MA2C03 Mathematics
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 41 (February 10, 2016)

David R. Wilkins

36. Vectors in Three-Dimensional Space

36.1. Vector Quantities

Vector quantities are objects that have attributes of magnitude and direction. Many physical quantities, such as velocity, acceleration, force, electric field and magnetic field are examples of vector quantities. Displacements between points of space may also be represented using vectors.

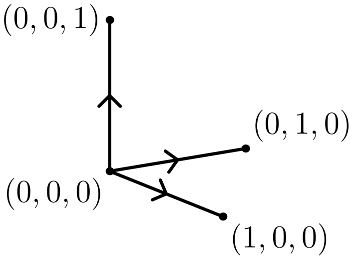
Quantities that do not have a sense of direction associated with them are known as *scalar quantities*. Such physical quantities as temperature and energy are scalar quantities. Scalar quantities are usually represented by real numbers.

36.2. Displacement Vectors

Points of three-dimensional space may be represented, in a Cartesian coordinate system, by ordered triples (x, y, z) of real numbers. Two ordered triples (x_1, y_1, z_1) and (x_2, y_2, z_2) of real numbers represent the same point of three-dimensional space if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$. The point whose Cartesian coordinates are given by the ordered triple $(0, 0, 0)$ is referred to as the *origin* of the Cartesian coordinate system.

It is usual to employ a Coordinate system such that the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are situated at a unit distance from the origin $(0, 0, 0)$, and so that the three lines that join the origin to these points are mutually perpendicular. Moreover it is customary to require that if the thumb of your right hand points in the direction from the origin to the point $(1, 0, 0)$, and if the first finger of that hand points in the direction from the origin to the point $(0, 1, 0)$, and if the second finger of that hand points in a direction perpendicular to the directions of the thumb and first finger, then that second finger points in the direction from the origin to the point $(0, 0, 1)$. (Thus if, at a point on the surface of the earth, away from the north and south pole, the point $(1, 0, 0)$ is located to the east of the origin, and the point $(0, 1, 0)$ is located to the north of the origin, then the point $(0, 0, 1)$ will be located above the origin.

36. Vectors in Three-Dimensional Space (continued)



Let P_1, P_2, P_3 and P_4 denote four points of three-dimensional space, represented in a Cartesian coordinate system by ordered triples as follows:

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2),$$

$$P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

The *displacement vector* $\overrightarrow{P_1, P_2}$ from the point P_1 to the point P_2 measures the distance and the direction in which one would have to travel in order to get from P_1 to P_2. This displacement vector may be represented by an ordered triple as follows:

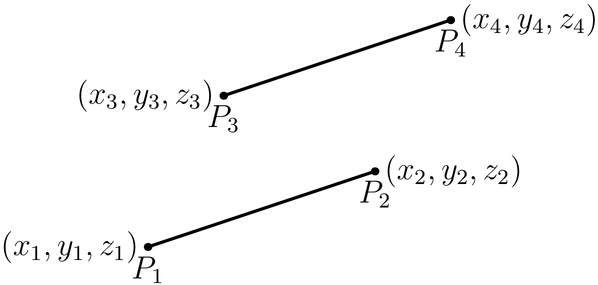
$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

The displacement vector $\overrightarrow{P_3P_4}$ is *equal* to the displacement vector $\overrightarrow{P_1P_2}$ if and only if

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3, \\ y_2 - y_1 &= y_4 - y_3, \\ z_2 - z_1 &= z_4 - z_3, \end{aligned}$$

in which case we represent the fact that these two displacement vectors are equal by writing

$$\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}.$$



Note: $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$ and therefore

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3, \\ y_2 - y_1 &= y_4 - y_3, \\ z_2 - z_1 &= z_4 - z_3, \end{aligned}$$

Now

$$\begin{aligned} & x_2 - x_1 = x_4 - x_3 \\ \iff \quad & x_2 + x_3 = x_1 + x_4 \\ \iff \quad & x_3 - x_1 = x_4 - x_2 \end{aligned}$$

Thus

$$x_2 - x_1 = x_4 - x_3 \quad \text{if and only if} \quad x_3 - x_1 = x_4 - x_2.$$

Similarly

$$y_2 - y_1 = y_4 - y_3 \quad \text{if and only if} \quad y_3 - y_1 = y_4 - y_2,$$

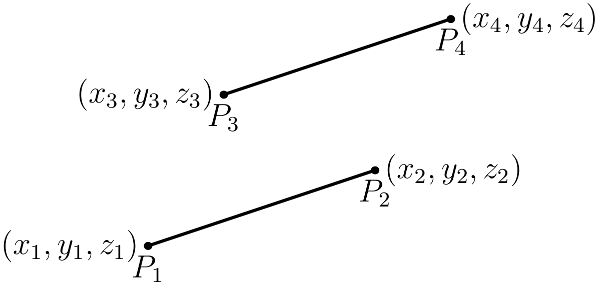
$$z_2 - z_1 = z_4 - z_3 \quad \text{if and only if} \quad z_3 - z_1 = z_4 - z_2.$$

Geometrically, these two displacement vectors are equal if and only if P_1, P_2, P_4 and P_3 are the vertices of a parallelogram in three-dimensional space, in which case

$$\begin{aligned} x_3 - x_1 &= x_4 - x_2, \\ y_3 - y_1 &= y_4 - y_2, \\ z_3 - z_1 &= z_4 - z_2, \end{aligned}$$

and thus

$$\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$



Note:

$$\left\{ \begin{array}{rcl} x_2 - x_1 & = & x_4 - x_3, \\ y_2 - y_1 & = & y_4 - y_3, \\ z_2 - z_1 & = & z_4 - z_3, \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{rcl} x_3 - x_1 & = & x_4 - x_2, \\ y_3 - y_1 & = & y_4 - y_2, \\ z_3 - z_1 & = & z_4 - z_2, \end{array} \right\}.$$

These displacement vectors may be regarded as objects in their own right, and denoted by symbols of their own: we use a symbol such as **u** to denote the displacement vector $\overrightarrow{P_1 P_2}$ from the point P_1 to the point P_2, and we write $\mathbf{u} = (u_x, u_y, u_z)$ where $u_x = x_2 - x_1$, $u_y = y_2 - y_1$ and $u_z = z_2 - z_1$.

Remark

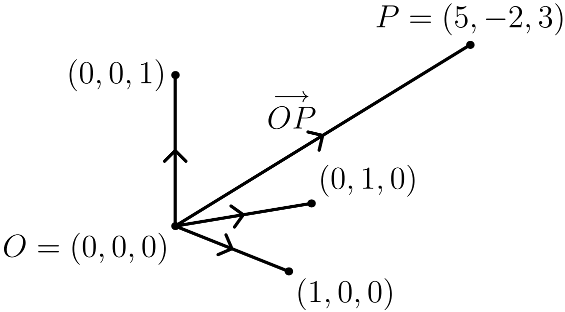
It is traditional in mathematics texts to denote vectors with boldface letters (e.g., **u**, **v**, **w**). The traditional way of writing the equivalent on paper or blackboards is to put a tilde underneath the letter (e.g., $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{w}$). When vectors are taught at second level, they are often written with an arrow on top (e.g., $\vec{u}$, $\vec{v}$, $\vec{w}$).

Vectors are used to record displacements and positions. Let P_1 and P_2 be points of three-dimensional Euclidean space with Cartesian coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. The *displacement vector* $\overrightarrow{P_1P_2}$ from P_1 to P_2 is the vector with components

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

that contains the information necessary to determine the distance of P_2 from P_1 and also the direction of P_2 in relation to P_1.

A Cartesian coordinate system in three-dimensional space determines an *origin* O that is the point whose Cartesian coordinates are $(0, 0, 0)$. The position of a point P of the plane with respect to the origin is specified by a vector $\mathbf{r}$, where $\mathbf{r} = \overrightarrow{OP}$. This vector $\mathbf{r}$ is the *position vector* of the point P. It represents the displacement of the point P from the origin of the Cartesian coordinate system.



Note: The position vector $\overrightarrow{OP}$ of the point P, where $P = (5, -2, 3)$.

36.3. The Parallelogram Law of Vector Addition

Let P_1, P_2, P_3 and P_4 denote four points of three-dimensional space, located such that $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$. Then (as we have seen) $\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}$ and the geometrical figure $P_1P_2P_4P_3$ is a parallelogram. Let

$$\mathbf{u} = \overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}, \quad \mathbf{v} = \overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$

Let

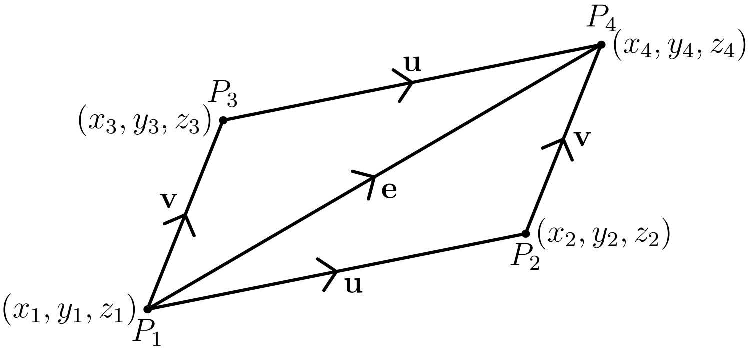
$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2),$$
$$P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

Then $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$, where

$$\begin{aligned} u_x &= x_2 - x_1 = x_4 - x_3, \\ u_y &= y_2 - y_1 = y_4 - y_3, \\ u_z &= z_2 - z_1 = z_4 - z_3, \\ v_x &= x_3 - x_1 = x_4 - x_2, \\ v_y &= y_3 - y_1 = y_4 - y_2, \\ v_z &= z_3 - z_1 = z_4 - z_2, \end{aligned}$$

Let $\mathbf{e} = \overrightarrow{P_1 P_4}$. Then $\mathbf{e} = (e_x, e_y, e_z)$, where

$$\begin{aligned} e_x &= x_4 - x_1 = u_x + v_x, \\ e_y &= y_4 - y_1 = u_y + v_y, \\ e_z &= z_4 - z_1 = u_z + v_z, \end{aligned}$$



Note: $\mathbf{u} = \overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$ and $\mathbf{v} = \overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}$, and

$$\begin{aligned} u_x &= x_2 - x_1 = x_4 - x_3 \quad \&c., \\ v_x &= x_3 - x_1 = x_4 - x_2 \quad \&c., \\ e_x &= x_4 - x_1 = u_x + v_x \quad \&c.. \end{aligned}$$

We say that the vector $\mathbf{e}$ is the *sum* of the vectors $\mathbf{u}$ and $\mathbf{v}$, and denote this fact by writing

$$\mathbf{e} = \mathbf{u} + \mathbf{v}.$$

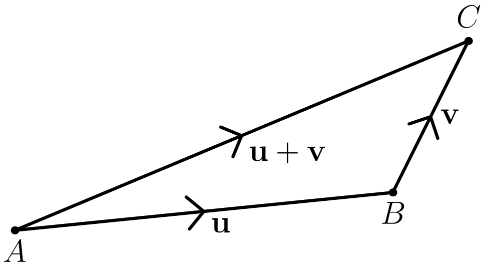
This rule for addition of vectors is known as the *parallelogram rule*, due to its association with the geometry of parallelograms. Note that vectors are added, by adding together the corresponding components of the two vectors. For example,

$$(0, 3, 2) + (4, 8, -5) = (4, 11, -3).$$

Note that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

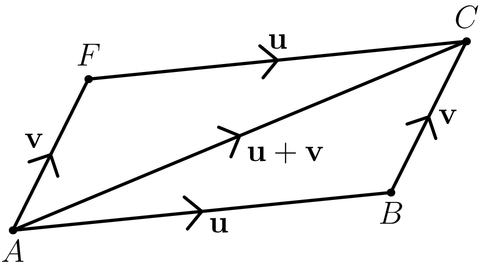
for all points A, B and C of space.



The identity

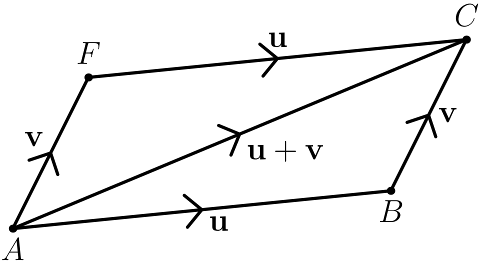
$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

holds for all vectors **u** and **v** in three-dimensional space.



The identity $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ may be interpreted geometrically as follows. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$, where A, B and C are points of three-dimensional space. Then there exists a point F in three-dimensional space such that $\overrightarrow{AF} = \overrightarrow{BC}$. Then $ABCF$ is a parallelogram, and $\overrightarrow{FC} = \overrightarrow{AB}$. It follows that

$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{u} + \mathbf{v}, \\ \overrightarrow{AC} &= \overrightarrow{AF} + \overrightarrow{FC} = \mathbf{v} + \mathbf{u}. \end{aligned}$$



In Cartesian coordinates

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}=(u_x+v_x, u_y+v_y, u_z+v_z),$$

where

$$\mathbf{u}=(u_x, u_y, u_z) \quad \text{and} \quad \mathbf{v}=(v_x, v_y, v_z).$$

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in three-dimensional space. Then

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

This identity may be verified algebraically as follows. Let

$$\mathbf{u}=(u_x,u_y,u_z), \quad \mathbf{v}=(v_x,v_y,v_z), \quad \mathbf{w}=(w_x,w_y,w_z).$$

Then

$$\mathbf{u}+\mathbf{v}=(u_x+v_x,u_y+v_y,u_z+v_z), \quad \mathbf{v}+\mathbf{w}=(v_x+w_x,v_y+w_y,v_z+w_z),$$

and therefore

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= (u_x+v_x+w_x,u_y+v_y+w_y,u_z+v_z+w_z)\\ &= \mathbf{u}+(\mathbf{v}+\mathbf{w}).\end{aligned}$$

This identity can be interpreted geometrically as follows. Let A be a point of three-dimensional space. Then there exist points B, C and D of three-dimensional space such that

$$\mathbf{u} = \overrightarrow{AB}, \quad \mathbf{v} = \overrightarrow{BC}, \quad \mathbf{w} = \overrightarrow{CD}.$$

Then

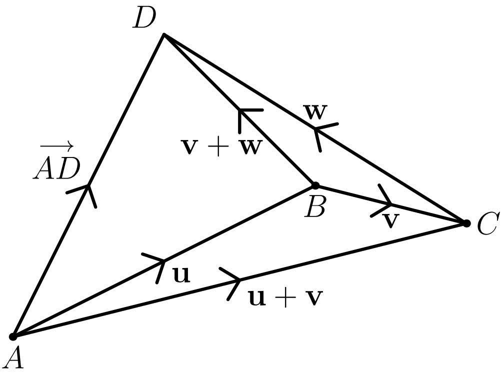
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{u} + \mathbf{v} \quad \text{and} \quad \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \mathbf{v} + \mathbf{w},$$

and hence

$$\begin{aligned} \overrightarrow{AD} &= \overrightarrow{AC} + \overrightarrow{CD} = (\mathbf{u} + \mathbf{v}) + \mathbf{w}, \\ \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD} = \mathbf{u} + (\mathbf{v} + \mathbf{w}). \end{aligned}$$

and thus

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \overrightarrow{AD} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$



Note:

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD} \\ &= \mathbf{u}+(\mathbf{v}+\mathbf{w}).\end{aligned}$$

The *zero vector* $\mathbf{0}$ is the vector $(0, 0, 0)$ that represents the displacement from any point in space to itself. The zero vector $\mathbf{0}$ has the property that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

for all vectors $\mathbf{u}$.

Given any vector $\mathbf{u}$, there exists a vector, denoted by $-\mathbf{u}$, characterized by the property that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

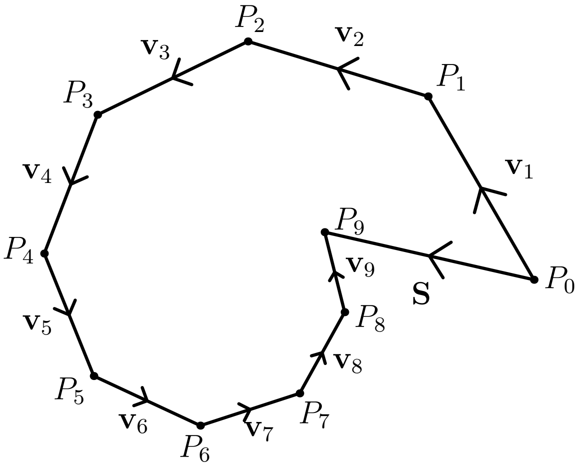
If $\mathbf{u} = (u_x, u_y, u_z)$, then $-\mathbf{u} = (-u_x, -u_y, -u_z)$.

We have shown that addition of vectors satisfies the Commutative Law and the Associative Law.
Given three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we define their sum $\mathbf{u} + \mathbf{v} + \mathbf{w}$ so that

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

More generally, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in three-dimensional space, and let P_0 be a point of three-dimensional space. Then there exist points $P_1, P_2, \ldots, P_k$ such that $\mathbf{v}_j = \overrightarrow{P_{j-1}, P_j}$ for $j = 1, 2, \ldots, n$. We define the sum of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ such that

$$\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_k = \overrightarrow{P_0 P_k}.$$



Note: case $k = 9$, with

$$\mathbf{S} = \overrightarrow{P_0 P_9} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_9,$$

where $\mathbf{v}_j = \overrightarrow{P_{j-1} P_j}$ for $j = 1, 2, \ldots, 9$.

Lemma 36.1

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *be vectors in three-dimensional space, where* $\mathbf{v}_j = (v_x^{(j)}, v_y^{(j)}, v_z^{(j)})$ *for* $j = 1, 2, \ldots, n$*, and let*

$$\mathbf{S} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k.$$

Then $\mathbf{S} = (S_x, S_y, S_z)$*, where*

$$S_x = \sum_{j-1}^{n} v_x^{(j)}, \quad S_y = \sum_{j-1}^{n} v_y^{(j)}, \quad S_z = \sum_{j-1}^{n} v_z^{(j)}.$$

Proof

Let points P_0 be a point in three-dimensional space, and let points $P_1, P_2, \ldots, P_k$ be successively constructed such that $\mathbf{v}_j = \overrightarrow{P_{j-1}P_j}$ for $j = 1, 2, \ldots, k$. Let $P_j = (x_j, y_j, z_j)$ for $j = 0, 1, 2, \ldots, k$. Then

$$\mathbf{v}_j = (x_j - x_{j-1}, y_j - y_{j-1}, z_j - z_{j-1})$$

for $j = 1, 2, \ldots, k$. Thus if

$$\mathbf{S} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k,$$

then

$$\mathbf{S} = \overrightarrow{P_0P_k} = (x_k - x_0, y_k - y_0, z_k - z_0) = (S_x, S_y, S_z)$$

where

$$\begin{aligned} S_x &= x_k - x_0 = \sum_{j=1}^{k} (x_j - x_{j-1}) = \sum_{j-1}^{n} v_x^{(j)}, \\ S_y &= y_k - y_0 = \sum_{j=1}^{k} (y_j - y_{j-1}) = \sum_{j-1}^{n} v_y^{(j)}, \\ S_z &= z_k - z_0 = \sum_{j=1}^{k} (z_j - z_{j-1}) = \sum_{j-1}^{n} v_z^{(j)}, \end{aligned}$$

as required. $\blacksquare$

36.4. Scalar Multiples of Vectors

Let $P_0, P_1, P_2, P_3, \ldots$ be an infinite sequence of points in three-dimensional space, where

$$\overrightarrow{P_0P_1} = \overrightarrow{P_1P_2} = \overrightarrow{P_2P_3} = \cdots .$$

Let $\mathbf{v} = \overrightarrow{P_0P_1}$, and let $\mathbf{v} = (v_x, v_y, v_z)$. Then $\overrightarrow{P_jP_{j+1}} = \mathbf{v}$ for all positive integers j. It then follows immediately from Lemma 36.1 that

$$\begin{aligned}
\mathbf{v} &= \overrightarrow{P_0P_1} = (v_x, v_y, v_z) \\
\mathbf{v} + \mathbf{v} &= \overrightarrow{P_0P_2} = (2v_x, 2v_y, 2v_z) \\
\mathbf{v} + \mathbf{v} + \mathbf{v} &= \overrightarrow{P_0P_3} = (3v_x, 3v_y, 3v_z) \\
\mathbf{v} + \mathbf{v} + \mathbf{v} + \mathbf{v} &= \overrightarrow{P_0P_4} = (4v_x, 4v_y, 4v_z) \\
&\vdots
\end{aligned}$$

It follows that

$$\mathbf{v} + \mathbf{v} = 2\mathbf{v}, \quad \mathbf{v} + \mathbf{v} + \mathbf{v} = 3\mathbf{v}, \quad \mathbf{v} + \mathbf{v} + \mathbf{v} + \mathbf{v} = 4\mathbf{v}, \quad \text{\&c.},$$

where

$$k\mathbf{v} = \overrightarrow{P_0 P_k} = (kv_x, kv_y, kv_z)$$

for all non-negative integers k.

More generally, let $\mathbf{v}$ be a vector, represented by the ordered triple (v_x, v_y, v_z), and let t be a real number. We define $t\mathbf{v}$ to be the vector represented by the ordered triple (tv_x, tv_y, tv_z). Thus $t\mathbf{v}$ is the vector obtained on multiplying each of the components of $\mathbf{v}$ by the real number t. The vector $t\mathbf{v}$ is said to be a *scalar multiple* of the vector $\mathbf{v}$, obtained by multiplying the vector $\mathbf{v}$ by the *scalar* t.

It follows from this definition of scalar multiples of vectors that

$$(s+t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}, \quad t(\mathbf{u}+\mathbf{v}) = t\mathbf{u} + t\mathbf{v}, \text{ and } s(t\mathbf{u}) = (st)\mathbf{u},$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ and real numbers s and t. Also $1\mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$.