MA2C03 Mathematics
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 51 (March 9, 2016)

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Let x and y be real variables, where the value of y is expressible as a function of the independent real variable x as x varies over some open interval I. We say that y satisfies a *ordinary differential equation* of *second order* in x if there exists a function H of four real variables with the property that

$$H\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0$$

for all real values of x in the appropriate range within which which the independent variable x takes its values.

We next prove results that determine all solutions of second order differential equations of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where b and c are constants. These results show that solutions can be determined directly once the roots of the *auxiliary polynomial* $z^2 + bz + c$ have been determined.

Proposition 39.2

Let b and c be real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

throughout the open interval I. Suppose that the quadratic polynomial $z^2 + bz + c$ has two distinct real roots r and s. Then there exist real constants A and B such that

$$y(x) = Ae^{rx} + Be^{sx}.$$

Proof

Let $u(x) = y(x)e^{-rx}$ for all $x \in I$. Then $y(x) = x(x)e^{rx}$ for all $x \in I$. Differentiating $y(x)$ with respect to x using the product rule, we find that

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{du}{dx} + ru\right) e^{rx}, \\ \frac{d^2y}{dx^2} &= \left(\frac{d^2u}{dx^2} + 2r\frac{du}{dx} + r^2u\right) e^{rx}. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy \\ &= \left(\frac{d^2u}{dx^2} + (2r+b)\frac{du}{dx} + (r^2 + br + c)u\right) e^{rx}. \end{aligned}$$

But r has been chosen so as to satisfy the quadratic equation $r^2 + br + c = 0$. It follows that

$$\frac{d^2u}{dx^2} + (2r + b)\frac{du}{dx} = 0.$$

Thus if $v = \dfrac{du}{dx}$ then

$$\frac{dv}{dx} + (2r + b)v = 0.$$

Now $z^2 + bz + c = (z - r)(z - s) = z^2 - (r + s)z + rs$. It follows that $b = -(r + s)$, and therefore $2r + b = r - s$. Thus

$$\frac{dv}{dx} - (s - r)v = 0.$$

It follows from Corollary 39.1 that there exists a constant B such that

$$v(x) = (s - r)Be^{(s-r)x}.$$

Integrative the function $v(x)$ in order to determine $u(x)$, we find that there exist constants A and B such that

$$u(x) = A + Be^{(s-r)x}.$$

But then

$$y(x) = Ae^{rx} + Be^{sx},$$

as required. $\blacksquare$

Proposition 39.3

Let b and c be real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

throughout the open interval I. Suppose that the quadratic polynomial $z^2 + bz + c$ has a repeated real root r. Then there exist real constants A and B such that

$$y(x) = (A + Bx)e^{rx}.$$

Proof

Let $u(x) = y(x)e^{-rx}$ for all $x \in I$. Then $y(x) = x(x)e^{rx}$ for all $x \in I$. Repeating the calculation in the proof of Proposition 39.2, we find that

$$\frac{d^2u}{dx^2} + (2r + b)\frac{du}{dx} = 0.$$

Moreover $z^2 + bz + c = (z - r)^2$ (because r is a repeated root of the quadratic polynomial on the left hand side of this equation) and therefore $b = -2r$. It follows that

$$\frac{d^2u}{dx^2} = 0,$$

and therefore $u(x) = A + Bx$, where A and B are real constants. It follows that $y(x) = (A + Bx)e^{rx}$, as required. ∎

Theorem 39.3

Let k be a positive real number, and let x be an independent real variable that takes values in an open interval I. Let y be a real variable, expressible as a twice-differentiable function of the independent real variable x, that satisfies the second order differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

throughout the open interval I. Then there exist real constants A and B such that

$$y = A\cos kx + B\sin kx$$

throughout the open interval I.

Proof

We first prove the result in the special case where $\cos kx \neq 0$ for all $x \in I$. In this case we can express y in terms of another real variable u, where u is a twice-differentiable function of x and $y(x) = u(x) \cos kx$ for all $x \in I$. Now

$$\frac{d}{dx}(\cos kx) = -k \sin kx \quad \text{and} \quad \frac{d}{dx}(\sin kx) = k \cos kx.$$

On applying the Product Rule of differential calculus, we find that if $y = u \cos kx$ then

$$\frac{dy}{dx} = \frac{du}{dx} \cos kx - ku \sin kx.$$

On differentiating again, we find that

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx}\cos kx\right) - k\frac{d}{dx}(u\sin kx) \\
&= \frac{d}{dx}\left(\frac{du}{dx}\right)\cos kx + \frac{du}{dx}\frac{d}{dx}(\cos kx) \\
&\qquad - k\frac{du}{dx}\sin kx - ku\frac{d}{dx}(\sin kx) \\
&= \frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx - k^2u\cos kx \\
&= \frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx - k^2y.
\end{aligned}
$$

Thus y satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2y = 0$$

if and only if $y = u \cos kx$, where

$$\frac{d^2u}{dx^2} \cos kx - 2k\frac{du}{dx} \sin kx = 0.$$

Now let

$$v = \frac{du}{dx} \cos^2 kx$$

(where $\cos^2 kx = (\cos kx)^2$). It then follows from the Product Rule of differential calculus that

$$\begin{aligned} \frac{dv}{dx} &= \frac{d^2 u}{dx^2} \cos^2 kx - 2k \frac{du}{dx} \cos kx \sin kx \\ &= \left(\frac{d^2 u}{dx^2} \cos kx - 2k \frac{du}{dx} \sin kx \right) \cos kx. \end{aligned}$$

Now $\cos kx \neq 0$ for all $x \in I$. It follows that

$$\frac{d^2u}{dx^2}\cos kx - 2k\frac{du}{dx}\sin kx = 0$$

for all $x \in I$ if and only if

$$\frac{dv}{dx} = 0$$

for all $x \in I$. However this is the case if and only if $v = Bk$ for all $x \in I$, where B is a real constant, in which case

$$\frac{du}{dx} = \frac{Bk}{\cos^2 kx}$$

for all $x \in I$.

We conclude that y satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2y = 0$$

on the open interval I, where $\cos kx \neq 0$ for all $x \in I$, if and only if $y = u \cos kx$ on I, where

$$\frac{du}{dx} = \frac{Bk}{\cos^2 kx}.$$

Now it follows from the Quotient Rule of differential calculus that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\sin kx}{\cos kx}\right) &= \frac{\dfrac{d}{dx}(\sin kx)\cos kx - \sin kx\,\dfrac{d}{dx}(\cos kx)}{\cos^2 kx} \\
&= \frac{k\cos^2 kx + k\sin^2 kx}{\cos^2 kx} = \frac{k}{\cos^2 kx}
\end{aligned}$$

(where we have used the fact that $\sin^2\theta + \cos^2\theta = 1$ for all real numbers θ).

It follows that a variable u expressible as a differentiable function of x on the open interval I satisfies

$$\frac{du}{dx} = \frac{Bk}{\cos^2 kx}$$

throughout that open interval if and only if

$$\frac{d}{dx}\left(u - \frac{B\sin kx}{\cos kx}\right) = 0,$$

in which case

$$u = A + \frac{B\sin kx}{\cos kx}$$

for some constant A.

We have thus shown that if k is a real number, and if y is a twice-differentiable function of an independent real variable x, where x varies over an open interval I and $\cos kx \neq 0$ for all $x \in I$, then y satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

if and only if

$$y = A\cos kx + B\sin kx$$

for all values of the independent variable x belonging to the open interval I.

We now extend the result to cases where the open interval I includes values of x for which $\cos kx = 0$. Let $s \in I$ satisfy $\cos ks = 0$, and let I_1 and I_2 be open subintervals of I that are of the form

$$I_1 = \{x \in \mathbb{R} : a < x < s\}, \quad I_2 = \{x \in \mathbb{R} : s < x < b\},$$

where a is chosen close enough to s to ensure that $\cos kx \neq 0$ for all $x \in I_1$ and b is chosen close enough to s to ensure that $\cos kx \neq 0$ for all $x \in I_2$. Let y be a twice-differentiable function of x for $a < x < b$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0.$$

Then there exist constants A_1, B_1, A_2 and B_2 such that

$$y = A_1 \cos kx + B_1 \sin kx \quad \text{for all } x \in I_1$$

$$y = A_2 \cos kx + B_2 \sin kx \quad \text{for all } x \in I_2$$

Then $\cos ks = 0$ and $\sin ks = \pm 1$. It follows from the continuity and differentiability of y with respect to x that

$$B_1 \sin ks = \lim_{x \to s} y = B_2 \sin ks$$

and

$$A_1 \sin ks = \lim_{x \to s} \frac{dy}{dx} = B_2 \sin ks,$$

and thus $A_1 = A_2$ and $B_1 = B_2$. We have thus shown that the coefficients of $\cos kx$ and $\sin kx$ that determine y as a function of x match up on both sides of points s of the open interval I at which $\cos ks = 0$. It follows that there exist constants A and B such that

$$y = A \cos kx + B \sin kx$$

for all values of the independent variable x belonging to the open interval I, as required. ∎