MA2C03 Mathematics
School of Mathematics, Trinity College
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Lecture 42 (February 10, 2016)

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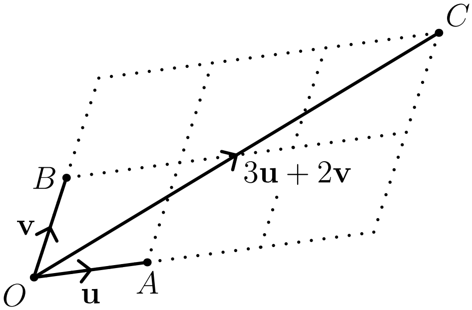
Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in three-dimensional space. A vector $\mathbf{v}$ is said to be a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there exist real numbers $t_1, t_2, \ldots, t_k$ such that

$$\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k.$$

Let O, A and B be distinct points of three-dimensional space that are not collinear (i.e., that do not all lie on any one line in that space). The displacement vector $\overrightarrow{OP}$ of a point P in three-dimensional space is a linear combination of the displacement vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ if and only if the point P lies in the unique plane that contains the points O, A and B.



Note: $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OB} = \mathbf{v}$, $\overrightarrow{OC} = 3\mathbf{u} + 2\mathbf{v}$.

36.6. Linear Dependence and Independence

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are said to be be *linearly dependent* if there exist real numbers $t_1, t_2, \ldots, t_k$, not all zero, such that

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are not linearly dependent, then they are said to be *linearly independent*.

Note that if any of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ is the zero vector, then those vectors are linearly dependent. Indeed if $\mathbf{v}_i = \mathbf{0}$ then these vectors satisfy a relation of the form

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

where $t_j = 0$ if $j \neq i$ and $t_i \neq 0$. We conclude that, in any list of linearly independent vectors, the vectors are all non-zero.

Also if any vector in the list $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ is a scalar multiple of some other vector in the list then these vectors are linearly dependent. Indeed suppose that $\mathbf{v}_k = t\mathbf{v}_j$, where $j \neq k$. Then $t\mathbf{v}_j - \mathbf{v}_k = \mathbf{0}$, and thus

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0},$$

where $t_j = t$, $t_k = -1$ and $t_i = 0$ whenever i is distinct from both j and k.

If a vector $\mathbf{v}$ is expressible as a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$ are linearly dependent. For there exist real numbers $s_1, \ldots, s_k$ such that

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k.$$

But then

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k - \mathbf{v} = \mathbf{0}.$$

Theorem 36.2

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be three vectors in three-dimensional space which are linearly independent. Then, given any vector* $\mathbf{s}$, *there exist unique real numbers* p, q *and* r *such that*

$$\mathbf{s} = p\mathbf{u} + q\mathbf{v} + r\mathbf{w}.$$

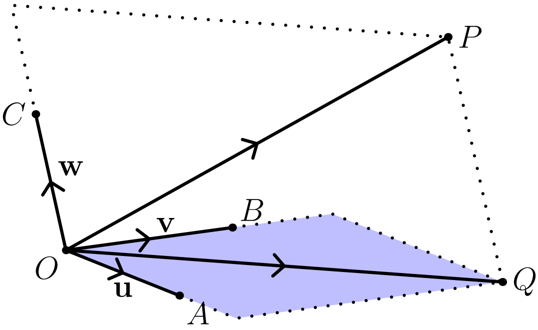
Proof

First we note that the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are all non-zero, and none of these vectors is a scalar multiple of another vector in the list. Let O denote the origin of a Cartesian coordinate system, and let A, B, C and P denote the points of three-dimensional space whose displacement vectors from the origin O are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{s}$ respectively. The points O, A, B and C are then all distinct, and there is a unique plane which contains the three points O, A and B. This plane OAB consists of all points whose displacement vector from the origin is expressible in the form $p\mathbf{u} + q\mathbf{v}$ for some real numbers p and q.

Now the vector $\mathbf{w}$ is not expressible as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and therefore the point C does not belong to the plane OAB. Therefore the line parallel to OC that passes through the point P is not parallel to the plane OAP. This line therefore intersects the plane in a single point Q. Now the displacement vector of the point Q from the origin is of the form $\mathbf{s} - r\mathbf{w}$ for some uniquely-determined real number r. But it is also expressible in the form $p\mathbf{u} + q\mathbf{v}$ for some uniquely-determined real numbers p and q, because Q lies in the plane OAB. Thus there exist real numbers p, q and r such that $\mathbf{s} - r\mathbf{w} = p\mathbf{u} + q\mathbf{v}$. But then

$$\mathbf{s} = p\mathbf{u} + q\mathbf{v} + r\mathbf{w}.$$

Moreover the point Q and thus the real numbers p, q and r are uniquely determined by $\mathbf{s}$, as required. ∎



Note:

$$\begin{aligned}\overrightarrow{OP} &= \mathbf{s} = 1.5\,\mathbf{u} + 1.6\,\mathbf{v} + 1.8\,\mathbf{w},\\ \overrightarrow{OQ} &= 1.5\,\mathbf{u} + 1.6\,\mathbf{v},\\ \overrightarrow{QP} &= 1.8\,\mathbf{w}.\end{aligned}$$

It follows from this theorem that no linearly independent list of vectors in three-dimensional space can contain more than three vectors, since were there a fourth vector in the list, then it would be expressible as a linear combination of the other three, and the vectors would not then be linearly independent.

36.7. Line Segments

Let O be the origin of Cartesian coordinates in three-dimensional Euclidean space, and let P and Q be points of three-dimensional space with position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$. We consider how to specify, in vector notation, the line segment joining the point P to a point Q.

Let R be a point on the line segment PQ whose endpoints are P and Q. Then the vectors $\overrightarrow{OP}$ and $\overrightarrow{OR}$ are collinear, and indeed $\overrightarrow{PR} = t\,\overrightarrow{PQ}$ for some real number t satisfying $0 \leq t \leq 1$. Now $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$ Thus a point with position vector $\mathbf{r}$ lies on the line segment joining P to Q if and only if

$$\mathbf{r} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$$

for some real number t satisfying $0 \leq t \leq 1$. It follows that the set of position vectors of points that lie on the line segment with endpoints P and Q is

$$\{\mathbf{r} : \mathbf{r} = (1 - t)\mathbf{p} + t\mathbf{q} \text{ for some } t \in \mathbb{R} \text{ satisfying } 0 \leq t \leq 1\}.$$

37. Real Vector Spaces

37.1. The Definition of a Real Vector Space

Definition

A *real vector space* consists of a set V on which are defined a binary operation of *vector addition*, adding any pair of elements $\mathbf{v}$ and $\mathbf{w}$ of V to yield an element $\mathbf{v} + \mathbf{w}$ of V, and an operation of multiplication-by-scalars, multiplying any element $\mathbf{v}$ of V by any real number t to yield an element $t\mathbf{v}$ of V, where these operations of vector addition and multiplication satisfy the following axioms:—

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ for all $\mathbf{v}, \mathbf{w} \in V$;
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there exists a zero element $\mathbf{0}$ of V characterized by the property that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;
4. given any element $\mathbf{v} \in V$, there exists an element $-\mathbf{v}$ of V characterized by the property that $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$,
5. $t(\mathbf{v} + \mathbf{w}) = t\mathbf{v} + t\mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers t;
6. $(s + t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}$ for all $\mathbf{v} \in V$ and for all real numbers s and t;
7. $s(t\mathbf{v}) = (st)\mathbf{v}$ for all $\mathbf{v} \in V$ and for all real numbers s and t;
8. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

The first four axioms in the definition of a vector space are equivalent to the requirement that a vector space be an Abelian group (or commutative group) with respect to the operation of vector addition. Thus a vector space is an Abelian group provided with an additional algebraic operation of multiplication-by-scalars that satisfies the last four axioms listed above.

All the real vector space axioms are satisfied by the set of vectors in three-dimensional Euclidean space, with the standard operations of vector addition and multiplication-by-scalars. Therefore vectors in three-dimensional space constitute a real vector space.

There is a corresponding real vector space whose elements are vectors in the Euclidean plane. Cartesian coordinates of points of the plane are represented as ordered pairs of real numbers. Given points P_1 and P_2 of the plane, where

$$P_1 = (x_1, y_1) \quad \text{and} \quad P_2 = (x_2, y_2),$$

the displacement vector $\overrightarrow{P_1 P_2}$ is represented by the ordered pair defined so that

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1).$$

Vector addition and multiplication-by-scalars is defined for vectors in two dimensions in the obvious fashion, so that

$$(v_x, v_y) + (w_x, w_y) = (v_x + w_x, v_y + w_y) \quad \text{and} \quad t(v_x, v_y) = (tv_x, tv_y)$$

for all two-dimensional vectors (v_x, v_y) and (w_x, w_y) and for all real numbers t.

Example

Let m be a positive integer, and let V_m be the set of all polynomials with real coefficients consisting of the zero polynomial together with all non-zero polynomials whose degree does not exceed m. (The degree of a polynomial is defined only for non-zero polynomials: it is the degree of the highest term for which the corresponding coefficient is non-zero.) If $p(x)$ and $q(x)$ are polynomials with real coefficients belonging to V_m then so is $p(x) + q(x)$. Also $tp(x)$ is a polynomial belonging to V_m for all (constant) real numbers t.

The operation of addition of two polynomials belonging to V_m to yield another polynomial belonging to V_m can be considered to be an operation of "vector addition" on the set V_m. Similarly the operation of multiplying a polynomial by a constant real number can be considered to be an operation of "multiplication by scalars". The set V_m, with all these algebraic operations, is a real vector space: all the axioms in the definition of a vector space as satisfied when the non-zero "vectors" are polynomials whose degree does not exceed m.

37.2. Linear Dependence and Independence in Vector Spaces

Let V be a real vector space. Elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of V are said to be be *linearly dependent* if there exist real numbers $t_1, t_2, \ldots, t_k$, not all zero, such that

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are not linearly dependent, then they are said to be *linearly independent*.

Note that if any of the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ of V is the zero element of V then those elements of V are linearly dependent. Indeed if $\mathbf{v}_i = \mathbf{0}$ then these vectors satisfy a relation of the form

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}.$$

where $t_j = 0$ if $j \neq i$ and $t_i \neq 0$. We conclude that, in any list of linearly independent elements of a real vector space V, the vectors are all non-zero.

If an element $\mathbf{v}$ of a real vector space V is expressible as a linear combination of elements $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of V then the elements $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$ are linearly dependent. For there exist real numbers $s_1, \ldots, s_k$ such that

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k.$$

But then

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k - \mathbf{v} = \mathbf{0}.$$