MA2C03 Mathematics
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David R. Wilkins

Example

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

where the real numbers b, c, g, h and m are constants and $m^2 + bm + c \neq 0$. In this case we look for a "particular integral" of the form

$$y_P = (u + vx)e^{mx}.$$

Differentiating using the Product Rule, we find that

$$\frac{dy_P}{dx} = ve^{mx} + m(u + vx)e^{mx} = (v + mu + mvx)e^{mx}$$

and

$$\frac{d^2y}{dx^2} = 2mve^{mx} + m^2(u + vx)e^{mx} = (2mv + m^2u + m^2vx)e^{mx}$$

and therefore

$$\begin{aligned}&\frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P \\ = &\Bigl(2mv + m^2u + bv + (bm + c)u + (m^2 + bm + c)vx\Bigr)e^{mx}.\end{aligned}$$

It follows that y_P solves the differential equation if and only if

$$(2m + b)v + (m^2 + bm + c)u = g$$

and

$$(m^2 + bm + c)v = h.$$

Solving the second of these equations for v, we find that

$$v = \frac{h}{m^2 + bm + c}.$$

Then solving the other equation for u, we find that

$$\begin{aligned} u &= \frac{1}{m^2+bm+c}(g-(2m+b)v) \\ &= \frac{(m^2+bm+c)g-(2m+b)h}{(m^2+bm+c)^2} \end{aligned}$$

Thus

$$y_P = \frac{(m^2+bm+c)(g+hx)-(2m+b)h}{(m^2+bm+c)^2}e^{mx}.$$

The general solution of the differential equation then takes the form

$$y = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx} + y_C(x).$$

where the complementary function y_C satisfies the differential equation

$$\frac{d^2 y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example

Consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = (3 - 2x)e^{4x}.$$

This equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

with $b = -2$, $c = 10$, $g = 3$, $h = -2$ and $m = 4$. We have shown that equations of this form have a particular integral y_P that takes the form

$$y_P = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx}.$$

Substituting the values of b, c, g, h and m into this equation, we find that

$$\begin{aligned} m^2 + bm + c &= 16 - 2 \times 4 + 10 = 18, \\ (2m + b)h &= (2 \times 4 - 2) \times (-2) = -12, \end{aligned}$$

and therefore

$$y_P = \frac{66 - 36x}{324} e^{4x} = \left(\frac{11}{54} - \frac{x}{9} \right) e^{4x}.$$

Now the auxiliary polynomial $z^2 - 2z + 10$ has roots $1 + \sqrt{-1}\,3$ and $1 - \sqrt{-1}3$. It follows that the complementary function y_C for this differential equation takes the form

$$y_C(x) = e^x(A\cos 3x + B\sin 3x).$$

The general solution to the differential equation thus takes the form

$$y = \left(\frac{11}{54} - \frac{x}{9}\right) e^{4x} + e^x(A\cos 3x + B\sin 3x).$$

Example

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\ \cos kx + h\ \sin kx$$

where the real numbers b, c, g, h and k are constants.

$$\frac{d}{dx}(\cos kx) = -k \sin kx \quad \text{and} \quad \frac{d}{dx}(\sin kx) = k \cos kx.$$

We look for a particular integral y_P of the form

$$y_P = u \cos kx + v \sin kx.$$

Differentiating, we find that

$$\frac{dy_P}{dx} = kv \cos kx - ku \sin kx$$

and

$$\frac{d^2 y_P}{dx^2} = -k^2 u \cos kx - k^2 v \sin kx,$$

and thus

$$\begin{aligned} &\frac{d^2 y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P \\ &\quad = \quad ((c - k^2)u + bkv) \cos kx + ((c - k^2)v - bku) \sin kx. \end{aligned}$$

Therefore u and v should be chosen to satisfy the equations

$$(c - k^2)u + bkv = g \quad \text{and} \quad (c - k^2)v - bku = h.$$

It follows that

$$\begin{aligned} & bkg + (c - k^2)h \\ & \quad = \ bk((c - k^2)u + bkv) + (c - k^2)((c - k^2)v - bku) \\ & \quad = \ (b^2k^2 + (c - k^2)^2)v \\ & (c - k^2)g - bkh \\ & \quad = \ (c - k^2)((c - k^2)u + bkv) - bk((c - k^2)v - bku) \\ & \quad = \ (b^2k^2 + (c - k^2)^2)u. \end{aligned}$$

Thus

$$u = \frac{(c - k^2)g - bkh}{b^2k^2 + (c - k^2)^2}$$

and

$$v = \frac{bkg + (c - k^2)h}{b^2k^2 + (c - k^2)^2},$$

and thus

$$y_P = \frac{1}{b^2k^2 + (c - k^2)^2}\Big(((c - k^2)g - bkh)\cos kx + (bkg + (c - k^2)h)\sin kx\Big).$$

It follows that the general solution of the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

takes the form

$$y = \frac{1}{b^2k^2 + (c-k^2)^2}\Bigl(((c-k^2)g - bkh)\cos kx + (bkg + (c-k^2)h)\sin kx\Bigr) + y_C,$$

where the complementary function y_C satisfies the differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example

Consider the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 3\cos 2x + 4\sin 2x.$$

This equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

with $b = -6$, $c = 9$, $k = 2$, $g = 3$ and $h = 4$. We have shown that equations of this form have a particular integral y_P that takes the form

$$\begin{aligned} y_P &= \frac{1}{b^2k^2 + (c-k^2)^2}\Bigl(((c-k^2)g - bkh)\cos kx \\ &\qquad + (bkg + (c-k^2)h)\sin kx\Bigr). \end{aligned}$$

Substituting the values of b, c, k, g and h into this equation, we find that

$$\begin{aligned}
bk &= -12 \\
c - k^2 &= 9 - 4 = 5 \\
b^2k^2 + (c - k^2)^2 &= 144 + 25 = 169, \\
(c - k^2)g - bkh &= 5 \times 3 - (-12) \times 4 = 15 + 48 = 63, \\
bkg + (c - k^2)h &= (-12) \times 3 + 5 \times 4 = -36 + 20 = -16.
\end{aligned}$$

and therefore

$$y_P = \frac{1}{169}\left(63\cos 2x - 16\sin 2x\right).$$

Now the auxiliary polynomial $z^2 - 6z + 9$ has a repeated root with value 3. It follows that the complementary function y_C for this differential equation takes the form

$$y_C(x) = (A + Bx)e^{3x}.$$

The general solution to the differential equation thus takes the form

$$y = \frac{1}{169}(63\cos 2x - 16\sin 2x) + (A + Bx)e^{3x}.$$