Module MA2C03: Discrete Mathematics
Hilary Term 2016
Section 39: Ordinary Differential Equations

D. R. Wilkins

Contents

39 Ordinary Differential Equations

We consider *differential equations* satisfied by real variables in situations where some real variable y is expressible as a differentiable function of a real variable x, where x takes values in some specified range. In such situations, the real variable x is said to be the *independent* real variable, and the variable y whose values are determined by the corresponding values of x is said to be a *dependent* real variable.

The independent real variable x will typically vary over an open interval. A subset I of the real numbers is said to be an *open interval* if it takes one of the following four forms:—

(i) $I = \mathbb{R}$;

(ii) $I = \{x \in \mathbb{R} : x > a\}$, where a is some specified real constant;

(iii) $I = \{x \in \mathbb{R} : x < b\}$, where b is some specified real constant;

(iv) $I = \{x \in \mathbb{R} : a < x < b\}$, where a and b are specified real constants.

Note that if I is an open interval, and if u and v are real numbers belonging to I, then $x \in I$ for all real numbers x satisfying $u < x < v$.

Let x and y be real variables, where the value of y depends on the the value of x, so $y = h(x)$ for all values of x in some specified range, where $h(x)$ is a differentiable function of x. In this situation we regard x as an *independent* real variable, and regard y to be a *dependent* real variable whose value depends on that of the independent variable x. For instance it may be the case that $y = x^3$ for all real values of the independent variable x, or $y = \dfrac{1}{x}$ for all positive values of the independent variable x. We say that y satisfies a *ordinary differential equation* of *first order* in x if there exists a function H of three real variables with the property that

$$H\left(\frac{dy}{dx}, y, x\right) = 0$$

for all real values of x in the appropriate range within which which the independent variable x takes its values.

Example Let $y = e^{3x}$ for all real numbers x. Then y satisfies the first order differential equation

$$\frac{dy}{dx} - 3y = 0.$$

Example Let $y = x^2$ for all real numbers x. Then y satisfies the first order differential equation

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0.$$

Example Let $y = \dfrac{1}{x^2}$ for all positive real numbers x. Then y satisfies the first order differential equation

$$\left(\frac{dy}{dx}\right)^2 - 4y^3 = 0,$$

where the independent real variable x ranges over the set of positive real numbers. Indeed

$$\frac{dy}{dx} = -\frac{2}{x^3},$$

and therefore

$$\left(\frac{dy}{dx}\right)^2 = \frac{4}{x^6} = 4y^3$$

for all positive real numbers x.

Example Let $y = \sin 4x$ for all real number x. Then y satisfies the first order differential equation

$$\left(\frac{dy}{dx}\right)^2 + 16y^2 - 16 = 0.$$

Indeed $\dfrac{dy}{dx} = 4\cos 4x$, and therefore

$$\left(\frac{dy}{dx}\right)^2 + 16y^2 - 16 = 16\cos^2 4x + 16\sin^2 4x - 16 = 0,$$

(We use here the trigonometrical identity that ensures that $\cos^2\theta + \sin^2\theta = 1$ for all real numbers θ.)

Let $f(x)$ be a continuous function of the independent real variable x, let c be a real number, and let y be a real variable that satisfies the differential equation

$$\frac{dy}{dx} + cy = f(x).$$

We seek to determine y as a function of the independent variable x.

Suppose that y is expressed in the form $y(x) = u(x)e^{rx}$, where r is a constant and u is a differentiable function of the independent variable x. It follows from the Product Rule of differential calculus that

$$\frac{dy}{dx} = \frac{du}{dx}e^{rx} + u\frac{d}{dx}(e^{rx}) = \frac{du}{dx}e^{rx} + rue^{rx}.$$

It follows that

$$\frac{dy}{dx} + cy = \left(\frac{du}{dx} + (c+r)u\right) e^{rx},$$

Thus the function y of x satisfies the given differential equation

$$\frac{dy}{dx} + cy = f(x).$$

if and only if $y(x) = u(x)e^{rx}$, where $u(x)$ is a differentiable function of the independent variable x that satisfies the differential equation

$$\frac{du}{dx} + (c+r)u = f(x)e^{-rx}.$$

The value of the constant r has not so far been chosen. Suppose we take $r = -c$. We conclude that y satisfies the given differential equation

$$\frac{dy}{dx} + cy = f(x).$$

if and only if $y(x) = u(x)e^{-cx}$, where u satisfies the differential equation

$$\frac{du}{dx} = f(x)e^{cx}.$$

Proposition 39.1 *Let I be an open interval, let x be an independent real variable which ranges over the open interval I, let c be a constant, let $f(x)$ be a continuous function of x on the interval I, and let y be a dependent variable expressible as a differentiable function of the independent variable x. Let $g(x)$ be a function of x that satisfies*

$$g(x) = \int f(x)e^{cx}\,dx.$$

Then the dependent variable y satisfies the differential equation

$$\frac{dy}{dx} + cy = f(x)$$

if and only if $y = g(x)e^{-cx} + Ae^{-cx}$ for all $x \in I$, where A is some real constant.

Proof Let

$$g(x) = \int f(x)e^{cx}\,dx.$$

(In other words, let $g(x)$ be any function of x whose derivative with respect to x is equal to the function $f(x)e^{cx}$.) Then u satisfies the differential equation

$$\frac{du}{dx} = f(x)e^{cx}$$

if and only if

$$\frac{d}{dx}\,(u(x) - g(x)) = 0,$$

and moreover this is the case if and only if $u(x) = g(x) + A$ for some real constant A.

It follows that the dependent variable y satisfies the differential equation

$$\frac{dy}{dx} + cy = f(x)$$

if and only if

$$y(x) = g(x)e^{-cx} + Ae^{-cx},$$

where A is a real constant. The result follows. ∎

Corollary 39.2 *Let y be a real variable expressible as a differentiable function of an independent real variable x. Then the dependent real variable y satisfies the differential equation*

$$\frac{dy}{dx} + cy = 0,$$

where c is a real constant, if and only if there exists some real constant A for which

$$y(x) = Ae^{-cx}.$$

Proof This follows from Proposition 39.1 on setting the function $f(x)$ in the statement of that proposition equal to the zero function. ∎

Corollary 39.3 *Let $f(x)$ be a continuous function of f defined over an open interval I, let c be a real constant, and Let y_1 and y_2 be real variables dependent on an independent real variable x that ranges over the open interval I. Suppose that the first order differential equation*

$$\frac{dy}{dx} + cy = f(x)$$

both when $y = y_1$ and also when $y = y_2$. Then there exists a real constant A such that $y_2 = y_1 + Ae^{-cx}$.

Proof The dependent variables y_1 and y_2 satisfy

$$\frac{dy_1}{dx} + cy_1 = f(x) \quad \text{and} \quad \frac{dy_2}{dx} + cy_2 = f(x).$$

Let $u = y_2 - y_1$. Then

$$\frac{du}{dx} + cu = \left(\frac{dy_2}{dx} + cy_2\right) - \left(\frac{dy_1}{dx} + cy_1\right) = 0.$$

It follows from Corollary 39.2 that there exists some real constant A such that $u = Ae^{-cx}$ for all $x \in I$. Then $y_2 = y_1 + Ae^{-cx}$, as required. ∎

Example Let us consider the differential equation

$$\frac{dy}{dx} + cy = g + hx + kx^2$$

where the real numbers c, g, h and k are constants and $c \neq 0$. This differential equation could be solved by applying the result of Proposition 39.1 and evaluating the resulting integral.

We shall however solve this differential equation by an alternative method, suitable in situations where the right hand side of the differential equation is a "forcing function" that is a polynomial in the independent variable x.

In this case we look for a "particular integral" that takes the form of a polynomial of the same degree as that occurring on the right hand side of the given differential equation.

Thus in this case we look for a solution y_P satisfying the differential equation

$$\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

that takes the form

$$y_P = u + vx + wx^2.$$

Differentiating, we find that

$$\frac{dy_P}{dx} = v + 2wx.$$

It follows that

$$\frac{dy_P}{dx} + cy_P = (v + cs) + (2w + cv)x + cwx^2.$$

Thus a quadratic polynomial y_P of the form $y_P = u + vx + wx^2$ satisfies the differential equation

$$\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

if and only if

$$v + cu + (2u + cv)x + cwx^2 = g + hx + kx^2$$

for all values of the independent variable x. This is the case if and only if the coefficients of the quadratic polynomial on the left hand side are equal to the corresponding coefficients of the quadratic polynomial on the right hand side. Thus y_P is the required "particular integral" if and only if

$$t + cs = g, \quad 2w + cv = h \quad \text{and} \quad cw = k.$$

Substituting $w = \dfrac{k}{c}$ into the equation $2w + cv = h$, we find that

$$v = \frac{1}{c}(h - 2w) = \frac{1}{c^2}(ch - 2k).$$

If we then substitute this formula for t into the equation $v + cu = g$, we find that

$$u = \frac{1}{c}(g - v) = \frac{1}{c^3}(c^2 g - ch + 2k).$$

Thus

$$y_P = \frac{1}{c^3}\left(c^2 g - ch + 2k + (c^2 h - 2ck)x + c^2 k x^2\right).$$

Now the quadratic polynomial y_P is just one of the solutions of the given differential equation. It follows from Corollary 39.3 that the other solutions of the differential equation

$$\frac{dy}{dx} + cy = g + hx + kx^2$$

take the form

$$y = y_P + Ae^{-cx},$$

where A is an arbitrary real constant. Thus the general solution of this differential equation takes the form

$$y(x) = \frac{1}{c^3}\left(c^2 g - ch + 2k + (c^2 h - 2ck)x + c^2 k x^2\right) + Ae^{-cx}.$$

The term Ae^{-cx} is often referred to as the "complementary function". It is the function that needs to be added to one solution to the differential equation to obtain other solutions. The general solution of the differential equation is the sum of the particular integral and the complementary function. The real constants c, g, h and k in the general solution are fixed constants determined by the differential equation. The real constant A takes different values for different solutions of the differential equation.

The solution can be verified on the Wolfram Alpha website at

```
http://www.wolframalpha.com/
```

by entering the string

```
y' + cy = g + hx + kx^2
```

into the search box.

The general solution of other differential equations of the form

$$\frac{dy}{dx} + cy = f(x)$$

can also be expressed as the sum of a particular integral and a complementary function.

Example Let us consider the differential equation

$$\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

where the real numbers c, g, h and m are constants and $m + c \neq 0$. In this case we look for a "particular integral" of the form

$$y_P = (u + vx)e^{mx}.$$

Differentiating using the Product Rule, we find that

$$\frac{dy_P}{dx} = ve^{mx} + m(u + vx)e^{mx} = (v + mu + mvx)e^{mx}$$

and therefore

$$\frac{dy_P}{dx} + cy_P = (v + (m + c)u + (m + c)vx)e^{mx}$$

It follows that y_P solves the differential equation if and only if

$$v + (m + c)u = g \quad \text{and} \quad (m + c)v = h.$$

Solving the second of these equations for v, we find that

$$v = \frac{h}{m + c}.$$

Then solving the other equation for u, we find that

$$u = \frac{1}{m + c}(g - v) = \frac{1}{(m + c)^2}((m + c)g - h).$$

Thus

$$y_P = \frac{1}{(m+c)^2}((m+c)(g+hx) - h)e^{mx}.$$

It follows that the general solution of the differential equation

$$\frac{dy}{dx} + cy = (g+hx)e^{mx}$$

(when $c + m \neq 0$) takes the form

$$y = \frac{1}{(m+c)^2}((m+c)(g+hx) - h)e^{mx} + Ae^{-cx}.$$

The solution can be verified on the Wolfram Alpha website at

`http://www.wolframalpha.com/`

by entering the string

```
y' + cy = (g + hx) e^(mx)
```

into the search box.

Example Let us consider the differential equation

$$\frac{dy}{dx} + cy = g\ \cos kx + h\ \sin kx$$

where the real numbers c, g, h and k are constants.

$$\frac{d}{dx}(\cos kx) = -k \sin kx \quad \text{and} \quad \frac{d}{dx}(\sin kx) = k \cos kx.$$

We look for a particular integral y_P of the form

$$y_P = u \cos kx + v \sin kx.$$

Differentiating, we find that

$$\frac{dy_P}{dx} + cy_P = (cu + kv) \cos kx + (cv - ku) \sin kx.$$

Therefore u and v should be chosen to satisfy the equations

$$cu + kv = g \quad \text{and} \quad cv - ku = h.$$

It follows that

$$kg + ch = k(cu + kv) + c(cv - ku) = (k^2 + c^2)v$$

and

$$cg - kh = c(cu + kv) - k(cv - ku) = (k^2 + c^2)u.$$

Thus

$$u = \frac{cg - kh}{k^2 + c^2} \quad \text{and} \quad v = \frac{kg + ch}{k^2 + c^2},$$

and thus

$$y_P = \frac{1}{k^2 + c^2} \left((cg - kh) \cos kx + (kg + ch) \sin kx \right).$$

It follows that the general solution of the differential equation

$$\frac{dy}{dx} + cy = g \, \cos kx + h \, \sin kx$$

takes the form

$$y = \frac{1}{k^2 + c^2} \left((cg - kh) \cos kx + (kg + ch) \sin kx \right) + Ae^{-cx}.$$

Let x and y be real variables, where the value of y is expressible as a function of the independent real variable x as x varies over some open interval I. We say that y satisfies a *ordinary differential equation* of *second order* in x if there exists a function H of four real variables with the property that

$$H\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x \right) = 0$$

for all real values of x in the appropriate range within which which the independent variable x takes its values.

We next prove results that determine all solutions of second order differential equations of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where b and c are constants. These results show that solutions can be determined directly once the roots of the *auxiliary polynomial* $z^2 + bz + c$ have been determined.

Proposition 39.4 *Let b and c be real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation*

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

throughout the open interval I. Suppose that the quadratic polynomial $z^2 + bz + c$ has two distinct real roots r and s. Then there exist real constants A and B such that

$$y(x) = Ae^{rx} + Be^{sx}.$$

Proof Let $u(x) = y(x)e^{-rx}$ for all $x \in I$. Then $y(x) = x(x)e^{rx}$ for all $x \in I$. Differentiating $y(x)$ with respect to x using the product rule, we find that

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{du}{dx} + ru\right) e^{rx}, \\ \frac{d^2y}{dx^2} &= \left(\frac{d^2u}{dx^2} + 2r\frac{du}{dx} + r^2u\right) e^{rx}. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy \\ &= \left(\frac{d^2u}{dx^2} + (2r+b)\frac{du}{dx} + (r^2 + br + c)u\right) e^{rx}. \end{aligned}$$

But r has been chosen so as to satisfy the quadratic equation $r^2+br+c = 0$. It follows that

$$\frac{d^2u}{dx^2} + (2r+b)\frac{du}{dx} = 0.$$

Thus if $v = \dfrac{du}{dx}$ then

$$\frac{dv}{dx} + (2r+b)v = 0.$$

Now $z^2 + bz + c = (z-r)(z-s) = z^2 - (r+s)z + rs$. It follows that $b = -(r+s)$, and therefore $2r + b = r - s$. Thus

$$\frac{dv}{dx} - (s-r)v = 0.$$

It follows from Corollary 39.2 that there exists a constant B such that

$$v(x) = (s-r)Be^{(s-r)x}.$$

Integrative the function $v(x)$ in order to determine $u(x)$, we find that there exist constants A and B such that

$$u(x) = A + Be^{(s-r)x}.$$

But then

$$y(x) = Ae^{rx} + Be^{sx},$$

as required. ∎

Proposition 39.5 *Let b and c be real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation*

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

throughout the open interval I. Suppose that the quadratic polynomial $z^2 + bz + c$ has a repeated real root r. Then there exist real constants A and B such that

$$y(x) = (A + Bx)e^{rx}.$$

Proof Let $u(x) = y(x)e^{-rx}$ for all $x \in I$. Then $y(x) = x(x)e^{rx}$ for all $x \in I$. Repeating the calculation in the proof of Proposition 39.4, we find that

$$\frac{d^2u}{dx^2} + (2r + b)\frac{du}{dx} = 0.$$

Moreover $z^2 + bz + c = (z - r)^2$ (because r is a repeated root of the quadratic polynomial on the left hand side of this equation) and therefore $b = -2r$. It follows that

$$\frac{d^2u}{dx^2} = 0,$$

and therefore $u(x) = A + Bx$, where A and B are real constants. It follows that $y(x) = (A + Bx)e^{rx}$, as required. ∎

Theorem 39.6 *Let k be a positive real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation*

$$\frac{d^2y}{dx^2} + k^2y = 0$$

throughout the open interval I. Then there exist real constants A and B such that

$$y = A\cos kx + B\sin kx$$

throughout the open interval I.

Proof We first prove the result in the special case where $\cos kx \neq 0$ for all $x \in I$. In this case we can express y in terms of another real variable u, where u is a twice-differentiable function of x and $y(x) = u(x)\cos kx$ for all $x \in I$. Now

$$\frac{d}{dx}(\cos kx) = -k\sin kx \quad \text{and} \quad \frac{d}{dx}(\sin kx) = k\cos kx.$$

On applying the Product Rule of differential calculus, we find that if $y = u\cos kx$ then

$$\frac{dy}{dx} = \frac{du}{dx}\cos kx - ku\sin kx.$$

On differentiating again, we find that

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx}\cos kx\right) - k\frac{d}{dx}(u\sin kx) \\
&= \frac{d}{dx}\left(\frac{du}{dx}\right)\cos kx + \frac{du}{dx}\frac{d}{dx}(\cos kx) \\
&\quad - k\frac{du}{dx}\sin kx - ku\frac{d}{dx}(\sin kx) \\
&= \frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx - k^2u\cos kx \\
&= \frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx - k^2y.
\end{aligned}$$

Thus y satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2y = 0$$

if and only if $y = u\cos kx$, where

$$\frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx = 0.$$

Now let

$$v = \frac{du}{dx}\cos^2 kx$$

(where $\cos^2 kx = (\cos kx)^2$). It then follows from the Product Rule of differential calculus that

$$\begin{aligned}
\frac{dv}{dx} &= \frac{d^2u}{dx^2}\cos^2 kx - 2k\frac{du}{dx}\cos kx\sin kx \\
&= \left(\frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx\right)\cos kx.
\end{aligned}$$

Now $\cos kx \neq 0$ for all $x \in I$. It follows that

$$\frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx = 0$$

for all $x \in I$ if and only if

$$\frac{dv}{dx} = 0$$

for all $x \in I$. However this is the case if and only if $v = Bk$ for all $x \in I$, where B is a real constant, in which case

$$\frac{du}{dx} = \frac{Bk}{\cos^2 kx}$$

for all $x \in I$.

We conclude that y satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

on the open interval I, where $\cos kx \neq 0$ for all $x \in I$, if and only if $y = u\cos kx$ on I, where

$$\frac{du}{dx} = \frac{Bk}{\cos^2 kx}.$$

Now it follows from the Quotient Rule of differential calculus that

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin kx}{\cos kx}\right) &= \frac{\dfrac{d}{dx}(\sin kx)\cos kx - \sin kx\,\dfrac{d}{dx}(\cos kx)}{\cos^2 kx} \\ &= \frac{k\cos^2 kx + k\sin^2 kx}{\cos^2 kx} = \frac{k}{\cos^2 kx}\end{aligned}$$

(where we have used the fact that $\sin^2\theta + \cos^2\theta = 1$ for all real numbers θ).

It follows that a variable u expressible as a differentiable function of x on the open interval I satisfies

$$\frac{du}{dx} = \frac{Bk}{\cos^2 kx}$$

throughout that open interval if and only if

$$\frac{d}{dx}\left(u - \frac{B\sin kx}{\cos kx}\right) = 0,$$

in which case

$$u = A + \frac{B\sin kx}{\cos kx}$$

for some constant A.

We have thus shown that if k is a real number, and if y is a twice-differentiable function of an independent real variable x, where x varies over an open interval I and $\cos kx \neq 0$ for all $x \in I$, then y satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

if and only if

$$y = A\cos kx + B\sin kx$$

for all values of the independent variable x belonging to the open interval I.

We now extend the result to cases where the open interval I includes values of x for which $\cos kx = 0$. Let $s \in I$ satisfy $\cos ks = 0$, and let I_1 and I_2 be open subintervals of I that are of the form

$$I_1 = \{x \in \mathbb{R} : a < x < s\}, \quad I_2 = \{x \in \mathbb{R} : s < x < b\},$$

where a is chosen close enough to s to ensure that $\cos kx \neq 0$ for all $x \in I_1$ and b is chosen close enough to s to ensure that $\cos kx \neq 0$ for all $x \in I_2$. Let y be a twice-differentiable function of x for $a < x < b$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0.$$

Then there exist constants A_1, B_1, A_2 and B_2 such that

$$y = A_1\cos kx + B_1\sin kx \quad \text{for all } x \in I_1$$

$$y = A_2\cos kx + B_2\sin kx \quad \text{for all } x \in I_2$$

Then $\cos ks = 0$ and $\sin ks = \pm 1$. It follows from the continuity and differentiability of y with respect to x that

$$B_1 \sin ks = \lim_{x\to s} y = B_2 \sin ks$$

and

$$A_1 \sin ks = \lim_{x\to s} \frac{dy}{dx} = B_2 \sin ks,$$

and thus $A_1 = A_2$ and $B_1 = B_2$. We have thus shown that the coefficients of $\cos kx$ and $\sin kx$ that determine y as a function of x match up on both sides of points s of the open interval I at which $\cos ks = 0$. It follows that there exist constants A and B such that

$$y = A\cos kx + B\sin kx$$

for all values of the independent variable x belonging to the open interval I, as required. ∎

Proposition 39.7 *Let b and c be real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation*

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

throughout the open interval I. Suppose that the quadratic polynomial $z^2 + bz + c$ has two non-real roots $p + \sqrt{-1}q$ and $p - \sqrt{-1}q$. real roots r and s. Then there exist real constants A and B such that

$$y(x) = e^{px}(A\cos qx + B\sin qx).$$

Proof Let $u(x) = y(x)e^{-px}$. Then

$$\begin{aligned} y(x) &= u(x)e^{px}, \\ \frac{dy(x)}{dx} &= \left(\frac{du(x)}{dx} + pu(x)\right)e^{px}, \\ \frac{d^2y(x)}{dx^2} &= \left(\frac{d^2u(x)}{dx^2} + 2p\frac{du(x)}{dx} + p^2u(x)\right)e^{px}. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy \\ &= \left(\frac{d^2u}{dx^2} + (b+2p)\frac{du}{dx} + (p^2 + bp + c)u\right)e^{px}. \end{aligned}$$

Moreover the quadratic polynomial $z^2 + bz + c$ satisfies

$$\begin{aligned} z^2 + bz + c &= (z - p - \sqrt{-1}\,q)(z - p + \sqrt{-1}\,q) \\ &= (z-p)^2 + q^2 = z^2 - 2pz + p^2 + q^2, \end{aligned}$$

and therefore, on equating coefficients of the variable z, we find that $2p = -b$ and $p^2 + q^2 = c$. It follows that

$$p^2 + bp + c = p^2 - 2p^2 + (p^2 + q^2) = q^2.$$

Therefore the dependent variable u satisfies the differential equation

$$\frac{d^2u}{dx^2} + q^2u = 0.$$

Theorem 39.6 therefore ensures that

$$u(x) = A\cos qx + B\sin qx,$$

where A and B are constants, and therefore

$$y(x) = e^{px}(A\cos qx + B\sin qx),$$

as required. ∎

39.1 Review of Solutions of $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

The following lemma is a basic result that enables us to superimpose solutions of second order differential equations of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Lemma 39.8 *Let u and v be solutions of the second order differential equation*

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where b and c are constants, and let A and B be real numbers. Then $Au+Bv$ is also a solution of the differential equation.

Proof Let $y = Au + Bv$. Then

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx}, \quad \frac{d^2y}{dx^2} = A\frac{d^2u}{d^2x} + B\frac{d^2v}{dx^2},$$

$$\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0 \quad \text{and} \quad \frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv = 0,$$

and therefore

$$\begin{aligned} \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= A\left(\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu\right) \\ &\quad + B\left(\frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv\right) \\ &= 0. \end{aligned}$$ ∎

We now verify the solutions of all second order differential equations of this type. Let r be a real root of the auxiliary polynomial $z^2 + bz + c$ of the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

and let $y = e^{rx}$. Then

$$\frac{dy}{dx} = re^{rx} = ry \quad \text{and} \quad \frac{d^2y}{dx^2} = r^2e^{rx} = r^2y,$$

and therefore

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (r^2 + br + c)y = 0.$$

Thus e^{rx} is a solution of the differential equation.

It follows that if the auxiliary polynomial $z^2 + bz + c$ of the differential equation of the differential equation has two distinct real roots r and s then $Ae^{rx} + Be^{sx}$ is a solution of the differential equation for all real numbers A and B. Proposition 39.4 then ensures that all solutions of the differential equation are of this form.

Next suppose that r is a repeated root of the auxiliary polynomial. Then

$$z^2 + bz + c = (z - r)^2,$$

and therefore $b = -2r$ and $c = r^2$. If $y = xe^{rx}$ then

$$\frac{dy}{dx} = (rx + 1)e^{rx} = ry + e^{rx}$$

and

$$\frac{d^2y}{dx^2} = (r^2x + 2r)e^{rx} = r^2y + 2re^{rx},$$

and therefore

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (r^2 + br + c)y + (2r + b)e^{rx} = 0.$$

Thus xe^{rx} is a solution of the differential equation.

We have already shown that e^{rx} is also a solution of this differential equation. It follows that $(A+Bx)e^{rx}$ is a solution of this differential equation for all real constants A and B. Proposition 39.5 then ensures that all solutions of the differential equation are of this form.

Finally suppose that $p+iq$ is a root of the auxiliary polynomial z^2+bz+c for the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where $q \neq 0$ and $i = \sqrt{-1}$. Then $p - iq$ is also a root of the auxiliary polynomial. It follows that

$$\begin{aligned} z^2 + bz + c &= (z - p - iq)(z - p + iq) = (z - p)^2 + q^2 \\ &= z^2 - 2pz + p^2 + q^2, \end{aligned}$$

and therefore $b = -2p$ and $c = p^2 + q^2$.

Let $u = e^{px} \cos qx$ and $v = e^{px} \sin qx$. Then

$$\frac{du}{dx} = e^{px}(p \cos qx - q \sin qx) = pu - qv$$

and

$$\frac{dv}{dx} = e^{px}(p \sin qx + q \cos qx) = pv + qu,$$

and therefore

$$\begin{aligned} \frac{d^2u}{dx^2} &= p\frac{du}{dx} - q\frac{dv}{dx} = p(pu - qv) - q(pv + qu) \\ &= (p^2 - q^2)u - 2pqv \\ \frac{d^2v}{dx^2} &= p\frac{dv}{dx} + q\frac{du}{dx} = p(pv + qu) + q(pu - qv) \\ &= (p^2 - q^2)v + 2pqu \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d^2u}{dx^2} + b\frac{du}{dx} + c &= ((p^2 - q^2)u - 2pqv) + b(pu - qv) + cu \\ &= (p^2 - q^2 + bp + c)u - (2pq + bq)v, \\ \frac{d^2v}{dx^2} + b\frac{dv}{dx} + c &= ((p^2 - q^2)v + 2pqu) + b(pv + qu) + cv \\ &= (p^2 - q^2 + bp + c)v + (2pq + bq)v. \end{aligned}$$

But

$$p^2 - q^2 + bp + c = p^2 - q^2 - 2p^2 + p^2 + q^2 = 0$$

and

$$2pq + bq = 2pq - 2pq = 0.$$

Therefore

$$\frac{d^2u}{dx^2} + b\frac{du}{dx} + c = 0$$

and

$$\frac{d^2v}{dx^2} + b\frac{dv}{dx} + c = 0.$$

Thus if

$$y = e^{px}(A\cos qx + B\sin qx)$$

where A and B are real constants, then $y = Au + Bv$ and therefore

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0.$$

Proposition 39.7 then ensures that all solutions of the differential equation are of this form.

39.2 Solutions of $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$.

Example Let us consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g + hx + kx^2$$

where the real numbers b, c, g, h and k are constants and $c \neq 0$.

In this case we look for a "particular integral" that takes the form of a polynomial of the same degree as that occurring on the right hand side of the given differential equation.

Thus in this case we look for a solution y_P satisfying the differential equation

$$\frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

that takes the form

$$y_P = u + vx + wx^2.$$

Differentiating, we find that

$$\frac{dy_P}{dx} = v + 2wx \quad \text{and} \quad \frac{d^2y_P}{dx^2} = 2w.$$

It follows that

$$\frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = 2w + bv + cu + (2bw + cv)x + cwx^2.$$

Thus a quadratic polynomial y_P of the form $y_P = u + vx + wx^2$ satisfies the differential equation

$$\frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

if and only if

$$2w + bv + cu + (2bw + cv)x + cwx^2 = g + hx + kx^2$$

for all values of the independent variable x. This is the case if and only if the coefficients of the quadratic polynomial on the left hand side are equal to the corresponding coefficients of the quadratic polynomial on the right hand side. Thus y_P is the required "particular integral" if and only if

$$2w + bv + cu = g, \quad 2bw + cv = h \quad \text{and} \quad cw = k.$$

Substituting $w = \dfrac{k}{c}$ into the equation $2bw + cv = h$, we find that

$$v = \frac{1}{c}(h - 2bw) = \frac{1}{c^2}(ch - 2bk).$$

If we then substitute this formula for v into the equation $2w + bv + cs = g$, we find that

$$u = \frac{1}{c}(g - 2w - bv) = \frac{1}{c^3}(c^2 g - 2ck - bch + 2b^2 k).$$

Thus

$$y_P = \frac{1}{c^3}\left(c^2 g - 2ck - bch + 2b^2 k + (c^2 h - 2bck)x + c^2 kx^2\right).$$

Now the quadratic polynomial y_P is just one of the solutions of the given differential equation. Other solutions are obtained by adding onto the particular integral y_P a *complementary function* y_C. Accordingly the general solution of the differential equation therefore takes the form

$$y = \frac{1}{c^3}\left(c^2 g - 2ck - bch + 2b^2 k + (c^2 h - 2bck)x + c^2 kx^2\right) + y_C(x),$$

where the complementary function y_C satisfies the differential equation

$$\frac{d^2 y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

The solution can be verified on the Wolfram Alpha website at

```
http://www.wolframalpha.com/
```

by entering the string

```
y'' + b y' + cy = g + hx + kx^2
```

into the search box.

Example Consider the differential equation

$$\frac{d^2y}{dy^2} + 5\frac{dy}{dx} + 6y = x^2 - 7.$$

The equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g + hx + kx^2,$$

with $b = 5$, $c = 6$, $g = -7$, $h = 0$ and $k = 1$. We have shown that equations of this form have a particular integral y_P that takes the form

$$y_P = \frac{1}{c^3}\left(c^2g - 2ck - bch + 2b^2k + (c^2h - 2bck)x + c^2kx^2\right).$$

Substituting the values of b, c, g, h and k into this equation, we find that

$$\begin{aligned} c^3 &= 216, \\ c^2g - 2ck - bch + 2b^2k &= 36 \times (-7) - 2 \times 6 \times 1 + 2 \times 25 \\ &= -252 - 12 + 50 = -214, \\ c^2h - 2bck &= 36 \times 0 - 2 \times 5 \times 6 \times 1 = -60, \\ c^2k &= 36 \times 1 = 36, \end{aligned}$$

and therefore

$$\begin{aligned} y_P &= \frac{-214 - 60x + 36x^2}{216}. \\ &= -\frac{107}{108} - \frac{5}{18}x + \frac{1}{6}x^2 \end{aligned}$$

Now the auxiliary polynomial $z^2 + 5z + 6$ has roots -2 and -3. The complementary function $y_C(x)$ therefore satisfies

$$y_C(x) = Ae^{-2x} + Be^{-3x}.$$

It follows that the general solution to the differential equation is given by

$$y = -\frac{107}{108} - \frac{5}{18}x + \frac{1}{6}x^2 + Ae^{-2x} + Be^{-3x},$$

where A and B are real constants.

The solution can be verified on the Wolfram Alpha website at

```
http://www.wolframalpha.com/
```

by entering the string

```
y'' + 5 y' + 6 y = x^2 - 7
```

into the search box.

Example Let us consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

where the real numbers b, c, g, h and m are constants and $m^2 + bm + c \neq 0$. In this case we look for a "particular integral" of the form

$$y_P = (u + vx)e^{mx}.$$

Differentiating using the Product Rule, we find that

$$\frac{dy_P}{dx} = ve^{mx} + m(u + vx)e^{mx} = (v + mu + mvx)e^{mx}$$

and

$$\frac{d^2y}{dx^2} = 2mve^{mx} + m^2(u + vx)e^{mx} = (2mv + m^2u + m^2vx)e^{mx}$$

and therefore

$$\begin{aligned} & \frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P \\ = & \Big(2mv + m^2u + bv + (bm + c)u + (m^2 + bm + c)vx\Big)e^{mx}. \end{aligned}$$

It follows that y_P solves the differential equation if and only if

$$(2m + b)v + (m^2 + bm + c)u = g$$

and

$$(m^2 + bm + c)v = h.$$

Solving the second of these equations for v, we find that

$$v = \frac{h}{m^2 + bm + c}.$$

Then solving the other equation for u, we find that

$$\begin{aligned} u &= \frac{1}{m^2 + bm + c}(g - (2m + b)v) \\ &= \frac{(m^2 + bm + c)g - (2m + b)h}{(m^2 + bm + c)^2} \end{aligned}$$

Thus

$$y_P = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx}.$$

The general solution of the differential equation then takes the form

$$y = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx} + y_C(x).$$

where the complementary function y_C satisfies the differential equation

$$\frac{d^2 y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example Consider the differential equation

$$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 10y = (3 - 2x)e^{4x}.$$

This equation is of the form

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

with $b = -2$, $c = 10$, $g = 3$, $h = -2$ and $m = 4$. We have shown that equations of this form have a particular integral y_P that takes the form

$$y_P = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx}.$$

Substituting the values of b, c, g, h and m into this equation, we find that

$$\begin{aligned} m^2 + bm + c &= 16 - 2 \times 4 + 10 = 18, \\ (2m + b)h &= (2 \times 4 - 2) \times (-2) = -12, \end{aligned}$$

and therefore

$$y_P = \frac{66 - 36x}{324} e^{4x} = \left(\frac{11}{54} - \frac{x}{9}\right) e^{4x}.$$

Now the auxiliary polynomial $z^2 - 2z + 10$ has roots $1 + \sqrt{-1}\,3$ and $1 - \sqrt{-1}3$. It follows that the complementary function y_C for this differential equation takes the form

$$y_C(x) = e^x(A\cos 3x + B\sin 3x).$$

The general solution to the differential equation thus takes the form

$$y = \left(\frac{11}{54} - \frac{x}{9}\right)e^{4x} + e^x(A\cos 3x + B\sin 3x).$$

Example Let us consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

where the real numbers b, c, g, h and k are constants.

$$\frac{d}{dx}(\cos kx) = -k\sin kx \quad \text{and} \quad \frac{d}{dx}(\sin kx) = k\cos kx.$$

We look for a particular integral y_P of the form

$$y_P = u\cos kx + v\sin kx.$$

Differentiating, we find that

$$\frac{dy_P}{dx} = kv\cos kx - ku\sin kx$$

and

$$\frac{d^2y_P}{dx^2} = -k^2u\cos kx - k^2v\sin kx,$$

and thus

$$\begin{aligned} &\frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P \\ &= ((c-k^2)u + bkv)\cos kx + ((c-k^2)v - bku)\sin kx. \end{aligned}$$

Therefore u and v should be chosen to satisfy the equations

$$(c-k^2)u + bkv = g \quad \text{and} \quad (c-k^2)v - bku = h.$$

It follows that

$$\begin{aligned}
&bkg + (c - k^2)h \\
&\quad = \quad bk((c - k^2)u + bkv) + (c - k^2)((c - k^2)v - bku) \\
&\quad = \quad (b^2k^2 + (c - k^2)^2)v \\
&(c - k^2)g - bkh \\
&\quad = \quad (c - k^2)((c - k^2)u + bkv) - bk((c - k^2)v - bku) \\
&\quad = \quad (b^2k^2 + (c - k^2)^2)u.
\end{aligned}$$

Thus

$$u = \frac{(c - k^2)g - bkh}{b^2k^2 + (c - k^2)^2}$$

and

$$v = \frac{bkg + (c - k^2)h}{b^2k^2 + (c - k^2)^2},$$

and thus

$$\begin{aligned}
y_P \quad = \quad & \frac{1}{b^2k^2 + (c - k^2)^2}\Big(((c - k^2)g - bkh)\cos kx \\
& \qquad + (bkg + (c - k^2)h)\sin kx\Big).
\end{aligned}$$

It follows that the general solution of the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

takes the form

$$\begin{aligned}
y \quad = \quad & \frac{1}{b^2k^2 + (c - k^2)^2}\Big(((c - k^2)g - bkh)\cos kx \\
& \qquad + (bkg + (c - k^2)h)\sin kx\Big) + y_C,
\end{aligned}$$

where the complementary function y_C satisfies the differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 3\cos 2x + 4\sin 2x.$$

This equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

with $b = -6$, $c = 9$, $k = 2$, $g = 3$ and $h = 4$. We have shown that equations of this form have a particular integral y_P that takes the form

$$\begin{aligned} y_P &= \frac{1}{b^2k^2 + (c-k^2)^2}\Big(((c-k^2)g - bkh)\cos kx \\ &\qquad + (bkg + (c-k^2)h)\sin kx\Big). \end{aligned}$$

Substituting the values of b, c, k, g and h into this equation, we find that

$$\begin{aligned} bk &= -12 \\ c - k^2 &= 9 - 4 = 5 \\ b^2k^2 + (c-k^2)^2 &= 144 + 25 = 169, \\ (c-k^2)g - bkh &= 5 \times 3 - (-12) \times 4 = 15 + 48 = 63, \\ bkg + (c-k^2)h &= (-12) \times 3 + 5 \times 4 = -36 + 20 = -16. \end{aligned}$$

and therefore

$$y_P = \frac{1}{169}\left(63\cos 2x - 16\sin 2x\right).$$

Now the auxiliary polynomial $z^2 - 6z + 9$ has a repeated root with value 3. It follows that the complementary function y_C for this differential equation takes the form

$$y_C(x) = (A + Bx)e^{3x}.$$

The general solution to the differential equation thus takes the form

$$y = \frac{1}{169}\left(63\cos 2x - 16\sin 2x\right) + (A + Bx)e^{3x}.$$