MA2C03 Mathematics
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 52 (March 11, 2016)

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Proposition 39.4

Let b and c be real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

throughout the open interval I. Suppose that the quadratic polynomial $z^2 + bz + c$ has two non-real roots $p + \sqrt{-1}q$ and $p - \sqrt{-1}q$. real roots r and s. Then there exist real constants A and B such that

$$y(x) = e^{px}(A\cos qx + B\sin qx).$$

Proof

Let $u(x) = y(x)e^{-px}$. Then

$$\begin{aligned} y(x) &= u(x)e^{px}, \\ \frac{dy(x)}{dx} &= \left(\frac{du(x)}{dx} + pu(x)\right)e^{px}, \\ \frac{d^2y(x)}{dx^2} &= \left(\frac{d^2u(x)}{dx^2} + 2p\frac{du(x)}{dx} + p^2u(x)\right)e^{px}. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy \\ &= \left(\frac{d^2u}{dx^2} + (b+2p)\frac{du}{dx} + (p^2+bp+c)u\right)e^{px}. \end{aligned}$$

Moreover the quadratic polynomial $z^2 + bz + c$ satisfies

$$\begin{aligned} z^2 + bz + c &= (z - p - \sqrt{-1}\, q)(z - p + \sqrt{-1}\, q) \\ &= (z-p)^2 + q^2 = z^2 - 2pz + p^2 + q^2, \end{aligned}$$

and therefore, on equating coefficients of the variable z, we find that $2p = -b$ and $p^2 + q^2 = c$. It follows that

$$p^2 + bp + c = p^2 - 2p^2 + (p^2 + q^2) = q^2.$$

Therefore the dependent variable u satisfies the differential equation

$$\frac{d^2 u}{dx^2} + q^2 u = 0.$$

Theorem 39.3 therefore ensures that

$$u(x) = A \cos qx + B \sin qx,$$

where A and B are constants, and therefore

$$y(x) = e^{px}(A \cos qx + B \sin qx),$$

as required. $\blacksquare$