MA2C03 Mathematics
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Example

We shall find the equation of the plane containing the points A, B and C where $A = (3, 4, 1)$, $B = (4, 6, 1)$ and $C = (3, 5, 3)$. Now if $\mathbf{u} = \overrightarrow{AB} = (1, 2, 0)$ and $\mathbf{v} = \overrightarrow{AC} = (0, 1, 2)$ then the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. It follows that the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to this plane. Now $\mathbf{u} \times \mathbf{v} = (4, -2, 1)$, and therefore the displacement vector between any two points of the plane must be perpendicular to the vector $(4, -2, 1)$. It follows that the function mapping the point (x, y, z) to the quantity $4x - 2y + z$ must be constant throughout the plane. Thus the equation of the plane takes the form

$$4x - 2y + z = k,$$

for some constant k.

We can calculate the value of k by substituting for x, y and z the coordinates of any chosen point of the plane. On taking this chosen point to be the point A, we find that $k = 4 \times 3 - 2 \times 4 + 1 = 5$. Thus the equation of the plane is the following:

$$4x - 2y + z = 5.$$

(We can check our result by verifying that the coordinates of the points A, B and C do indeed satisfy this equation.)

38.4. Scalar Triple Products

Given three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in three-dimensional space, we can form the *scalar triple product* $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$. This quantity can be expressed as the determinant of a 3×3 matrix whose rows contain the Cartesian components of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Indeed

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\; v_3 w_1 - v_1 w_3,\; v_1 w_2 - v_2 w_1),$$

and thus

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1).$$

The quantity on the right hand side of this equality defines the determinant of the 3×3 matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

We have therefore obtained the following result.

Lemma 38.2

Let **u**, **v** *and* **w** *be vectors in three-dimensional space. Then*

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Using basic properties of determinants, or by direct calculation, one can easily obtain the identities

$$\begin{aligned} \mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \,.\, (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \,.\, (\mathbf{u} \times \mathbf{v}) \\ &= -\mathbf{u} \,.\, (\mathbf{w} \times \mathbf{v}) = -\mathbf{v} \,.\, (\mathbf{u} \times \mathbf{w}) = -\mathbf{w} \,.\, (\mathbf{v} \times \mathbf{u}) \end{aligned}$$

One can show that the absolute value of the scalar triple product $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$ is the volume of the parallelepiped in three-dimensional space whose vertices are the points whose displacement vectors from some fixed point O are $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} + \mathbf{w}$, $\mathbf{v} + \mathbf{w}$ and $\mathbf{u} + \mathbf{v} + \mathbf{w}$. (A *parallelepiped* is a solid like a brick, but whereas the faces of a brick are rectangles, the faces of the parallelepiped are parallelograms.)

Example

We shall find the volume of the parallelepiped in 3-dimensional space with vertices at $(0,0,0)$, $(1,2,0)$, $(-4,2,-5)$, $(0,1,1)$, $(-3,4,-5)$, $(1,3,1)$, $(-4,3,-4)$ and $(-3,5,-4)$. The volume of this parallelepiped is the absolute value of the scalar triple product $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$, where

$$\mathbf{u} = (1,2,0), \quad \mathbf{v} = (-4,2,-5), \quad \mathbf{w} = (0,1,1).$$

Now

$$\begin{aligned} \mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) &= (1,2,0) \,.\, (\,(-4,2,-5) \times (0,1,1)\,) \\ &= (1,2,0) \,.\, (7,4,-4) = 7 + 2 \times 4 = 15. \end{aligned}$$

Thus the volume of the paralellepiped is 15 units.

38.5. The Vector Triple Product Identity

Proposition 38.3

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w}.$$

Proof

Let $\mathbf{q} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$, and $\mathbf{q} = (q_1, q_2, q_3)$. Then

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

and hence $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{q} = (q_1, q_2, q_3)$, where

$$\begin{aligned} q_1 &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= (u_2 w_2 + u_3 w_3) v_1 - (u_2 v_2 + u_3 v_3) w_1 \\ &= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_1 \\ &= (\mathbf{u} \,.\, \mathbf{w}) v_1 - (\mathbf{u} \,.\, \mathbf{v}) w_1 \end{aligned}$$

Similarly

$$q_2 = (\mathbf{u} \,.\, \mathbf{w}) v_2 - (\mathbf{u} \,.\, \mathbf{v}) w_2$$

and

$$q_3 = (\mathbf{u} \,.\, \mathbf{w}) v_3 - (\mathbf{u} \,.\, \mathbf{v}) w_3$$

(In order to verify the formula for q_2 with an minimum of calculation, take the formulae above involving q_1, and cyclicly permute the subcripts 1, 2 and 3, replacing 1 by 2, 2 by 3, and 3 by 1. A further cyclic permutation of these subscripts yields the formula for q_3.) It follows that

$$\mathbf{q} = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w},$$

as required, since we have shown that the Cartesian components of the vectors on either side of this identity are equal. ∎

38.6. Orthonormal Triads of Unit Vectors

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors (i.e., vectors of length one) that are perpendicular to each other, and let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. It follows immediately from Proposition 38.2 that $|\mathbf{w}| = |\mathbf{u}|\,|\mathbf{v}| = 1$, and that this unit vector $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Then

$$\mathbf{u} \,.\, \mathbf{u} = \mathbf{v} \,.\, \mathbf{v} = \mathbf{w} \,.\, \mathbf{w} = 1$$

and

$$\mathbf{u} \,.\, \mathbf{v} = \mathbf{v} \,.\, \mathbf{w} = \mathbf{w} \,.\, \mathbf{u} = 0.$$

On applying the Vector Triple Product Identity (Proposition 38.3) we find that

$$\mathbf{v} \times \mathbf{w} = \mathbf{v} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \,.\, \mathbf{v})\,\mathbf{u} - (\mathbf{v} \,.\, \mathbf{u})\,\mathbf{v} = \mathbf{u},$$

and

$$\mathbf{w} \times \mathbf{u} = -\mathbf{u} \times \mathbf{w} = -\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = -(\mathbf{u} \,.\, \mathbf{v})\,\mathbf{u} + (\mathbf{u} \,.\, \mathbf{u})\,\mathbf{v} = \mathbf{v},$$

Therefore

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u} = \mathbf{w}, \quad \mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v} = \mathbf{u}, \quad \mathbf{w}\times\mathbf{u} = -\mathbf{u}\times\mathbf{w} = \mathbf{v},$$

Three unit vectors, such as the vectors **u**, **v** and **w** above, that are mutually perpendicular, are referred to as an *orthonormal triad* of vectors in three-dimensional space. The vectors **u**, **v** and **w** in any orthonormal triad are linearly independent. It follows directly from Theorem 36.2 that any vector in three-dimensional space may be expressed, uniquely, as a linear combination of the form

$$p\mathbf{u} + q\mathbf{v} + r\mathbf{w}.$$

Any Cartesian coordinate system on three-dimensional space determines an orthonormal triad **i**, **j** and **k**, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

The scalar and vector products of these vectors satisfy the same relations as the vectors **u**, **v** and **w** above. A vector represented in these Cartesian components by an ordered triple (x, y, z) then satisfies the identity

$$(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$