MA2C03 Mathematics
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40. Introduction to Harmonic Analysis

40.1. Basic Trigonometrical Identities and Integrals

The following trigonometric identities satisfied by the sine and cosine functions are basic and well-known:—

$$
\begin{aligned}
\cos^2 A + \sin^2 A &= 1, \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B, \\
\cos 2A &= \cos^2 A - \sin^2 A, \\
\sin(A+B) &= \sin A \cos B + \cos A \sin B, \\
\sin 2A &= 2 \sin A \cos A, \\
\cos^2 A &= \tfrac{1}{2}(1 + \cos 2A), \\
\sin^2 A &= \tfrac{1}{2}(1 - \cos 2A), \\
2 \cos A \cos B &= \cos(A+B) + \cos(A-B), \\
2 \sin A \cos B &= \sin(A+B) + \sin(A-B), \\
2 \sin A \sin B &= \cos(A-B) - \cos(A+B),
\end{aligned}
$$

On differentiating the sine and cosine function, we find that

$$\begin{aligned} \frac{d}{dx} \sin qx &= q \cos qx \\ \frac{d}{dx} \cos qx &= -q \sin qx. \end{aligned}$$

for all real numbers q.
It follows that

$$\begin{aligned} \int \sin qx &= -\frac{1}{q} \cos qx + C \\ \int \cos qx &= \frac{1}{q} \sin qx + C, \end{aligned}$$

for all non-zero real numbers q, where C is a constant of integration.

Proposition 40.1

Let j and k be positive integers. Then

$$\begin{aligned}
\int_0^{2\pi} \cos jx \, dx &= 0, \\
\int_0^{2\pi} \sin jx \, dx &= 0, \\
\int_0^{2\pi} \cos jx \, \cos kx \, dx &= \begin{cases} \pi & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \\
\int_0^{2\pi} \sin jx \, \sin kx \, dx &= \begin{cases} \pi & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \\
\int_0^{2\pi} \sin jx \, \cos kx \, dx &= 0.
\end{aligned}$$

Proof

First we note that

$$\int_0^{2\pi} \cos jx \, dx = \left[\frac{1}{j} \sin jx \right]_0^{2\pi} = \frac{1}{j} (\sin 2j\pi - 0) = 0$$

and

$$\int_0^{2\pi} \sin jx \, dx = \left[-\frac{1}{j} \cos jx \right]_0^{2\pi} = -\frac{1}{j} (\cos 2j\pi - 1) = 0$$

for all non-zero integers j, since $\cos 2j\pi = 1$ and $\sin 2j\pi = 0$ for all integers j.

Let j and k be positive integers. It follows from basic trigonometrical identities that

$$\int_0^{2\pi} \cos jx \, \cos kx \, dx = \tfrac{1}{2} \int_0^{2\pi} (\cos((j-k)x) + \cos((j+k)x)) \, dx.$$

and

$$\int_0^{2\pi} \sin jx \, \sin kx \, dx = \tfrac{1}{2} \int_0^{2\pi} (\cos((j-k)x) - \cos((j+k)x)) \, dx$$

But

$$\int_0^{2\pi} \cos((j+k)x) \, dx = 0$$

(since $j + k$ is a positive integer, and is thus non-zero).

Also

$$\int_0^{2\pi} \cos((j-k)x)\,dx = 0 \text{ if } j \neq k,$$

and

$$\int_0^{2\pi} \cos((j-k)x)\,dx = 2\pi \text{ if } j = k$$

(since $\cos((j-k)x) = 1$ when $j = k$). It follows that

$$\begin{aligned}\int_0^{2\pi} \cos jx \cos kx\,dx &= \int_0^{2\pi} \sin jx \sin kx\,dx = \tfrac{1}{2}\int_0^{2\pi} \cos((j-k)x)\,dx \\ &= \begin{cases} \pi & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}\end{aligned}$$

Also

$$\int_0^{2\pi} \sin jx \cos kx\,dx = \tfrac{1}{2}\int_0^{2\pi} (\sin((j+k)x) + \sin((j-k)x))\,dx = 0$$

for all positive integers m and n. (Note that $\sin((j-k)x) = 0$ in the case when $j = k$). ∎

40.2. Fourier Coefficients

We consider the theory of *harmonic analysis*, in which functions are approximated by sums of trigonometric functions.
Let p and q be real numbers satisfying $p < q$. Let us denote by $\mathcal{I}(p, q)$ the set whose elements are those real-valued functions on the interval

$$\{x \in \mathbb{R} : p \leq x \leq q\}$$

that are integrable and that have finitely many points of discontinuity in the interval.

We restrict attention to the case where $p = 0$ and $q = 2\pi$. Given $f, g \in \mathcal{I}(0, 2\pi)$, we define

$$(f, g) = \frac{1}{\pi} \int_0^{2\pi} f(x) g(x) \, dx$$

Note that

$$(f + h, g) = (f, g) + (h, g) \quad \text{and} \quad (f, g + h) = (f, g) + (f, h)$$

for all $f, g, h \in \mathcal{I}(0, 2\pi)$. Moreover $(f, g) = (g, f)$, and

$$(cf, g) = (f, cg) = c(f, g)$$

for all $f, g \in \mathcal{I}(0, 2\pi)$ and for all real numbers c.
Also let

$$\|f\| = \sqrt{(f, f)} = \left(\frac{1}{\pi} \int_0^{2\pi} f(x)^2 \, dx \right)^{\frac{1}{2}}.$$

If $f \in \mathcal{I}(0, 2\pi)$, and if $\|f\| = 0$ then either $f(x) = 0$ for all real numbers x satisfying $0 \leq x \leq l$ or else the set of values of x for which $f(x) \neq 0$ is a finite set whose elements are points of discontinuity of the function f. It follows that if $f, g \in \mathcal{I}(0, 2\pi)$ and if $\|f - g\| = 0$ then either $f(x) = g(x)$ for all real numbers x satisfying $0 \leq x \leq l$ or else the set of values of x for which $f(x) \neq g(x)$ is a finite set whose elements are points of discontinuity either of the function f or else of the function g.

In general $\|f - g\|$ can be regarded as a measure of the "closeness" of the functions f and g. It is but one of many such measures of closeness in widespread use by mathematicians.

let $c_j(x) = \cos jx$ for all non-negative integers j, and let $s_j(x) = \sin jx$ for all positive integers j. Then $c_0(x) = 1$ for all x, and therefore

$$(c_0, c_0) = \frac{1}{\pi} \int_0^{2\pi} (c_0(x))^2 \, dx = 2.$$

Also if j is a positive integer then

$$\begin{aligned} (c_0, c_j) &= (c_j, c_0) = \frac{1}{\pi} \int_0^{2\pi} \cos jx \, dx = 0, \\ (c_0, s_j) &= (s_j, c_0) = \frac{1}{\pi} \int_0^{2\pi} \sin jx \, dx = 0. \end{aligned}$$

Next let j and k be positive integers. It follows from Proposition 40.1 that

$$\begin{aligned}
(c_j, c_k) &= \frac{1}{\pi}\int_0^{2\pi} \cos jx \; coskx \, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \\
(s_j, s_k) &= \frac{1}{\pi}\int_0^{2\pi} \sin jx \; sinkx \, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \\
(s_j, c_k) &= (c_j, s_k) = 0
\end{aligned}$$

Proposition 40.2

Let $f(x)$ be a real-valued function of the real variable x defined for $0 \leq x \leq 2\pi$. Suppose that there exist constants $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ such that

$$f(x) = \tfrac{1}{2}a_0 + \sum_{j=1}^{N} a_j \cos jx + \sum_{j=1}^{N} b_j \sin jx$$

for all x. Then

$$a_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cosjx\, dx$$

for $j = 0, 1, \ldots, N$ and

$$b_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\, sinjx\, dx$$

for $j = 1, 2, \ldots, N$.

Proof

The function f satisfies

$$f(x) = \frac{1}{2} a_0 c_0 + \sum_{k=1}^{N} a_k c_k(x) + \sum_{k=1}^{N} b_k s_k(x),$$

where the functions $c_0, c_1, \ldots, c_N$ and $s_1, s_2, \ldots, s_N$ are defined as described above. It follows that

$$(f(x), c_0) = \tfrac{1}{2} a_0 (c_0, c_0) + \sum_{k=1}^{N} a_j (c_k, c_0) + \sum_{k=1}^{N} b_k (s_k, c_0).$$

But $(c_k, c_0) = 0$ and $(s_k, c_0) = 0$ for all positive integers k. It follows that

$$(f(x), c_0) = \tfrac{1}{2} a_0 (c_0, c_0) = a_0.$$

Next let j be a positive integer. Then

$$(f(x), c_j) = \tfrac{1}{2} a_0 (c_0, c_j) + \sum_{k=1}^{N} a_k (c_k, c_j) + \sum_{k=1}^{N} b_k (s_k, c_j).$$

But $(c_0, c_j) = 0$, $(s_k, c_j) = 0$ for all integers k, and $(c_k, c_j) = 0$ unless $j = k$. It follows that

$$(f(x), c_j) = a_j.$$

Similarly

$$(f(x), s_j) = \tfrac{1}{2} a_0 (c_0, s_j) + \sum_{k=1}^{N} a_k (c_k, s_j) + \sum_{k=1}^{N} b_k (s_k, s_j) = b_j.$$

The result follows. ∎

Now let $f(x)$ be an integrable function, defined for values of the real variable x satisfying $0 \leq x \leq 2\pi$, that is either continuous throughout its domain or else has at most finitely many points of discontinuity there. Let

$$p(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{N} a_k c_k(x) + \sum_{k=1}^{N} b_k s_k(x),$$

where $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ are the Fourier coefficients of f, determined so that $a_0 = (f, c_0)$, $a_k = (f, c_k)$ and $b_k = (f, s_k)$ for $k = 1, 2, \ldots, N$. Then

$$\begin{aligned}
(f - p, c_0) &= (f, c_0) - \tfrac{1}{2}a_0(c_0, c_0) = (f, c_0) - a_0 = 0,\\
(f - p, c_j) &= (f, c_j) - (p, c_j) = (f, c_j) - a_j = 0,\\
(f - p, s_j) &= (f, s_j) - (p, s_j) = (f, s_j) - b_j = 0.
\end{aligned}$$

Let $u_0, u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ be arbitary real numbers, and let

$$q(x) = \tfrac{1}{2} u_0 + \sum_{k=1}^{N} u_k c_k(x) + \sum_{k=1}^{N} v_k s_k(x).$$

Then

$$(f-p, q) = \tfrac{1}{2} u_0 (f-p, c_0) + \sum_{k=1}^{N} u_k (f-p, c_k) + \sum_{k=1}^{N} v_k (f-p, s_k) = 0,$$

and $(q, f-p) = (f-p, q) = 0$. It follows that

$$\begin{aligned} &(f-p-q, f-p-q) \\ &\quad = (f-p, f-p) - (f-p, q) - (q, f-p) + (q, q) \\ &\quad = (f-p, f-p) + (q, q). \end{aligned}$$

Thus

$$\|f - p - q\|^2 = \|f - p\|^2 + \|q\|^2.$$

Now, taking $\|f - p - q\|$ as a measure of the closeness of the function $p + q$ to the function f, we see that the function $p + q$ is closest to f with respect to this measure when $q = 0$.

Thus if we seek to approximate f by a function of the form

$$p(x) = \tfrac{1}{2}a_0 + \sum_{j=1}^{N} a_j \cos jx + \sum_{j=1}^{N} b_j \sin jx,$$

where coefficients $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ are to be determined to as to achieve a good fit, we see that the values of these coefficients that result in an approximating function that is closest to the function f, where distance from f is measured by the quantity $\|f - p\|$, precisely when the coefficients $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ are the Fourier coefficients of f, defined such that

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f(x)\, cosjx\, dx$$

for $j = 0, 1, \ldots, N$ and

$$b_j = \frac{1}{\pi} \int_0^{2\pi} f(x)\, sinjx\, dx$$

for $j = 1, 2, \ldots, N$.