MA2C03 Mathematics
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39. Ordinary Differential Equations

We consider *differential equations* satisfied by real variables in situations where some real variable y is expressible as a differentiable function of a real variable x, where x takes values in some specified range. In such situations, the real variable x is said to be the *independent* real variable, and the variable y whose values are determined by the corresponding values of x is said to be a *dependent* real variable.

The independent real variable x will typically vary over an open interval. A subset I of the real numbers is said to be an *open interval* if it takes one of the following four forms:—

(i) $I = \mathbb{R}$;

(ii) $I = \{x \in \mathbb{R} : x > a\}$, where a is some specified real constant;

(iii) $I = \{x \in \mathbb{R} : x < b\}$, where b is some specified real constant;

(iv) $I = \{x \in \mathbb{R} : a < x < b\}$, where a and b are specified real constants.

Note that if I is an open interval, and if u and v are real numbers belonging to I, then $x \in I$ for all real numbers x satisfying $u < x < v$.

Let x and y be real variables, where the value of y depends on the the value of x, so $y = h(x)$ for all values of x in some specified range, where $h(x)$ is a differentiable function of x. In this situation we regard x as an *independent* real variable, and regard y to be a *dependent* real variable whose value depends on that of the independent variable x. For instance it may be the case that $y = x^3$ for all real values of the independent variable x, or $y = \frac{1}{x}$ for all positive values of the independent variable x. We say that y satisfies a *ordinary differential equation* of *first order* in x if there exists a function H of three real variables with the property that

$$H\left(\frac{dy}{dx}, y, x\right) = 0$$

for all real values of x in the appropriate range within which which the independent variable x takes its values.

Example

Let $y = e^{3x}$ for all real numbers x. Then y satisfies the first order differential equation

$$\frac{dy}{dx} - 3y = 0.$$

Example

Let $y = x^2$ for all real numbers x. Then y satisfies the first order differential equation

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0.$$

Example

Let $y = \dfrac{1}{x^2}$ for all positive real numbers x. Then y satisfies the first order differential equation

$$\left(\frac{dy}{dx}\right)^2 - 4y^3 = 0,$$

where the independent real variable x ranges over the set of positive real numbers. Indeed

$$\frac{dy}{dx} = -\frac{2}{x^3},$$

and therefore

$$\left(\frac{dy}{dx}\right)^2 = \frac{4}{x^6} = 4y^3$$

for all positive real numbers x.

Example

Let $y = \sin 4x$ for all real number x. Then y satisfies the first order differential equation

$$\left(\frac{dy}{dx}\right)^2 + 16y^2 - 16 = 0.$$

Indeed $\dfrac{dy}{dx} = 4\cos 4x$, and therefore

$$\left(\frac{dy}{dx}\right)^2 + 16y^2 - 16 = 16\cos^2 4x + 16\sin^2 4x - 16 = 0,$$

(We use here the trigonometrical identity that ensures that $\cos^2\theta + \sin^2\theta = 1$ for all real numbers θ.)

Let $f(x)$ be a continuous function of the independent real variable x, let c be a real number, and let y be a real variable that satisfies the differential equation

$$\frac{dy}{dx} + cy = f(x).$$

We seek to determine y as a function of the independent variable x.

Suppose that y is expressed in the form $y(x) = u(x)e^{rx}$, where r is a constant and u is a differentiable function of the independent variable x. It follows from the Product Rule of differential calculus that

$$\frac{dy}{dx} = \frac{du}{dx}e^{rx} + u\frac{d}{dx}(e^{rx}) = \frac{du}{dx}e^{rx} + rue^{rx}.$$

It follows that

$$\frac{dy}{dx} + cy = \left(\frac{du}{dx} + (c+r)u\right) e^{rx},$$

Thus the function y of x satisfies the given differential equation

$$\frac{dy}{dx} + cy = f(x).$$

if and only if $y(x) = u(x)e^{rx}$, where $u(x)$ is a differentiable function of the independent variable x that satisfies the differential equation

$$\frac{du}{dx} + (c+r)u = f(x)e^{-rx}.$$

The value of the constant r has not so far been chosen. Suppose we take $r = -c$. We conclude that y satisfies the given differential equation

$$\frac{dy}{dx} + cy = f(x).$$

if and only if $y(x) = u(x)e^{-cx}$, where u satisfies the differential equation

$$\frac{du}{dx} = f(x)e^{cx}.$$

Proposition 39.1

Let I be an open interval, let x be an independent real variable which ranges over the open interval I, let c be a constant, let $f(x)$ be a continuous function of x on the interval I, and let y be a dependent variable expressible as a differentiable function of the independent variable x. Let $g(x)$ be a function of x that satisfies

$$g(x) = \int f(x)e^{cx}\,dx.$$

Then the dependent variable y satisfies the differential equation

$$\frac{dy}{dx} + cy = f(x)$$

if and only if $y = g(x)e^{-cx} + Ae^{-cx}$ for all $x \in I$, where A is some real constant.

Proof

Let

$$g(x) = \int f(x)e^{cx}\,dx.$$

(In other words, let $g(x)$ be any function of x whose derivative with respect to x is equal to the function $f(x)e^{cx}$.) Then u satisfies the differential equation

$$\frac{du}{dx} = f(x)e^{cx}$$

if and only if

$$\frac{d}{dx}\left(u(x) - g(x)\right) = 0,$$

and moreover this is the case if and only if $u(x) = g(x) + A$ for some real constant A.

It follows that the dependent variable y satisfies the differential equation

$$\frac{dy}{dx} + cy = f(x)$$

if and only if

$$y(x) = g(x)e^{-cx} + Ae^{-cx},$$

where A is a real constant. The result follows. ∎

Corollary 39.1

Let y be a real variable expressible as a differentiable function of an independent real variable x. Then the dependent real variable y satisfies the differential equation

$$\frac{dy}{dx} + cy = 0,$$

where c is a real constant, if and only if there exists some real constant A for which

$$y(x) = Ae^{-cx}.$$

Proof

This follows from Proposition 39.1 on setting the function $f(x)$ in the statement of that proposition equal to the zero function. ∎

Corollary 39.2

Let $f(x)$ be a continuous function of f defined over an open interval I, let c be a real constant, and Let y_1 and y_2 be real variables dependent on an independent real variable x that ranges over the open interval I. Suppose that the first order differential equation

$$\frac{dy}{dx} + cy = f(x)$$

both when $y = y_1$ and also when $y = y_2$. Then there exists a real constant A such that $y_2 = y_1 + Ae^{-cx}$.

Proof

The dependent variables y_1 and y_2 satisfy

$$\frac{dy_1}{dx} + cy_1 = f(x) \quad \text{and} \quad \frac{dy_2}{dx} + cy_2 = f(x).$$

Let $u = y_2 - y_1$. Then

$$\frac{du}{dx} + cu = \left(\frac{dy_2}{dx} + cy_2\right) - \left(\frac{dy_1}{dx} + cy_1\right) = 0.$$

It follows from Corollary 39.1 that there exists some real constant A such that $u = Ae^{-cx}$ for all $x \in I$. Then $y_2 = y_1 + Ae^{-cx}$, as required. ∎

Example

Let us consider the differential equation

$$\frac{dy}{dx} + cy = g + hx + kx^2$$

where the real numbers c, g, h and k are constants and $c \neq 0$. This differential equation could be solved by applying the result of Proposition 39.1 and evaluating the resulting integral.

We shall however solve this differential equation by an alternative method, suitable in situations where the right hand side of the differential equation is a "forcing function" that is a polynomial in the independent variable x.

In this case we look for a "particular integral" that takes the form of a polynomial of the same degree as that occurring on the right hand side of the given differential equation.

Thus in this case we look for a solution y_P satisfying the differential equation

$$\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

that takes the form

$$y_P = u + vx + wx^2.$$

Differentiating, we find that

$$\frac{dy_P}{dx} = v + 2wx.$$

It follows that

$$\frac{dy_P}{dx} + cy_P = (v + cs) + (2w + cv)x + cwx^2.$$

Thus a quadratic polynomial y_P of the form $y_P = u + vx + wx^2$ satisfies the differential equation

$$\frac{dy_P}{dx} + cy_P = g + hx + kx^2$$

if and only if

$$v + cu + (2u + cv)x + cwx^2 = g + hx + kx^2$$

for all values of the independent variable x. This is the case if and only if the coefficients of the quadratic polynomial on the left hand side are equal to the corresponding coefficients of the quadratic polynomial on the right hand side. Thus y_P is the required "particular integral" if and only if

$$t + cs = g, \quad 2w + cv = h \quad \text{and} \quad cw = k.$$

Substituting $w = \dfrac{k}{c}$ into the equation $2w + cv = h$, we find that

$$v = \frac{1}{c}(h - 2w) = \frac{1}{c^2}(ch - 2k).$$

If we then substitute this formula for t into the equation $v + cu = g$, we find that

$$u = \frac{1}{c}(g - v) = \frac{1}{c^3}(c^2 g - ch + 2k).$$

Thus

$$y_P = \frac{1}{c^3}\left(c^2 g - ch + 2k + (c^2 h - 2ck)x + c^2 k x^2\right).$$

Now the quadratic polynomial y_P is just one of the solutions of the given differential equation. It follows from Corollary 39.2 that the other solutions of the differential equation

$$\frac{dy}{dx} + cy = g + hx + kx^2$$

take the form

$$y = y_P + Ae^{-cx},$$

where A is an arbitrary real constant. Thus the general solution of this differential equation takes the form

$$y(x) = \frac{1}{c^3}\left(c^2 g - ch + 2k + (c^2 h - 2ck)x + c^2 kx^2\right) + Ae^{-cx}.$$

The term Ae^{-cx} is often referred to as the "complementary function". It is the function that needs to be added to one solution to the differential equation to obtain other solutions. The general solution of the differential equation is the sum of the particular integral and the complementary function. The real constants c, g, h and k in the general solution are fixed constants determined by the differential equation. The real constant A takes different values for different solutions of the differential equation.

The solution can be verified on the Wolfram Alpha website at

`http://www.wolframalpha.com/`

by entering the string

`y' + cy = g + hx + kx^2`

into the search box.

The general solution of other differential equations of the form

$$\frac{dy}{dx} + cy = f(x)$$

can also be expressed as the sum of a particular integral and a complementary function.

Example

Let us consider the differential equation

$$\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

where the real numbers c, g, h and m are constants and $m + c \neq 0$. In this case we look for a "particular integral" of the form

$$y_P = (u + vx)e^{mx}.$$

Differentiating using the Product Rule, we find that

$$\frac{dy_P}{dx} = ve^{mx} + m(u + vx)e^{mx} = (v + mu + mvx)e^{mx}$$

and therefore

$$\frac{dy_P}{dx} + cy_P = (v + (m + c)u + (m + c)vx)e^{mx}$$

It follows that y_P solves the differential equation if and only if

$$v + (m + c)u = g \quad \text{and} \quad (m + c)v = h.$$

Solving the second of these equations for v, we find that

$$v = \frac{h}{m + c}.$$

Then solving the other equation for u, we find that

$$u = \frac{1}{m + c}(g - v) = \frac{1}{(m + c)^2}((m + c)g - h).$$

Thus

$$y_P = \frac{1}{(m + c)^2}((m + c)(g + hx) - h)e^{mx}.$$

It follows that the general solution of the differential equation

$$\frac{dy}{dx} + cy = (g + hx)e^{mx}$$

(when $c + m \neq 0$) takes the form

$$y = \frac{1}{(m+c)^2}((m+c)(g+hx) - h)e^{mx} + Ae^{-cx}.$$

The solution can be verified on the Wolfram Alpha website at

`http://www.wolframalpha.com/`

by entering the string

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y' + cy = (g + hx) e^(mx)
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into the search box.

Example

Let us consider the differential equation

$$\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

where the real numbers c, g, h and k are constants.

$$\frac{d}{dx}(\cos kx) = -k\sin kx \quad \text{and} \quad \frac{d}{dx}(\sin kx) = k\cos kx.$$

We look for a particular integral y_P of the form

$$y_P = u\cos kx + v\sin kx.$$

Differentiating, we find that

$$\frac{dy_P}{dx} + cy_P = (cu + kv)\cos kx + (cv - ku)\sin kx.$$

Therefore u and v should be chosen to satisfy the equations

$$cu + kv = g \quad \text{and} \quad cv - ku = h.$$

It follows that

$$kg + ch = k(cu + kv) + c(cv - ku) = (k^2 + c^2)v$$

and

$$cg - kh = c(cu + kv) - k(cv - ku) = (k^2 + c^2)u.$$

Thus

$$u = \frac{cg - kh}{k^2 + c^2} \quad \text{and} \quad v = \frac{kg + ch}{k^2 + c^2},$$

and thus

$$y_P = \frac{1}{k^2 + c^2}\left((cg - kh)\cos kx + (kg + ch)\sin kx\right).$$

It follows that the general solution of the differential equation

$$\frac{dy}{dx} + cy = g\,\cos kx + h\,\sin kx$$

takes the form

$$y = \frac{1}{k^2 + c^2}\left((cg - kh)\cos kx + (kg + ch)\sin kx\right) + Ae^{-cx}.$$