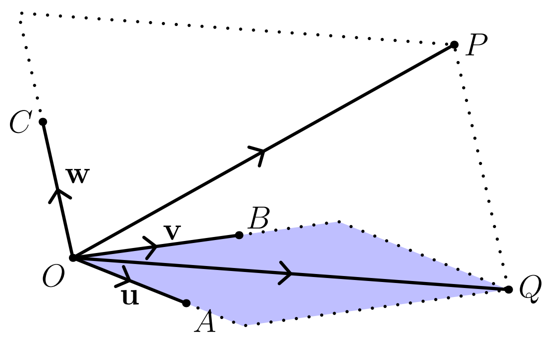
$$\mathbf{s} = p\mathbf{u} + q\mathbf{v} + r\mathbf{w}.$$

Proof First we note that the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are all non-zero, and none of these vectors is a scalar multiple of another vector in the list. Let O denote the origin of a Cartesian coordinate system, and let A, B, C and P denote the points of three-dimensional space whose displacement vectors from the origin O are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{s}$ respectively. The points O, A, B and C are then all distinct, and there is a unique plane which contains the three points O, A and B. This plane OAB consists of all points whose displacement vector from the origin is expressible in the form $p\mathbf{u} + q\mathbf{v}$ for some real numbers p and q.

Now the vector $\mathbf{w}$ is not expressible as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and therefore the point C does not belong to the plane OAB. Therefore the line parallel to OC that passes through the point P is not parallel to the plane OAP. This line therefore intersects the plane in a single point Q. Now the displacement vector of the point Q from the origin is of the form $\mathbf{s} - r\mathbf{w}$ for some uniquely-determined real number r. But it is also expressible in the form $p\mathbf{u} + q\mathbf{v}$ for some uniquely-determined real numbers p and q, because Q lies in the plane OAB. Thus there exist real numbers p, q and r such that $\mathbf{s} - r\mathbf{w} = p\mathbf{u} + q\mathbf{v}$. But then

$$\mathbf{s} = p\mathbf{u} + q\mathbf{v} + r\mathbf{w}.$$

Moreover the point Q and thus the real numbers p, q and r are uniquely determined by $\mathbf{s}$, as required. ∎



Note:

$$\begin{aligned} \overrightarrow{OP} &= \mathbf{s} = 1.5\,\mathbf{u} + 1.6\,\mathbf{v} + 1.8\,\mathbf{w}, \\ \overrightarrow{OQ} &= 1.5\,\mathbf{u} + 1.6\,\mathbf{v}, \\ \overrightarrow{QP} &= 1.8\,\mathbf{w}. \end{aligned}$$

It follows from this theorem that no linearly independent list of vectors in three-dimensional space can contain more than three vectors, since were there a fourth vector in the list, then it would be expressible as a linear combination of the other three, and the vectors would not then be linearly independent.

36.7 Line Segments

Let O be the origin of Cartesian coordinates in three-dimensional Euclidean space, and let P and Q be points of three-dimensional space with position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$. We consider how to specify, in vector notation, the line segment joining the point P to a point Q.

Let R be a point on the line segment PQ whose endpoints are P and Q. Then the vectors $\overrightarrow{OP}$ and $\overrightarrow{OR}$ are collinear, and indeed $\overrightarrow{PR} = t\,\overrightarrow{PQ}$ for some real number t satisfying $0 \leq t \leq 1$. Now $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$ Thus a point with position vector $\mathbf{r}$ lies on the line segment joining P to Q if and only if

$$\mathbf{r} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$$

for some real number t satisfying $0 \leq t \leq 1$. It follows that the set of position vectors of points that lie on the line segment with endpoints P and Q is

$$\{\mathbf{r} : \mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q} \text{ for some } t \in \mathbb{R} \text{ satisfying } 0 \leq t \leq 1\}.$$

37 Real Vector Spaces

37.1 The Definition of a Real Vector Space

Definition A *real vector space* consists of a set V on which are defined a binary operation of *vector addition*, adding any pair of elements $\mathbf{v}$ and $\mathbf{w}$ of V to yield an element $\mathbf{v}+\mathbf{w}$ of V, and an operation of multiplication-by-scalars, multiplying any element $\mathbf{v}$ of V by any real number t to yield an element $t\mathbf{v}$ of V, where these operations of vector addition and multiplication satisfy the following axioms:—

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ for all $\mathbf{v}, \mathbf{w} \in V$;
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there exists a zero element $\mathbf{0}$ of V characterized by the property that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;
4. given any element $\mathbf{v} \in V$, there exists an element $-\mathbf{v}$ of V characterized by the property that $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$,
5. $t(\mathbf{v} + \mathbf{w}) = t\mathbf{v} + t\mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers t;
6. $(s + t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}$ for all $\mathbf{v} \in V$ and for all real numbers s and t;
7. $s(t\mathbf{v}) = (st)\mathbf{v}$ for all $\mathbf{v} \in V$ and for all real numbers s and t;
8. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

The first four axioms in the definition of a vector space are equivalent to the requirement that a vector space be an Abelian group (or commutative group) with respect to the operation of vector addition. Thus a vector space is an Abelian group provided with an additional algebraic operation of multiplication-by-scalars that satisfies the last four axioms listed above.

All the real vector space axioms are satisfied by the set of vectors in three-dimensional Euclidean space, with the standard operations of vector addition and multiplication-by-scalars. Therefore vectors in three-dimensional space constitute a real vector space.

There is a corresponding real vector space whose elements are vectors in the Euclidean plane. Cartesian coordinates of points of the plane are represented as ordered pairs of real numbers. Given points P_1 and P_2 of the plane, where

$$P_1 = (x_1, y_1) \quad \text{and} \quad P_2 = (x_2, y_2),$$

the displacement vector $\overrightarrow{P_1P_2}$ is represented by the ordered pair defined so that

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1).$$

Vector addition and multiplication-by-scalars is defined for vectors in two dimensions in the obvious fashion, so that

$$(v_x, v_y) + (w_x, w_y) = (v_x + w_x, v_y + w_y) \quad \text{and} \quad t(v_x, v_y) = (tv_x, tv_y)$$

for all two-dimensional vectors (v_x, v_y) and (w_x, w_y) and for all real numbers t.

Example Let m be a positive integer, and let V_m be the set of all polynomials with real coefficients consisting of the zero polynomial together with all non-zero polynomials whose degree does not exceed m. (The degree of a polynomial is defined only for non-zero polynomials: it is the degree of the highest term for which the corresponding coefficient is non-zero.) If $p(x)$ and $q(x)$ are polynomials with real coefficients belonging to V_m then so is $p(x) + q(x)$. Also $tp(x)$ is a polynomial belonging to V_m for all (constant) real numbers t. The operation of addition of two polynomials belonging to V_m to yield another polynomial belonging to V_m can be considered to be an operation of "vector addition" on the set V_m. Similarly the operation of multiplying a polynomial by a constant real number can be considered to be an operation of "multiplication by scalars". The set V_m, with all these algebraic operations, is a real vector space: all the axioms in the definition of a vector space as satisfied when the non-zero "vectors" are polynomials whose degree does not exceed m.

37.2 Linear Dependence and Independence in Vector Spaces

Let V be a real vector space. Elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of V are said to be be *linearly dependent* if there exist real numbers $t_1, t_2, \ldots, t_k$, not all zero, such that

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are not linearly dependent, then they are said to be *linearly independent*.

Note that if any of the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ of V is the zero element of V then those elements of V are linearly dependent. Indeed if $\mathbf{v}_i = \mathbf{0}$ then these vectors satisfy a relation of the form

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

where $t_j = 0$ if $j \neq i$ and $t_i \neq 0$. We conclude that, in any list of linearly independent elements of a real vector space V, the vectors are all non-zero.

If an element $\mathbf{v}$ of a real vector space V is expressible as a linear combination of elements $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of V then the elements $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$ are linearly dependent. For there exist real numbers $s_1, \ldots, s_k$ such that

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k.$$

But then

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k - \mathbf{v} = \mathbf{0}.$$

38 Scalar and Vector Products in Three Dimensions

38.1 The Length of Three-Dimensional Vectors

Let P_1 and P_2 be points in space, and let $\mathbf{u}$ denote the displacement vector $\overrightarrow{P_1P_2}$ from the point P_1 to the point P_2. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ then $\mathbf{u} = (u_x, u_y, u_z)$ where $u_x = x_2 - x_1$, $u_y = y_2 - y_1$ and $u_z = z_2 - z_1$.

The *length* (or *magnitude*) of the vector $\mathbf{u}$ is defined to be the distance from the point P_1 to the point P_2. This distance may be calculated using Pythagoras's Theorem. Let $Q = (x_2, y_2, z_1)$ and $R = (x_2, y_1, z_1)$. If the points P_1 and P_2 are distinct, and if $z_1 \neq z_2$, then the triangle P_1QP_2 is a right-angled triangle with hypotenuse P_1P_2, and it follows from Pythagoras's Theorem that

$$P_1P_2^2 = P_1Q^2 + QP_2^2 = P_1Q^2 + (z_2 - z_1)^2.$$

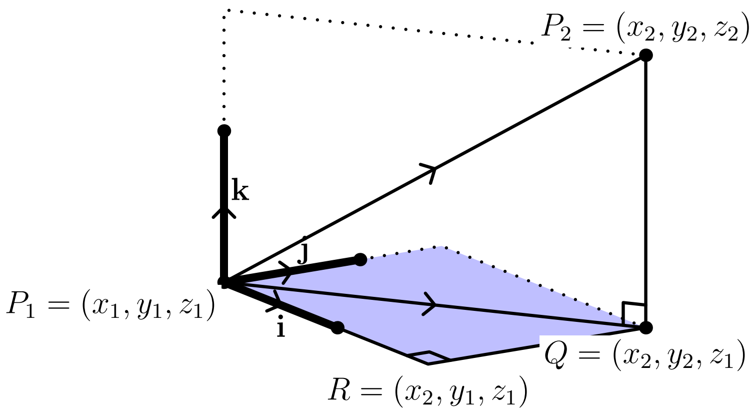
This identity also holds when $P_1 = P_2$, and when $z_1 = z_2$, and therefore holds wherever the points P_1 and P_2 are located.

Similarly

$$P_1Q^2 = P_1R^2 + RQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(since P_1RQ is a right-angled triangle with hypotenuse P_1Q whenever the points P_1, R and Q are distinct), and therefore the length $|\mathbf{u}|$ of the displacement vector $\mathbf{u}$ from the point P_1 to the point P_2 satisfies the equation

$$\begin{aligned} |\mathbf{u}|^2 = P_1P_2^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= u_x^2 + u_y^2 + u_z^2. \end{aligned}$$



Note:

$$\overrightarrow{P_1R} = (x_2 - x_1, 0, 0), \quad \overrightarrow{RQ} = (0, y_2 - y_1, 0), \quad \overrightarrow{QP_2} = (0, 0, z_2 - z_1).$$

In general we define the *length*, or *magnitude*, $|\mathbf{v}|$ of any vector quantity $\mathbf{v}$ by the formula

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

where $\mathbf{v} = (v_x, v_y, v_z)$. This ensures that the length of any displacement vector is equal to the distance between the two points that determine the displacement.

Example The vector $(3, 4, 12)$ is of length 13, since

$$3^2 + 4^2 + 12^2 = 5^2 + 12^2 = 13^2.$$

A vector whose length is equal to one is said to be a *unit vector*.

Let $\mathbf{v}$ be a non-zero vector in three-dimensional space, and let t be a real number.

Note that if $t > 0$ then $t\mathbf{v}$ is a vector, pointing in the same direction as $\mathbf{v}$, whose length is obtained on multiplying the length of $\mathbf{v}$ by the positive real number t.

Similarly if $t < 0$ then $t\mathbf{v}$ is a vector, pointing in the opposite direction to $\mathbf{v}$, whose length is obtained on multiplying the length of $\mathbf{v}$ by the positive real number $|t|$.

38.2 The Scalar Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. The *scalar product* of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be the real number $\mathbf{u} \,.\, \mathbf{v}$ defined by the formula

$$\mathbf{u} \,.\, \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

In particular,

$$\mathbf{u} \,.\, \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2,$$

for any vector $\mathbf{u}$, where $|\mathbf{u}|$ denotes the length of the vector $\mathbf{u}$.

Note that $\mathbf{u} \,.\, \mathbf{v} = \mathbf{v} \,.\, \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Also

$$\begin{aligned} (s\mathbf{u} + t\mathbf{v}) \,.\, \mathbf{w} &= s\mathbf{u} \,.\, \mathbf{w} + t\mathbf{v} \,.\, \mathbf{w}, \\ \mathbf{u} \,.\, (s\mathbf{v} + t\mathbf{w}) &= s\mathbf{u} \,.\, \mathbf{v} + t\mathbf{u} \,.\, \mathbf{w} \end{aligned}$$

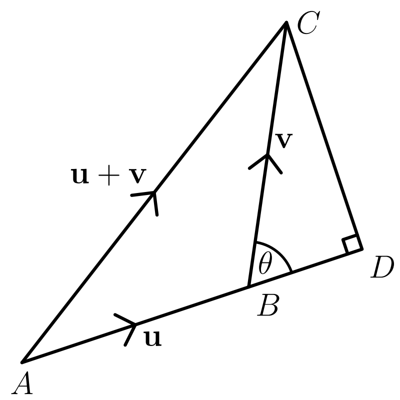
for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and real numbers s and t.

Proposition 38.1 *Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in three-dimensional space. Then their scalar product $\mathbf{u} \,.\, \mathbf{v}$ is given by the formula*

$$\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta,$$

where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Proof Suppose first that the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ is an acute angle, so that $0 < \theta < \frac{1}{2}\pi$. Let us consider a triangle ABC, where $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, and thus $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$. Let ADC be the right-angled triangle constructed as depicted in the figure below, so that the line AD extends AB and the angle at D is a right angle.



Note:

$$\begin{aligned} AD &= |\mathbf{u}| + |\mathbf{v}|\cos\theta, \\ CD &= |\mathbf{v}|\sin\theta, \\ |\mathbf{u}+\mathbf{v}|^2 &= AC^2 = AD^2 + CD^2 \quad \text{(Pythagoras)}. \end{aligned}$$

Then the lengths of the line segments AB, BC, AC, BD and CD may be expressed in terms of the lengths $|\mathbf{u}|$, $|\mathbf{v}|$ and $|\mathbf{u}+\mathbf{v}|$ of the displacement vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ and the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ by means of the following equations:

$$AB = |\mathbf{u}|, \quad BC = |\mathbf{v}|, \quad AC = |\mathbf{u}+\mathbf{v}|,$$

$$BD = |\mathbf{v}|\,\cos\theta \quad \text{and} \quad DC = |\mathbf{v}|\,\sin\theta.$$

Then

$$AD = AB + BD = |\mathbf{u}| + |\mathbf{v}|\cos\theta.$$

The triangle ADC is a right-angled triangle with hypotenuse AC. It follows from Pythagoras' Theorem that

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= AC^2 = AD^2 + DC^2 = (|\mathbf{u}| + |\mathbf{v}|\,\cos\theta)^2 + |\mathbf{v}|\,\sin^2\theta \\ &= |\mathbf{u}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\,\cos\theta + |\mathbf{v}|^2\,\cos^2\theta + |\mathbf{v}|^2\,\sin^2\theta \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\,\cos\theta, \end{aligned}$$

because $\cos^2\theta + \sin^2\theta = 1$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

and therefore

$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 &= (u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2\\ &= u_1^2+2u_1v_1+v_1^2+u_2^2+2u_2v_2+v_2^2+u_3^2+2u_3v_3+v_3^2\\ &= |\mathbf{u}|^2+|\mathbf{v}|^2+2(u_1v_1+u_2v_2+u_3v_3)\\ &= |\mathbf{u}|^2+|\mathbf{v}|^2+2\mathbf{u}.\mathbf{v}.\end{aligned}$$

On comparing the expressions for $|\mathbf{u}+\mathbf{v}|^2$ given by the above equations, we see that $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ when $0 < \theta < \frac{1}{2}\pi$.

The identity $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ clearly holds when $\theta = 0$ and $\theta = \pi$. Pythagoras' Theorem ensures that it also holds when the angle θ is a right angle (so that $\theta = \frac{1}{2}\pi$. Suppose that $\frac{1}{2}\pi < \theta < \pi$, so that the angle θ is obtuse. Then the angle between the vectors $\mathbf{u}$ and $-\mathbf{v}$ is acute, and is equal to $\pi - \theta$. Moreover $\cos(\pi-\theta) = -\cos\theta$ for all angles θ. It follows that

$$\mathbf{u}.\mathbf{v} = -\mathbf{u}.(-\mathbf{v}) = -|\mathbf{u}|\,|\mathbf{v}|\,\cos(\pi-\theta) = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$$

when $\frac{1}{2}\pi < \theta < \pi$. We have therefore verified that the identity $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ holds for all non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, as required. ∎

Corollary 38.2 *Two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in three-dimensional space are perpendicular if and only if* $\mathbf{u}\,.\,\mathbf{v} = 0$.

Proof It follows directly from Proposition 38.1 that $\mathbf{u}\,.\,\mathbf{v} = 0$ if and only if $\cos\theta = 0$, where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. This is the case if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. ∎

Example We can use the scalar product to calculate the angle θ between the vectors $(2,2,0)$ and $(0,3,3)$ in three-dimensional space. Let $\mathbf{u} = (2,2,0)$ and $\mathbf{v} = (0,3,3)$. Then $|\mathbf{u}|^2 = 2^2+2^2 = 8$ and $|\mathbf{v}|^2 = 3^2+3^2 = 18$. It follows that $(|\mathbf{u}|\,|\mathbf{v}|)^2 = 8\times 18 = 144$, and thus $|\mathbf{u}|\,|\mathbf{v}| = 12$. Now $\mathbf{u}\,.\,\mathbf{v} = 6$. It follows that

$$6 = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta = 12\cos\theta.$$

Therefore $\cos\theta = \frac{1}{2}$, and thus $\theta = \frac{1}{3}\pi$.

Example We can use the scalar product to find the distance between points on a sphere. Now the Cartesian coordinates of a point P on the unit sphere about the origin O in three-dimensional space may be expressed in terms of angles θ and φ as follows:

$$P = (\sin\theta\,\cos\varphi,\,\sin\theta\,\sin\varphi,\,\cos\theta).$$

The angle θ is that between the displacement vector $\overrightarrow{OP}$ and the vectical vector $(0,0,1)$. Thus the angle $\frac{1}{2}\pi - \theta$ represents the 'latitude' of the point P, when we regard the point $(0,0,1)$ as the 'north pole' of the sphere. The angle φ measures the 'longitude' of the point P.

Now let P_1 and P_2 be points on the unit sphere, where

$$\begin{aligned} P_1 &= (\sin\theta_1\,\cos\varphi_1,\, \sin\theta_1\,\sin\varphi_1,\, \cos\theta_1), \\ P_2 &= (\sin\theta_2\,\cos\varphi_2,\, \sin\theta_2\,\sin\varphi_2,\, \cos\theta_2). \end{aligned}$$

We wish to find the angle ψ between the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ of the points P_1 and P_2 from the origin. Now $|\overrightarrow{OP_1}| = 1$ and $|\overrightarrow{OP_2}| = 1$. On applying Proposition 38.1, we see that

$$\begin{aligned} \cos\psi &= \overrightarrow{OP_1}\,.\,\overrightarrow{OP_2} \\ &= \sin\theta_1\,\sin\theta_2\,\cos\varphi_1\,\cos\varphi_2 + \sin\theta_1\,\sin\theta_2\,\sin\varphi_1\,\sin\varphi_2 \\ &\qquad + \cos\theta_1\,\cos\theta_2 \\ &= \sin\theta_1\,\sin\theta_2\,(\cos\varphi_1\,\cos\varphi_2 + \sin\varphi_1\,\sin\varphi_2) + \cos\theta_1\,\cos\theta_2 \\ &= \sin\theta_1\,\sin\theta_2\,\cos(\varphi_1 - \varphi_2) + \cos\theta_1\,\cos\theta_2. \end{aligned}$$

Example Let X be a plane in three-dimensional space, and let $\mathbf{p}$ be a vector that is perpendicular to the plane X. Let O be the origin of a Cartesian coordinate system in three-dimensional space, and let $\mathbf{v}$ and $\mathbf{w}$ be the position vectors $\overrightarrow{OV}$ and $\overrightarrow{OW}$ of points V and W respectively lying in the plane X. Then the vector $\mathbf{p}$ is perpendicular to the displacement vector $\overrightarrow{VW}$. Now $\overrightarrow{VW} = \mathbf{w} - \mathbf{v}$. It follows that

$$(\mathbf{w} - \mathbf{v})\,.\,\mathbf{p} = 0$$

(see Corollary 38.2), and therefore $\mathbf{v}.\mathbf{p} = \mathbf{w}.\mathbf{p}$. Identifying the points of the plane X with their position vectors $\mathbf{r}$ with respect to the origin O of the Cartesian coordinate system, we find that It follows from this that there exists a real number k such that

$$X = \{\mathbf{r} \in \mathbb{R}^3 : \mathbf{r}\,.\,\mathbf{p} = k\}.$$

Let $\mathbf{r} = (x,y,z)$ and $\mathbf{p} = (a,b,c)$. The point $\mathbf{r}$ belongs to the plane X if and only if $\mathbf{r}\,.\,\mathbf{p} = k$. It follows that

$$X = \{(x,y,z) \in \mathbb{R}^3 : ax + by + cz = k\}.$$

Suppose that the vector $\mathbf{r}$ is the position vector of an arbitrary point R of three-dimensional space. We wish to determine the distance from this point to the plane X. Now the line through the point $\mathbf{r}$ parallel to the vector $\mathbf{p}$ cuts the plane X in a single point. Therefore there exists a unique real number t for which $\mathbf{r} + t\mathbf{p} \in X$. For this value of t the equation

$$(\mathbf{r} + t\mathbf{p}) \cdot \mathbf{p} = k$$

is satisfied. Then

$$\mathbf{r} \cdot \mathbf{p} = t|\mathbf{p}|^2 = k,$$

and therefore

$$t = \frac{1}{|\mathbf{p}|^2}(k - \mathbf{r} \cdot \mathbf{p}).$$

Let $\mathbf{w} = \mathbf{r} + t\mathbf{p}$, where t has the value determined above that ensures that $\mathbf{w} \in X$. Let $\mathbf{v}$ be an arbitrary point that lies on the plane X. Then the displacement vector $\mathbf{v} - \mathbf{w}$ from W to V is perpendicular to the vector $\mathbf{p}$. Now

$$\mathbf{v} - \mathbf{r} = t\mathbf{p} + (\mathbf{v} - \mathbf{w}).$$

It follows, either directly from Pythagoras' Theorem, or else from an equivalent calculation using scalar products (using the result of Corollary 38.2) that

$$|\mathbf{v} - \mathbf{r}|^2 = t^2|\mathbf{p}|^2 + |\mathbf{v} - \mathbf{w}|^2.$$

It follows that

$$|\mathbf{v} - \mathbf{r}| \geq t|\mathbf{p}|,$$

and that

$$|\mathbf{v} - \mathbf{r}| = t|\mathbf{p}| \iff \mathbf{v} = \mathbf{w}.$$

Thus the point $\mathbf{w}$ is the closest point of the plane X to the point R with position vector $\mathbf{r}$. It follows that the distance $d(\mathbf{r}, X)$ from the point R to the plane X is the length $|\mathbf{w} - \mathbf{r}|$ of the vector $\mathbf{w} - \mathbf{r}$. Thus

$$d(\mathbf{r}, X) = t|\mathbf{p}| = \frac{1}{|\mathbf{p}|}|k - \mathbf{r} \cdot \mathbf{p}|.$$

Let $\mathbf{r} = (x, y, z)$ and $p = (a, b, c)$. Then

$$d(\mathbf{r}, X) = \frac{|k - ax - by - cz|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example Suppose that we wish to determine the equation of a cone in three-dimensional space. Let O be the origin of a Cartesian coordinate system, let V be the apex of the cone, let $\mathbf{v}$ be the position vector of V, so that $\mathbf{v} = \overrightarrow{OV}$, and let $\mathbf{b}$ be a vector pointed into the axis of the cone. Let θ be a fixed angle between zero and a right angle. The cone consists of those points R for which the displacement vector $\overrightarrow{VR}$ makes an angle θ with the vector $\mathbf{b}$. It follows from Proposition 38.1 that $\mathbf{r}$ is the position vector of a point lying on the cone if and only if

$$(\mathbf{r} - \mathbf{v}) \,.\, \mathbf{b} = |\mathbf{r} - \mathbf{v}|\, |\mathbf{b}| \cos\theta.$$

Squaring both sides of this identity, we find that

$$((\mathbf{r} - \mathbf{v}) \,.\, \mathbf{b})^2 = |\mathbf{r} - \mathbf{v}|^2\, |\mathbf{b}|^2 \cos^2\theta.$$

Let

$$\mathbf{r} = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z) \quad \text{and} \quad \mathbf{b} = (b_x, b_y, b_z).$$

Then the equation of the cone becomes

$$\begin{aligned} &((x - v_x)b_x + (y - v_y)b_y + (z - v_z)b_z)^2 \\ &\quad = \; C\left((x - v_x)^2 + (y - v_y)^2 + (z - v_z)^2\right), \end{aligned}$$

where $C = |\mathbf{b}|^2 \cos^2\theta$. Note that this constant C must satisfy the inequalities $0 \leq C < |\mathbf{b}|^2$.

38.3 The Vector Product

Definition Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space, with Cartesian components given by the formulae $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{v} = (b_1, b_2, b_3)$. The *vector product* $\mathbf{u} \times \mathbf{v}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ is the vector defined by the formula

$$\mathbf{u} \times \mathbf{v} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1).$$

Note that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Also $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ for all vectors $\mathbf{u}$. It follows easily from the definition of the vector product that

$$(s\mathbf{u} + t\mathbf{v}) \times \mathbf{w} = s\mathbf{u} \times \mathbf{w} + t\mathbf{v} \times \mathbf{w}, \quad \mathbf{u} \times (s\mathbf{v} + t\mathbf{w}) = s\mathbf{u} \times \mathbf{v} + t\mathbf{u} \times \mathbf{w}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and real numbers s and t.

Proposition 38.3 *Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space $\mathbb{R}^3$. Then their vector product $\mathbf{u} \times \mathbf{v}$ is a vector of length $|\mathbf{u}|\, |\mathbf{v}|\, |\sin\theta|$, where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Moreover the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the vectors $\mathbf{u}$ and $\mathbf{v}$.*

Proof Let $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{v} = (b_1, b_2, b_3)$, and let l denote the length $|\mathbf{u} \times \mathbf{v}|$ of the vector $\mathbf{u} \times \mathbf{v}$. Then

$$
\begin{aligned}
l^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 \\
&\quad + a_3^2b_1^2 + a_1^2b_3^2 - 2a_3a_1b_3b_1 \\
&\quad + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 \\
&= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) \\
&\quad - 2a_2a_3b_2b_3 - 2a_3a_1b_3b_1 - 2a_1a_2b_1b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 - 2a_2b_2a_3b_3 - 2a_3b_3a_1b_1 - 2a_1b_1a_2b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \,.\, \mathbf{v})^2
\end{aligned}
$$

since

$$|\mathbf{u}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\mathbf{v}|^2 = b_1^2 + b_2^2 + b_3^2, \quad \mathbf{u} \,.\, \mathbf{v} = a_1b_1 + a_2b_2 + a_3b_3$$

But $\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ (Proposition 38.1). Therefore

$$l^2 = |\mathbf{u}|^2|\mathbf{v}|^2(1 - \cos^2\theta) = |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta$$

(since $\sin^2\theta + \cos^2\theta = 1$ for all angles θ) and thus $l = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|$. Also

$$\mathbf{u} \,.\, (\mathbf{u} \times \mathbf{v}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0$$

and

$$\mathbf{v} \,.\, (\mathbf{u} \times \mathbf{v}) = b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) = 0$$

and therefore the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ (Corollary 38.2), as required. ∎

Using elementary geometry, and the formula for the length of the vector product $\mathbf{u} \times \mathbf{v}$ given by Proposition 38.3 it is not difficult to show that the length of this vector product is equal to the area of a parallelogram in three-dimensional space whose sides are represented, in length and direction, by the vectors $\mathbf{u}$ and $\mathbf{v}$.

Remark Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors that are not colinear (i.e., so that they do not point in the same direction, or in opposite directions). The direction of $\mathbf{u} \times \mathbf{v}$ may be determined, using the thumb and first two fingers of your right hand, as follows. Orient your right hand such that the thumb points in the direction of the vector $\mathbf{u}$ and the first finger points in the direction of the vector $\mathbf{v}$, and let your second finger point outwards from the palm of your hand so that it is perpendicular to both the thumb and the first finger. Then the second finger points in the direction of the vector product $\mathbf{u} \times \mathbf{v}$.

Indeed it is customary to describe points of three-dimensional space by Cartesian coordinates (x, y, z) oriented so that if the positive x-axis and positive y-axis are pointed in the directions of the thumb and first finger respectively of your right hand, then the positive z-axis is pointed in the direction of the second finger of that hand, when the thumb and first two fingers are mutually perpendicular. For example, if the positive x-axis points towards the East, and the positive y-axis points towards the North, then the positive z-axis is chosen so that it points upwards. Moreover if $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ then these vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors pointed in the direction of the positive x-axis and positive y-axis respectively, and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, where $\mathbf{k} = (0, 0, 1)$, and the vector $\mathbf{k}$ points in the direction of the positive z-axis. Thus the 'right-hand' rule for determining the direction of the vector product $\mathbf{u} \times \mathbf{v}$ using the fingers of your right hand is valid when $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$.

If the directions of the vectors $\mathbf{u}$ and $\mathbf{v}$ are allowed to vary continuously, in such a way that these vectors never point either in the same direction or in opposite directions, then their vector product $\mathbf{u} \times \mathbf{v}$ will always be a non-zero vector, whose direction will vary continuously with the directions of $\mathbf{u}$ and $\mathbf{v}$. It follows from this that if the 'right-hand rule' for determining the direction of $\mathbf{u} \times \mathbf{v}$ applies when $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$, then it will also apply whatever the directions of $\mathbf{u}$ and $\mathbf{v}$, since, if your right hand is moved around in such a way that the thumb and first finger never point in the same direction, and if the second finger is always perpendicular to the thumb and first finger, then the direction of the second finger will vary continuously, and will therefore always point in the direction of the vector product of two vectors pointed in the direction of the thumb and first finger respectively.

Example We shall find the area of the parallelogram $OACB$ in three-dimensional space, where

$$O = (0, 0, 0), \quad A = (1, 2, 0),$$

$$B = (-4, 2, -5), \quad C = (-3, 4, -5).$$

Note that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. Let $\mathbf{u} = \overrightarrow{OA} = (1, 2, 0)$ and $\mathbf{v} = \overrightarrow{OB} = (-4, 2, -5)$. Then $\mathbf{u} \times \mathbf{v} = (-10, 5, 10)$. Now $(-10, 5, 10) = 5(-2, 1, 2)$, and $|(-2, 1, 2)| = \sqrt{9} = 3$. It follows that

$$\text{area}\, OACB = |\mathbf{u} \times \mathbf{v}| = 15.$$

Note also that the vector $(-2, 1, 2)$ is perpendicular to the parallelogram $OACB$.

Example We shall find the equation of the plane containing the points A, B and C where $A = (3, 4, 1)$, $B = (4, 6, 1)$ and $C = (3, 5, 3)$. Now if $\mathbf{u} = \overrightarrow{AB} = (1, 2, 0)$ and $\mathbf{v} = \overrightarrow{AC} = (0, 1, 2)$ then the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. It follows that the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to this plane. Now $\mathbf{u} \times \mathbf{v} = (4, -2, 1)$, and therefore the displacement vector between any two points of the plane must be perpendicular to the vector $(4, -2, 1)$. It follows that the function mapping the point (x, y, z) to the quantity $4x - 2y + z$ must be constant throughout the plane. Thus the equation of the plane takes the form

$$4x - 2y + z = k,$$

for some constant k.

We can calculate the value of k by substituting for x, y and z the coordinates of any chosen point of the plane. On taking this chosen point to be the point A, we find that $k = 4 \times 3 - 2 \times 4 + 1 = 5$. Thus the equation of the plane is the following:

$$4x - 2y + z = 5.$$

(We can check our result by verifying that the coordinates of the points A, B and C do indeed satisfy this equation.)

38.4 Scalar Triple Products

Given three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in three-dimensional space, we can form the *scalar triple product* $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$. This quantity can be expressed as the determinant of a 3×3 matrix whose rows contain the Cartesian components of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Indeed

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1),$$

and thus

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1).$$

The quantity on the right hand side of this equality defines the determinant of the 3×3 matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

We have therefore obtained the following result.

Lemma 38.4 *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Using basic properties of determinants, or by direct calculation, one can easily obtain the identities

$$\begin{aligned} \mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \,.\, (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \,.\, (\mathbf{u} \times \mathbf{v}) \\ &= -\mathbf{u} \,.\, (\mathbf{w} \times \mathbf{v}) = -\mathbf{v} \,.\, (\mathbf{u} \times \mathbf{w}) = -\mathbf{w} \,.\, (\mathbf{v} \times \mathbf{u}) \end{aligned}$$

One can show that the absolute value of the scalar triple product $\mathbf{u}.(\mathbf{v}\times\mathbf{w})$ is the volume of the parallelepiped in three-dimensional space whose vertices are the points whose displacement vectors from some fixed point O are $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{w}$, $\mathbf{v}+\mathbf{w}$ and $\mathbf{u}+\mathbf{v}+\mathbf{w}$. (A *parallelepiped* is a solid like a brick, but whereas the faces of a brick are rectangles, the faces of the parallelepiped are parallelograms.)

Example We shall find the volume of the parallelepiped in 3-dimensional space with vertices at $(0,0,0)$, $(1,2,0)$, $(-4,2,-5)$, $(0,1,1)$, $(-3,4,-5)$, $(1,3,1)$, $(-4,3,-4)$ and $(-3,5,-4)$. The volume of this parallelepiped is the absolute value of the scalar triple product $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$, where

$$\mathbf{u} = (1,2,0), \quad \mathbf{v} = (-4,2,-5), \quad \mathbf{w} = (0,1,1).$$

Now

$$\begin{aligned} \mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) &= (1,2,0) \,.\, (\,(-4,2,-5) \times (0,1,1)\,) \\ &= (1,2,0) \,.\, (7,4,-4) = 7 + 2 \times 4 = 15. \end{aligned}$$

Thus the volume of the paralellepiped is 15 units.

38.5 The Vector Triple Product Identity

Proposition 38.5 *Let* **u**, **v** *and* **w** *be vectors in three-dimensional space. Then*

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\, \mathbf{w}.$$

Proof Let $\mathbf{q} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$, and $\mathbf{q} = (q_1, q_2, q_3)$. Then

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

and hence $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{q} = (q_1, q_2, q_3)$, where

$$\begin{aligned} q_1 &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= (u_2 w_2 + u_3 w_3) v_1 - (u_2 v_2 + u_3 v_3) w_1 \\ &= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_1 \\ &= (\mathbf{u} \,.\, \mathbf{w}) v_1 - (\mathbf{u} \,.\, \mathbf{v}) w_1 \end{aligned}$$

Similarly

$$q_2 = (\mathbf{u} \,.\, \mathbf{w}) v_2 - (\mathbf{u} \,.\, \mathbf{v}) w_2$$

and

$$q_3 = (\mathbf{u} \,.\, \mathbf{w}) v_3 - (\mathbf{u} \,.\, \mathbf{v}) w_3$$

(In order to verify the formula for q_2 with an minimum of calculation, take the formulae above involving q_1, and cyclicly permute the subcripts 1, 2 and 3, replacing 1 by 2, 2 by 3, and 3 by 1. A further cyclic permutation of these subscripts yields the formula for q_3.) It follows that

$$\mathbf{q} = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\, \mathbf{w},$$

as required, since we have shown that the Cartesian components of the vectors on either side of this identity are equal. ∎

38.6 Orthonormal Triads of Unit Vectors

Let **u** and **v** be unit vectors (i.e., vectors of length one) that are perpendicular to each other, and let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. It follows immediately from Proposition 38.3 that $|\mathbf{w}| = |\mathbf{u}|\, |\mathbf{v}| = 1$, and that this unit vector **w** is perpendicular to both **u** and **v**. Then

$$\mathbf{u} \,.\, \mathbf{u} = \mathbf{v} \,.\, \mathbf{v} = \mathbf{w} \,.\, \mathbf{w} = 1$$

and

$$\mathbf{u} \,.\, \mathbf{v} = \mathbf{v} \,.\, \mathbf{w} = \mathbf{w} \,.\, \mathbf{u} = 0.$$

On applying the Vector Triple Product Identity (Proposition 38.5) we find that

$$\mathbf{v} \times \mathbf{w} = \mathbf{v} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \,.\, \mathbf{v})\,\mathbf{u} - (\mathbf{v} \,.\, \mathbf{u})\,\mathbf{v} = \mathbf{u},$$

and

$$\mathbf{w} \times \mathbf{u} = -\mathbf{u} \times \mathbf{w} = -\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = -(\mathbf{u} \,.\, \mathbf{v})\,\mathbf{u} + (\mathbf{u} \,.\, \mathbf{u})\,\mathbf{v} = \mathbf{v},$$

Therefore

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} = \mathbf{w}, \quad \mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} = \mathbf{u}, \quad \mathbf{w} \times \mathbf{u} = -\mathbf{u} \times \mathbf{w} = \mathbf{v},$$

Three unit vectors, such as the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ above, that are mutually perpendicular, are referred to as an *orthonormal triad* of vectors in three-dimensional space. The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in any orthonormal triad are linearly independent. It follows directly from Theorem 36.2 that any vector in three-dimensional space may be expressed, uniquely, as a linear combination of the form

$$p\mathbf{u} + q\mathbf{v} + r\mathbf{w}.$$

Any Cartesian coordinate system on three-dimensional space determines an orthonormal triad $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

The scalar and vector products of these vectors satisfy the same relations as the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ above. A vector represented in these Cartesian components by an ordered triple (x, y, z) then satisfies the identity

$$(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$