MA2C03 Mathematics
School of Mathematics, Trinity College
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Lecture 44 (February 17, 2016)

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38. Scalar and Vector Products in Three Dimensions

38.1. The Length of Three-Dimensional Vectors

Let P_1 and P_2 be points in space, and let $\mathbf{u}$ denote the displacement vector $\overrightarrow{P_1P_2}$ from the point P_1 to the point P_2. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ then $\mathbf{u} = (u_x, u_y, u_z)$ where $u_x = x_2 - x_1$, $u_y = y_2 - y_1$ and $u_z = z_2 - z_1$.

The *length* (or *magnitude*) of the vector $\mathbf{u}$ is defined to be the distance from the point P_1 to the point P_2. This distance may be calculated using Pythagoras's Theorem. Let $Q = (x_2, y_2, z_1)$ and $R = (x_2, y_1, z_1)$. If the points P_1 and P_2 are distinct, and if $z_1 \neq z_2$, then the triangle P_1QP_2 is a right-angled triangle with hypotenuse P_1P_2, and it follows from Pythagoras's Theorem that

$$P_1P_2^2 = P_1Q^2 + QP_2^2 = P_1Q^2 + (z_2 - z_1)^2.$$

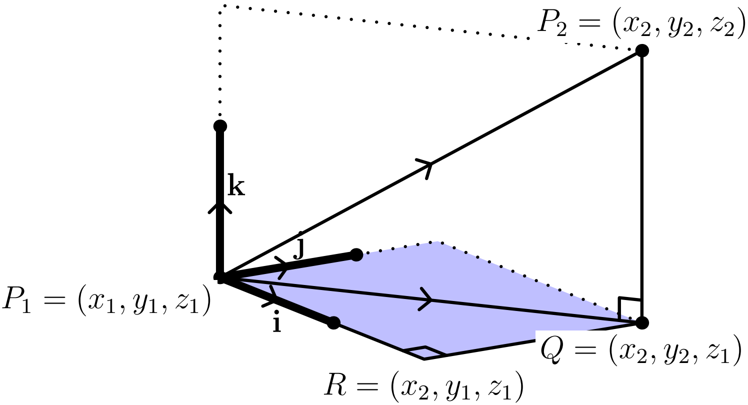
This identity also holds when $P_1 = P_2$, and when $z_1 = z_2$, and therefore holds wherever the points P_1 and P_2 are located.

Similarly

$$P_1Q^2 = P_1R^2 + RQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(since P_1RQ is a right-angled triangle with hypotenuse P_1Q whenever the points P_1, R and Q are distinct), and therefore the length $|\mathbf{u}|$ of the displacement vector $\mathbf{u}$ from the point P_1 to the point P_2 satisfies the equation

$$\begin{aligned} |\mathbf{u}|^2 = P_1P_2^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= u_x^2 + u_y^2 + u_z^2. \end{aligned}$$



Note:

$\overrightarrow{P_1R} = (x_2 - x_1, 0, 0)$, $\quad \overrightarrow{RQ} = (0, y_2 - y_1, 0)$, $\quad \overrightarrow{QP_2} = (0, 0, z_2 - z_1)$.

In general we define the *length*, or *magnitude*, $|\mathbf{v}|$ of any vector quantity $\mathbf{v}$ by the formula

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

where $\mathbf{v} = (v_x, v_y, v_z)$. This ensures that the length of any displacement vector is equal to the distance between the two points that determine the displacement.

Example

The vector $(3, 4, 12)$ is of length 13, since

$$3^2 + 4^2 + 12^2 = 5^2 + 12^2 = 13^2.$$

A vector whose length is equal to one is said to be a *unit vector*.

Let $\mathbf{v}$ be a non-zero vector in three-dimensional space, and let t be a real number.

Note that if $t > 0$ then $t\mathbf{v}$ is a vector, pointing in the same direction as $\mathbf{v}$, whose length is obtained on multiplying the length of $\mathbf{v}$ by the positive real number t.

Similarly if $t < 0$ then $t\mathbf{v}$ is a vector, pointing in the opposite direction to $\mathbf{v}$, whose length is obtained on multiplying the length of $\mathbf{v}$ by the positive real number $|t|$.