MA2C03 Mathematics
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38.3. The Vector Product

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space, with Cartesian components given by the formulae $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{v} = (b_1, b_2, b_3)$. The *vector product* $\mathbf{u} \times \mathbf{v}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ is the vector defined by the formula

$$\mathbf{u} \times \mathbf{v} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

Note that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Also $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ for all vectors $\mathbf{u}$. It follows easily from the definition of the vector product that

$$(s\mathbf{u} + t\mathbf{v}) \times \mathbf{w} = s\mathbf{u} \times \mathbf{w} + t\mathbf{v} \times \mathbf{w}, \quad \mathbf{u} \times (s\mathbf{v} + t\mathbf{w}) = s\mathbf{u} \times \mathbf{v} + t\mathbf{u} \times \mathbf{w}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and real numbers s and t.

Proposition 38.2

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space $\mathbb{R}^3$. Then their vector product $\mathbf{u} \times \mathbf{v}$ is a vector of length $|\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|$, where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Moreover the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the vectors $\mathbf{u}$ and $\mathbf{v}$.

Proof

Let $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{v} = (b_1, b_2, b_3)$, and let l denote the length $|\mathbf{u} \times \mathbf{v}|$ of the vector $\mathbf{u} \times \mathbf{v}$. Then

$$
\begin{aligned}
l^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 \\
&\quad + a_3^2b_1^2 + a_1^2b_3^2 - 2a_3a_1b_3b_1 \\
&\quad + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 \\
&= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) \\
&\quad - 2a_2a_3b_2b_3 - 2a_3a_1b_3b_1 - 2a_1a_2b_1b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 - 2a_2b_2a_3b_3 - 2a_3b_3a_1b_1 - 2a_1b_1a_2b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \,.\, \mathbf{v})^2
\end{aligned}
$$

since

$$|\mathbf{u}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\mathbf{v}|^2 = b_1^2 + b_2^2 + b_3^2, \quad \mathbf{u}\,.\,\mathbf{v} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

But $\mathbf{u}\,.\,\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ (Proposition 38.1). Therefore

$$l^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 (1 - \cos^2\theta) = |\mathbf{u}|^2 |\mathbf{v}|^2 \sin^2\theta$$

(since $\sin^2\theta + \cos^2\theta = 1$ for all angles θ) and thus
$l = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|$. Also

$$\mathbf{u}\,.\,(\mathbf{u}\times\mathbf{v}) = a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) = 0$$

and

$$\mathbf{v}\,.\,(\mathbf{u}\times\mathbf{v}) = b_1(a_2 b_3 - a_3 b_2) + b_2(a_3 b_1 - a_1 b_3) + b_3(a_1 b_2 - a_2 b_1) = 0$$

and therefore the vector $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ (Corollary 38.1), as required. ∎

Using elementary geometry, and the formula for the length of the vector product $\mathbf{u} \times \mathbf{v}$ given by Proposition 38.2 it is not difficult to show that the length of this vector product is equal to the area of a parallelogram in three-dimensional space whose sides are represented, in length and direction, by the vectors $\mathbf{u}$ and $\mathbf{v}$.

Remark

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors that are not colinear (i.e., so that they do not point in the same direction, or in opposite directions). The direction of $\mathbf{u} \times \mathbf{v}$ may be determined, using the thumb and first two fingers of your right hand, as follows. Orient your right hand such that the thumb points in the direction of the vector $\mathbf{u}$ and the first finger points in the direction of the vector $\mathbf{v}$, and let your second finger point outwards from the palm of your hand so that it is perpendicular to both the thumb and the first finger. Then the second finger points in the direction of the vector product $\mathbf{u} \times \mathbf{v}$.

Indeed it is customary to describe points of three-dimensional space by Cartesian coordinates (x, y, z) oriented so that if the positive x-axis and positive y-axis are pointed in the directions of the thumb and first finger respectively of your right hand, then the positive z-axis is pointed in the direction of the second finger of that hand, when the thumb and first two fingers are mutually perpendicular. For example, if the positive x-axis points towards the East, and the positive y-axis points towards the North, then the positive z-axis is chosen so that it points upwards. Moreover if $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ then these vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors pointed in the direction of the positive x-axis and positive y-axis respectively, and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, where $\mathbf{k} = (0, 0, 1)$, and the vector $\mathbf{k}$ points in the direction of the positive z-axis. Thus the 'right-hand' rule for determining the direction of the vector product $\mathbf{u} \times \mathbf{v}$ using the fingers of your right hand is valid when $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$.

If the directions of the vectors $\mathbf{u}$ and $\mathbf{v}$ are allowed to vary continuously, in such a way that these vectors never point either in the same direction or in opposite directions, then their vector product $\mathbf{u} \times \mathbf{v}$ will always be a non-zero vector, whose direction will vary continuously with the directions of $\mathbf{u}$ and $\mathbf{v}$. It follows from this that if the 'right-hand rule' for determining the direction of $\mathbf{u} \times \mathbf{v}$ applies when $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$, then it will also apply whatever the directions of $\mathbf{u}$ and $\mathbf{v}$, since, if your right hand is moved around in such a way that the thumb and first finger never point in the same direction, and if the second finger is always perpendicular to the thumb and first finger, then the direction of the second finger will vary continuously, and will therefore always point in the direction of the vector product of two vectors pointed in the direction of the thumb and first finger respectively.

Example

We shall find the area of the parallelogram $OACB$ in three-dimensional space, where

$$O = (0, 0, 0), \quad A = (1, 2, 0),$$

$$B = (-4, 2, -5), \quad C = (-3, 4, -5).$$

Note that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. Let $\mathbf{u} = \overrightarrow{OA} = (1, 2, 0)$ and $\mathbf{v} = \overrightarrow{OB} = (-4, 2, -5)$. Then $\mathbf{u} \times \mathbf{v} = (-10, 5, 10)$. Now $(-10, 5, 10) = 5(-2, 1, 2)$, and $|(-2, 1, 2)| = \sqrt{9} = 3$. It follows that

$$\text{area}\, OACB = |\mathbf{u} \times \mathbf{v}| = 15.$$

Note also that the vector $(-2, 1, 2)$ is perpendicular to the parallelogram $OACB$.