MA2C03 Mathematics
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38.2. The Scalar Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. The *scalar product* of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be the real number $\mathbf{u} \,.\, \mathbf{v}$ defined by the formula

$$\mathbf{u} \,.\, \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

In particular,

$$\mathbf{u} \,.\, \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2,$$

for any vector $\mathbf{u}$, where $|\mathbf{u}|$ denotes the length of the vector $\mathbf{u}$.

Note that $\mathbf{u} \,.\, \mathbf{v} = \mathbf{v} \,.\, \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Also

$$\begin{aligned} (s\mathbf{u} + t\mathbf{v}) \,.\, \mathbf{w} &= s\mathbf{u} \,.\, \mathbf{w} + t\mathbf{v} \,.\, \mathbf{w}, \\ \mathbf{u} \,.\, (s\mathbf{v} + t\mathbf{w}) &= s\mathbf{u} \,.\, \mathbf{v} + t\mathbf{u} \,.\, \mathbf{w} \end{aligned}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and real numbers s and t.

Proposition 38.1

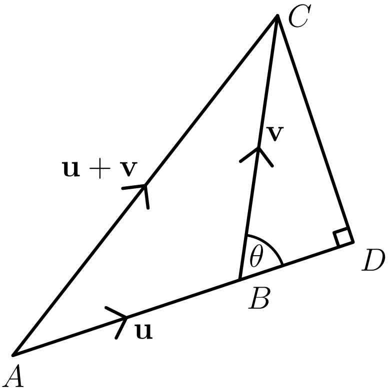
Let **u** *and* **v** *be non-zero vectors in three-dimensional space. Then their scalar product* **u** . **v** *is given by the formula*

$$\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\, |\mathbf{v}|\, \cos\theta,$$

where θ *denotes the angle between the vectors* **u** *and* **v***.*

Proof

Suppose first that the angle θ between the vectors **u** and **v** is an acute angle, so that $0 < \theta < \frac{1}{2}\pi$. Let us consider a triangle ABC, where $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, and thus $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$. Let ADC be the right-angled triangle constructed as depicted in the figure below, so that the line AD extends AB and the angle at D is a right angle.



Note:

$$\begin{aligned} AD &= |\mathbf{u}| + |\mathbf{v}| \cos\theta, \\ CD &= |\mathbf{v}| \sin\theta, \\ |\mathbf{u}+\mathbf{v}|^2 &= AC^2 = AD^2 + CD^2 \quad \text{(Pythagoras)}. \end{aligned}$$

Then the lengths of the line segments AB, BC, AC, BD and CD may be expressed in terms of the lengths $|\mathbf{u}|$, $|\mathbf{v}|$ and $|\mathbf{u}+\mathbf{v}|$ of the displacement vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ and the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ by means of the following equations:

$$AB = |\mathbf{u}|, \quad BC = |\mathbf{v}|, \quad AC = |\mathbf{u}+\mathbf{v}|,$$

$$BD = |\mathbf{v}|\ \cos\theta \quad \text{and} \quad DC = |\mathbf{v}|\ \sin\theta.$$

Then

$$AD = AB + BD = |\mathbf{u}| + |\mathbf{v}|\cos\theta.$$

The triangle ADC is a right-angled triangle with hypotenuse AC. It follows from Pythagoras' Theorem that

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= AC^2 = AD^2 + DC^2 = (|\mathbf{u}| + |\mathbf{v}|\ \cos\theta)^2 + |\mathbf{v}|\ \sin^2\theta \\ &= |\mathbf{u}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\ \cos\theta + |\mathbf{v}|^2\ \cos^2\theta + |\mathbf{v}|^2\ \sin^2\theta \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\ \cos\theta, \end{aligned}$$

because $\cos^2\theta + \sin^2\theta = 1$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

and therefore

$$\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 &= (u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2 \\ &= u_1^2 + 2u_1 v_1 + v_1^2 + u_2^2 + 2u_2 v_2 + v_2^2 + u_3^2 + 2u_3 v_3 + v_3^2 \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u}.\mathbf{v}. \end{aligned}$$

On comparing the expressions for $|\mathbf{u} + \mathbf{v}|^2$ given by the above equations, we see that $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$ when $0 < \theta < \frac{1}{2}\pi$.

38. Scalar and Vector Products in Three Dimensions (continued)

The identity $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ clearly holds when $\theta = 0$ and $\theta = \pi$. Pythagoras' Theorem ensures that it also holds when the angle θ is a right angle (so that $\theta = \frac{1}{2}\pi$. Suppose that $\frac{1}{2}\pi < \theta < \pi$, so that the angle θ is obtuse. Then the angle between the vectors $\mathbf{u}$ and $-\mathbf{v}$ is acute, and is equal to $\pi - \theta$. Moreover $\cos(\pi - \theta) = -\cos\theta$ for all angles θ. It follows that

$$\mathbf{u}.\mathbf{v} = -\mathbf{u}.(-\mathbf{v}) = -|\mathbf{u}|\,|\mathbf{v}|\,\cos(\pi - \theta) = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$$

when $\frac{1}{2}\pi < \theta < \pi$. We have therefore verified that the identity $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ holds for all non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, as required. ∎

Corollary 38.1

Two non-zero vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in three-dimensional space are perpendicular if and only if* $\mathbf{u} \,.\, \mathbf{v} = 0$.

Proof

It follows directly from Proposition 38.1 that $\mathbf{u} \,.\, \mathbf{v} = 0$ if and only if $\cos\theta = 0$, where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. This is the case if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. ∎

Example

We can use the scalar product to calculate the angle θ between the vectors $(2, 2, 0)$ and $(0, 3, 3)$ in three-dimensional space. Let $\mathbf{u} = (2, 2, 0)$ and $\mathbf{v} = (0, 3, 3)$. Then $|\mathbf{u}|^2 = 2^2 + 2^2 = 8$ and $|\mathbf{v}|^2 = 3^2 + 3^2 = 18$. It follows that $(|\mathbf{u}|\,|\mathbf{v}|)^2 = 8 \times 18 = 144$, and thus $|\mathbf{u}|\,|\mathbf{v}| = 12$. Now $\mathbf{u} \,.\, \mathbf{v} = 6$. It follows that

$$6 = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta = 12\cos\theta.$$

Therefore $\cos\theta = \frac{1}{2}$, and thus $\theta = \frac{1}{3}\pi$.

Example

We can use the scalar product to find the distance between points on a sphere. Now the Cartesian coordinates of a point P on the unit sphere about the origin O in three-dimensional space may be expressed in terms of angles θ and φ as follows:

$$P = (\sin\theta\,\cos\varphi,\,\sin\theta\,\sin\varphi,\,\cos\theta).$$

The angle θ is that between the displacement vector $\overrightarrow{OP}$ and the vectical vector $(0,0,1)$. Thus the angle $\frac{1}{2}\pi - \theta$ represents the 'latitude' of the point P, when we regard the point $(0,0,1)$ as the 'north pole' of the sphere. The angle φ measures the 'longitude' of the point P.

Now let P_1 and P_2 be points on the unit sphere, where

$$\begin{aligned} P_1 &= (\sin\theta_1 \, \cos\varphi_1, \sin\theta_1 \, \sin\varphi_1, \cos\theta_1), \\ P_2 &= (\sin\theta_2 \, \cos\varphi_2, \sin\theta_2 \, \sin\varphi_2, \cos\theta_2). \end{aligned}$$

We wish to find the angle ψ between the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ of the points P_1 and P_2 from the origin. Now $|\overrightarrow{OP_1}| = 1$ and $|\overrightarrow{OP_2}| = 1$. On applying Proposition 38.1, we see that

$$\begin{aligned} \cos\psi &= \overrightarrow{OP_1} \,.\, \overrightarrow{OP_2} \\ &= \sin\theta_1 \, \sin\theta_2 \, \cos\varphi_1 \, \cos\varphi_2 + \sin\theta_1 \, \sin\theta_2 \, \sin\varphi_1 \, \sin\varphi_2 \\ &\qquad + \cos\theta_1 \, \cos\theta_2 \\ &= \sin\theta_1 \, \sin\theta_2 \, (\cos\varphi_1 \, \cos\varphi_2 + \sin\varphi_1 \, \sin\varphi_2) + \cos\theta_1 \, \cos\theta_2 \\ &= \sin\theta_1 \, \sin\theta_2 \, \cos(\varphi_1 - \varphi_2) + \cos\theta_1 \, \cos\theta_2. \end{aligned}$$

Example

Let X be a plane in three-dimensional space, and let $\mathbf{p}$ be a vector that is perpendicular to the plane X. Let O be the origin of a Cartesian coordinate system in three-dimensional space, and let $\mathbf{v}$ and $\mathbf{w}$ be the position vectors $\overrightarrow{OV}$ and $\overrightarrow{OW}$ of points V and W respectively lying in the plane X. Then the vector $\mathbf{p}$ is perpendicular to the displacement vector $\overrightarrow{VW}$. Now $\overrightarrow{VW} = \mathbf{w} - \mathbf{v}$. It follows that

$$(\mathbf{w} - \mathbf{v}) \,.\, \mathbf{p} = 0$$

(see Corollary 38.1), and therefore $\mathbf{v}.\mathbf{p} = \mathbf{w}.\mathbf{p}$. Identifying the points of the plane X with their position vectors $\mathbf{r}$ with respect to the origin O of the Cartesian coordinate system, we find that It follows from this that there exists a real number k such that

$$X = \{\mathbf{r} \in \mathbb{R}^3 : \mathbf{r} \,.\, \mathbf{p} = k\}.$$

Let $\mathbf{r} = (x, y, z)$ and $\mathbf{p} = (a, b, c)$. The point $\mathbf{r}$ belongs to the plane X if and only if $\mathbf{r} \,.\, \mathbf{p} = k$. It follows that

$$X = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = k\}.$$

Suppose that the vector $\mathbf{r}$ is the position vector of an arbitrary point R of three-dimensional space. We wish to determine the distance from this point to the plane X. Now the line through the point $\mathbf{r}$ parallel to the vector $\mathbf{p}$ cuts the plane X in a single point. Therefore there exists a unique real number t for which $\mathbf{r} + t\mathbf{p} \in X$. For this value of t the equation

$$(\mathbf{r} + t\mathbf{p}) \,.\, \mathbf{p} = k$$

is satisfied. Then

$$\mathbf{r} \,.\, \mathbf{p} = t|\mathbf{p}|^2 = k,$$

and therefore

$$t = \frac{1}{|\mathbf{p}|^2}(k - \mathbf{r} \,.\, \mathbf{p}).$$

Let $\mathbf{w} = \mathbf{r} + t\mathbf{p}$, where t has the value determined above that ensures that $\mathbf{w} \in X$. Let $\mathbf{v}$ be an arbitrary point that lies on the plane X. Then the displacement vector $\mathbf{v} - \mathbf{w}$ from W to V is perpendicular to the vector $\mathbf{p}$. Now

$$\mathbf{v} - \mathbf{r} = t\mathbf{p} + (\mathbf{v} - \mathbf{w}).$$

It follows, either directly from Pythagoras' Theorem, or else from an equivalent calculation using scalar products (using the result of Corollary 38.1) that

$$|\mathbf{v} - \mathbf{r}|^2 = t^2|\mathbf{p}|^2 + |\mathbf{v} - \mathbf{w}|^2.$$

It follows that

$$|\mathbf{v} - \mathbf{r}| \geq t|\mathbf{p}|,$$

and that

$$|\mathbf{v} - \mathbf{r}| = t|\mathbf{p}| \iff \mathbf{v} = \mathbf{w}.$$

Thus the point $\mathbf{w}$ is the closest point of the plane X to the point R with position vector $\mathbf{r}$. It follows that the distance $d(\mathbf{r}, X)$ from the point R to the plane X is the length $|\mathbf{w} - \mathbf{r}|$ of the vector $\mathbf{w} - \mathbf{r}$. Thus

$$d(\mathbf{r}, X) = t|\mathbf{p}| = \frac{1}{|\mathbf{p}|}|k - \mathbf{r} \,.\, \mathbf{p}|.$$

Let $\mathbf{r} = (x, y, z)$ and $p = (a, b, c)$. Then

$$d(\mathbf{r}, X) = \frac{|k - ax - by - cz|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example

Suppose that we wish to determine the equation of a cone in three-dimensional space. Let O be the origin of a Cartesian coordinate system, let V be the apex of the cone, let $\mathbf{v}$ be the position vector of V, so that $\mathbf{v} = \overrightarrow{OV}$, and let $\mathbf{b}$ be a vector pointed into the axis of the cone. Let θ be a fixed angle between zero and a right angle. The cone consists of those points R for which the displacement vector $\overrightarrow{VR}$ makes an angle θ with the vector $\mathbf{b}$. It follows from Proposition 38.1 that $\mathbf{r}$ is the position vector of a point lying on the cone if and only if

$$(\mathbf{r} - \mathbf{v}) \,.\, \mathbf{b} = |\mathbf{r} - \mathbf{v}|\,|\mathbf{b}|\,\cos\theta.$$

Squaring both sides of this identity, we find that

$$((\mathbf{r} - \mathbf{v}) \,.\, \mathbf{b})^2 = |\mathbf{r} - \mathbf{v}|^2 \, |\mathbf{b}|^2 \, \cos^2\theta.$$

Let

$$\mathbf{r} = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z) \quad \text{and} \quad \mathbf{b} = (b_x, b_y, b_z).$$

Then the equation of the cone becomes

$$\begin{aligned} &((x - v_x)b_x + (y - v_y)b_y + (z - v_z)b_z)^2 \\ &\qquad = \quad C\left((x - v_x)^2 + (y - v_y)^2 + (z - v_z)^2\right), \end{aligned}$$

where $C = |\mathbf{b}|^2 \cos^2\theta$. Note that this constant C must satisfy the inequalities $0 \leq C < |\mathbf{b}|^2$.