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An *undirected graph* can be thought of as consisting of a finite set V of points, referred to as the *vertices* of the graph, together with a finite set E of *edges*, where each edge joins two distinct vertices of the graph.

We now proceed to formulate the definition of an undirected graph in somewhat more formal language.

Let V be a set. We denote by V_2 the set consisting of all subsets of V with exactly two elements. Thus, for any set V,

$$V_2 = \{A \in \mathcal{P}V : \#(A) = 2\},$$

where $\mathcal{P}V$ denotes the power set of V (i.e., the set consisting of all subsets of V), and $\#(A)$ denotes the number of elements in a subset A of V.

Definition

An *undirected graph* (V, E) consists of a finite set V together with a subset E of V_2 (where V_2 is the set consisting of all subsets of V with exactly two elements). The elements of V are the *vertices* of the graph; the elements of E are the *edges* of the graph.

Definition

A graph is said to be *trivial* if it consists of a single vertex.

We may denote a graph by a single letter such as G. Writing $G = (V, E)$ indicates that V is the set of vertices and E is the set of edges of some graph G.

Definition

If v is a vertex of some graph, if e is an edge of the graph, and if $e = v\,v'$ for some vertex v' of the graph, then the vertex v is said to be *incident* to the edge e, and the edge e is said to be *incident* to the vertex v.

Definition

Two distinct vertices v and v' of a graph (V, E) are said to be *adjacent* if and only if $v\,v' \in E$.

Definition

Let (V, E) and (V', E') be graphs. The graph (V', E') is said to be a *subgraph* of (V, E) if and only if $V' \subset V$ and $E' \subset E$ (i.e., if and only if the vertices and edges of (V', E') are all vertices and edges of (V, E)).

Definition

Let (V, E) be a graph. The *degree* $\deg v$ of a vertex v of this graph is defined to be the number of edges of the graph that are incident to v (i.e., the number of edges of the graph which have v as one of their endpoints).

Definition

A vertex of a graph of degree 0 is said to be an *isolated* vertex.

Definition

A vertex of a graph of degree 1 is said to be an *pendent* vertex.

Definition

Let (V, E) be a graph. A *walk* $v_0\, v_1\, v_2\, \ldots\, v_n$ of length n in the graph from a vertex a to a vertex b is determined by a finite sequence $v_0, v_1, v_2, \ldots, v_n$ of vertices of the graph such that $v_0 = a$, $v_n = b$ and $v_{i-1}\, v_i$ is an edge of the graph for $i = 1, 2, \ldots, n$.

A walk $v_0\, v_1\, v_2\, \ldots\, v_n$ in a graph is said to *traverse* the edges $v_{i-1}v_i$ for $i = 1, 2, \ldots, n$ and to *pass through* the vertices $v_0, v_1, \ldots, v_n$. Each vertex v in a graph determines a walk of length of length zero in the graph, consisting of the single vertex v; such a walk is said to be *trivial*.

Definition

Let (V, E) be a graph. A *trail* $v_0\, v_1\, v_2\, \ldots\, v_n$ of length n in the graph from a vertex a to a vertex b is a walk of length n from a to b with the property that the edges $v_{i-1}v_i$ are distinct for $i = 1, 2, \ldots, n$.

A trail in a graph is thus a walk in the graph which traverses edges of the graph at most once.

Definition

Let (V, E) be a graph. A *path* $v_0\, v_1\, v_2\, \ldots\, v_n$ of length n in the graph from a vertex a to a vertex b is a walk of length n from a to b with the property that the vertices $v_0, v_1, \ldots, v_n$ are distinct.

A path in a graph is thus a walk in the graph which passes through vertices of the graph at most once.

Definition

An undirected graph is said to be *connected* if, given any two vertices u and v of the graph, there exists a path in the graph from u to v.

Definition

Let (V, E) be a graph. A *walk* $v_0\, v_1\, v_2\, \ldots\, v_n$ in the graph is said to be *closed* if $v_0 = v_n$.

Thus a walk in a graph is closed if and only if it starts and ends at the same vertex.

Definition

Let (V, E) be a graph. A *circuit* in the graph is a non-trivial closed trail in the graph.

Definition

A circuit $v_0\, v_1\, v_2\, \ldots\, v_{n-1}\, v_0$ in a graph is said to be *simple* if the vertices $v_0, v_1, v_2, \ldots\, v_{n-1}$ are distinct.

Theorem 29.1

If a graph has no isolated or pendant vertices then it contains at least one simple circuit.

Theorem 29.2

Let u *and* v *be vertices of a graph, where* $u \neq v$. *Suppose that there exist at least two distinct paths in the graph from* u *to* v. *Then the graph contains at least one simple circuit.*

32. Forests and Trees

Definition

A graph is said to be *acyclic* if it contains no circuits.

Definition

A *forest* is an acyclic graph.

Definition

A *tree* is a connected forest.

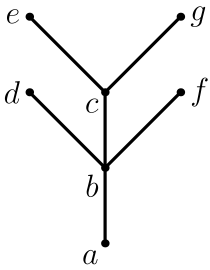
Note that the components of any forest are trees.

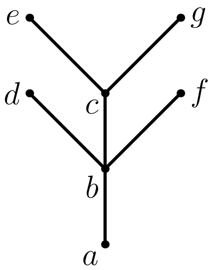
Example

The graph (V, E), where

$$\begin{aligned} V &= \{a,\ b,\ c,\ d,\ e,\ f,\ g\}, \\ E &= \{a\,b,\ b\,c,\ b\,d,\ c\,e,\ b\,f,\ c\,g\}, \end{aligned}$$

is a tree.





The vertices a, d, e, f and g are pendent vertices (i.e., each of these vertices is incident to exactly one edge of the graph, and is therefore of degree one.) The tree has 7 vertices and 6 edges.

Theorem 32.1

Every forest contains at least one isolated or pendent vertex.

Proof

If a graph has no isolated or pendent vertices, then it contains a circuit (Theorem 29.1). But a forest contains no circuits. Therefore must have at least one isolated or pendent vertex. ∎

Theorem 32.2

A non-trivial tree contains at least one pendent vertex.

Proof

A non-trivial graph has more than one vertex. If a non-trivial graph has an isolated vertex then there does not exist any path or walk from that vertex to any other vertex of the graph, and therefore the graph is not connected. But a tree is by definition connected. Therefore a non-trivial tree cannot have any isolated vertex. However a tree is a forest, and therefore contains at least one vertex that is either an isolated vertex or a pendent vertex (Theorem 32.1). Such a vertex must then be a pendent vertex. ∎

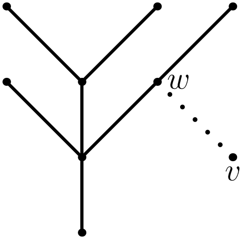
Theorem 32.3

Let (V, E) be a tree. Then $\#(E) = \#(V) - 1$, where $\#(V)$ and $\#(E)$ denote respectively the number of vertices and the number of edges of the tree.

Proof

We can prove the result by induction on the number $\#(V)$ of vertices of the tree. The result is clearly true when the tree is trivial, since it then consists of one vertex and no edges.

Suppose that every tree with m vertices has $m - 1$ edges. Let (V, E) be a tree with $m + 1$ vertices. At least one of these vertices is a pendent vertex (Theorem 32.2). Let v be a pendent vertex, let w be the vertex that is adjacent to v, let $V' = V \setminus \{v\}$, and let $E' = E \setminus \{v\, w\}$. Then (V', E') is a subgraph of (V, E), and this subgraph has m vertices. (This subgraph is obtained from the original graph by deleting the vertex v and the edge $v\, w$ from that graph.) We claim that this subgraph (V', E') is in fact a tree.



w
v

First we show that (V', E') is connected. Now, given any two vertices in V', there exists a path in (V, E) from one vertex to the other. This path could not pass through the vertex v, since otherwise the path would have to pass through w twice (going out to v and then returning from v), which is impossible since a path by definition has no repeated vertices. Therefore this path is in fact a path in (V', E'). We conclude that (V', E') is connected.
Now the tree (V, E) does not contain any circuits. It follows immediately that the connected subgraph (V', E') does not contain any circuits, and is thus a tree. It has m vertices.
The induction hypothesis now ensures that the tree (V', E') has $m - 1$ edges, and therefore the tree (V, E) has m edges. The required result therefore follows by the Principle of Mathematical Induction. ∎

Theorem 32.4

Given two distinct vertices of a tree, there exists a unique path in the tree from the first vertex to the the second.

Proof

Let u and v be distinct vertices of the tree. There must exist at least one path in the tree from u to v, since any tree is connected. Were there to exist more than one, then it would follow from Theorem 29.2 that there would exist at least one circuit in the tree, which is impossible, since that a tree cannot contain any circuits. Therefore there must exist exactly one path in the tree from u to v. ∎