MA2C03 Mathematics
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 53 (March 16, 2016)

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39.1. Review of Solutions of $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$.

The following lemma is a basic result that enables us to superimpose solutions of second order differential equations of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Lemma 39.4

Let u and v be solutions of the second order differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where b and c are constants, and let A and B be real numbers. Then $Au + Bv$ is also a solution of the differential equation.

39. Ordinary Differential Equations (continued)

Proof

Let $y = Au + Bv$. Then

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx}, \quad \frac{d^2y}{dx^2} = A\frac{d^2u}{d^2x} + B\frac{d^2v}{dx^2},$$

$$\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0 \quad \text{and} \quad \frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv = 0,$$

and therefore

$$\begin{aligned} \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= A\left(\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu\right) \\ &\quad + B\left(\frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv\right) \\ &= 0. \quad \blacksquare \end{aligned}$$

We now verify the solutions of all second order differential equations of this type. Let r be a real root of the auxiliary polynomial $z^2 + bz + c$ of the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

and let $y = e^{rx}$. Then

$$\frac{dy}{dx} = re^{rx} = ry \quad \text{and} \quad \frac{d^2y}{dx^2} = r^2e^{rx} = r^2y,$$

and therefore

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (r^2 + br + c)y = 0.$$

Thus e^{rx} is a solution of the differential equation.

It follows that if the auxiliary polynomial $z^2 + bz + c$ of the differential equation of the differential equation has two distinct real roots r and s then $Ae^{rx} + Be^{sx}$ is a solution of the differential equation for all real numbers A and B. Proposition 39.2 then ensures that all solutions of the differential equation are of this form.

Next suppose that r is a repeated root of the auxiliary polynomial. Then

$$z^2 + bz + c = (z - r)^2,$$

and therefore $b = -2r$ and $c = r^2$. If $y = xe^{rx}$ then

$$\frac{dy}{dx} = (rx + 1)e^{rx} = ry + e^{rx}$$

and

$$\frac{d^2y}{dx^2} = (r^2x + 2r)e^{rx} = r^2y + 2re^{rx},$$

and therefore

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (r^2 + br + c)y + (2r + b)e^{rx} = 0.$$

Thus xe^{rx} is a solution of the differential equation.

We have already shown that e^{rx} is also a solution of this differential equation. It follows that $(A + Bx)e^{rx}$ is a solution of this differential equation for all real constants A and B. Proposition 39.3 then ensures that all solutions of the differential equation are of this form.

Finally suppose that $p + iq$ is a root of the auxiliary polynomial $z^2 + bz + c$ for the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where $q \neq 0$ and $i = \sqrt{-1}$. Then $p - iq$ is also a root of the auxiliary polynomial. It follows that

$$\begin{aligned} z^2 + bz + c &= (z - p - iq)(z - p + iq) = (z - p)^2 + q^2 \\ &= z^2 - 2pz + p^2 + q^2, \end{aligned}$$

and therefore $b = -2p$ and $c = p^2 + q^2$.

Let $u = e^{px}\cos qx$ and $v = e^{px}\sin qx$. Then

$$\frac{du}{dx} = e^{px}(p\cos qx - q\sin qx) = pu - qv$$

and

$$\frac{dv}{dx} = e^{px}(p\sin qx + q\cos qx) = pv + qu,$$

and therefore

$$\begin{aligned}
\frac{d^2u}{dx^2} &= p\frac{du}{dx} - q\frac{dv}{dx} = p(pu - qv) - q(pv + qu) \\
&= (p^2 - q^2)u - 2pqv \\
\frac{d^2v}{dx^2} &= p\frac{dv}{dx} + q\frac{du}{dx} = p(pv + qu) + q(pu - qv) \\
&= (p^2 - q^2)v + 2pqu
\end{aligned}$$

39. Ordinary Differential Equations (continued)

It follows that

$$\begin{aligned}
\frac{d^2u}{dx^2} + b\frac{du}{dx} + c &= ((p^2 - q^2)u - 2pqv) + b(pu - qv) + cu \\
&= (p^2 - q^2 + bp + c)u - (2pq + bq)v, \\
\frac{d^2v}{dx^2} + b\frac{dv}{dx} + c &= ((p^2 - q^2)v + 2pqu) + b(pv + qu) + cv \\
&= (p^2 - q^2 + bp + c)v + (2pq + bq)v.
\end{aligned}$$

But

$$p^2 - q^2 + bp + c = p^2 - q^2 - 2p^2 + p^2 + q^2 = 0$$

and

$$2pq + bq = 2pq - 2pq = 0.$$

Therefore

$$\frac{d^2u}{dx^2} + b\frac{du}{dx} + c = 0$$

and

$$\frac{d^2v}{dx^2} + b\frac{dv}{dx} + c = 0.$$

Thus if

$$y = e^{px}(A\cos qx + B\sin qx)$$

where A and B are real constants, then $y = Au + Bv$ and therefore

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0.$$

Proposition 39.4 then ensures that all solutions of the differential equation are of this form.

39.2. Solutions of $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$.

Example

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g + hx + kx^2$$

where the real numbers b, c, g, h and k are constants and $c \neq 0$. In this case we look for a "particular integral" that takes the form of a polynomial of the same degree as that occurring on the right hand side of the given differential equation.

Thus in this case we look for a solution y_P satisfying the differential equation

$$\frac{d^2 y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

that takes the form

$$y_P = u + vx + wx^2.$$

Differentiating, we find that

$$\frac{dy_P}{dx} = v + 2wx \quad \text{and} \quad \frac{d^2 y_P}{dx^2} = 2w.$$

It follows that

$$\frac{d^2 y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = 2w + bv + cu + (2bw + cv)x + cwx^2.$$

Thus a quadratic polynomial y_P of the form $y_P = u + vx + wx^2$ satisfies the differential equation

$$\frac{d^2 y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

if and only if

$$2w + bv + cu + (2bw + cv)x + cwx^2 = g + hx + kx^2$$

for all values of the independent variable x. This is the case if and only if the coefficients of the quadratic polynomial on the left hand side are equal to the corresponding coefficients of the quadratic polynomial on the right hand side. Thus y_P is the required "particular integral" if and only if

$$2w + bv + cu = g, \quad 2bw + cv = h \quad \text{and} \quad cw = k.$$

Substituting $w = \frac{k}{c}$ into the equation $2bw + cv = h$, we find that

$$v = \frac{1}{c}(h - 2bw) = \frac{1}{c^2}(ch - 2bk).$$

If we then substitute this formula for v into the equation $2w + bv + cs = g$, we find that

$$u = \frac{1}{c}(g - 2w - bv) = \frac{1}{c^3}(c^2 g - 2ck - bch + 2b^2 k).$$

Thus

$$y_P = \frac{1}{c^3}\left(c^2 g - 2ck - bch + 2b^2 k + (c^2 h - 2bck)x + c^2 kx^2\right).$$

Now the quadratic polynomial y_P is just one of the solutions of the given differential equation. Other solutions are obtained by adding onto the particular integral y_P a *complementary function* y_C. Accordingly the general solution of the differential equation therefore takes the form

$$y = \frac{1}{c^3}\left(c^2 g - 2ck - bch + 2b^2 k + (c^2 h - 2bck)x + c^2 kx^2\right) + y_C(x),$$

where the complementary function y_C satisfies the differential equation

$$\frac{d^2 y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

The solution can be verified on the Wolfram Alpha website at

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http://www.wolframalpha.com/
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by entering the string

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y'' + b y' + cy = g + hx + kx^2
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into the search box.

Example

Consider the differential equation

$$\frac{d^2y}{dy^2} + 5\frac{dy}{dx} + 6y = x^2 - 7.$$

The equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g + hx + kx^2,$$

with $b = 5$, $c = 6$, $g = -7$, $h = 0$ and $k = 1$. We have shown that equations of this form have a particular integral y_P that takes the form

$$y_P = \frac{1}{c^3}\left(c^2g - 2ck - bch + 2b^2k + (c^2h - 2bck)x + c^2kx^2\right).$$

Substituting the values of b, c, g, h and k into this equation, we find that

$$\begin{aligned} c^3 &= 216, \\ c^2 g - 2ck - bch + 2b^2 k &= 36 \times (-7) - 2 \times 6 \times 1 + 2 \times 25 \\ &= -252 - 12 + 50 = -214, \\ c^2 h - 2bck &= 36 \times 0 - 2 \times 5 \times 6 \times 1 = -60, \\ c^2 k &= 36 \times 1 = 36, \end{aligned}$$

and therefore

$$\begin{aligned} y_P &= \frac{-214 - 60x + 36x^2}{216}. \\ &= -\frac{107}{108} - \frac{5}{18}x + \frac{1}{6}x^2 \end{aligned}$$

Now the auxiliary polynomial $z^2 + 5z + 6$ has roots -2 and -3. The complementary function $y_C(x)$ therefore satisfies

$$y_C(x) = Ae^{-2x} + Be^{-3x}.$$

It follows that the general solution to the differential equation is given by

$$y = -\frac{107}{108} - \frac{5}{18}x + \frac{1}{6}x^2 + Ae^{-2x} + Be^{-3x},$$

where A and B are real constants.

The solution can be verified on the Wolfram Alpha website at

`http://www.wolframalpha.com/`

by entering the string

`y'' + 5 y' + 6 y = x^2 - 7`

into the search box.