Course MA2C03: Michaelmas Term 2014.

Assignment I.

To be handed in by Friday 21st November, 2014.
Please include both name and student number on any work handed in.

1. *Let A, B and C be sets. Prove that*

$$(A \setminus B) \cap C = A \cap (C \setminus B).$$

Let $x \in (A \setminus B) \cap C$. Then $x \in A \setminus B$ and $x \in C$. Now $x \in A$ and $x \notin B$. It follows that $x \in C \setminus B$, because $x \in C$ and $x \notin B$. But also $x \in A$. It follows that $x \in A \cap (C \setminus B)$.

Now let $x \in A \cap (C \setminus B)$. Then $x \in A$ and $x \in C \setminus B$. Then $x \in C$ and $x \notin B$, but $x \in A$, and therefore $x \in A \setminus B$, and thus $x \in (A \setminus B) \cap C$.

We deduce that $(A \setminus B) \cap C = A \cap (C \setminus B)$ because every element of one set is an element of the other.

2. *Let S be the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xSy if and only if $x^2 + y^2$ is divisible by 2. Determine*

   (i) whether or not the relation S is reflexive*,*

   (ii) whether or not the relation S is symmetric*,*

   (iii) whether or not the relation S is anti-symmetric*,*

   (iv) whether or not the relation S is transitive*,*

   (v) whether or not the relation S is a equivalence relation*,*

   (vi) whether or not the relation S is a partial order*.*

[Justify your answers with short proofs and/or counterexamples.]

The relation S is reflexive. Indeed if integers x and y satisfy $x = y$ then $x^2 + y^2 = 2x^2$, and therefore $x^2 + y^2$ is divisible by 2.

The relation S is symmetric, because integers x and y satisfy xSy if and only if $x^2 + y^2$ is divisible by 2 in which case $y^2 + x^2$ is divisible by 2 and thus ySx.

The relation S is not anti-symmetric. In particular $1S3$ and $3S1$ but $1 \neq 3$.

The relation S is transitive. Indeed suppose that integers x, y and z satisfy xSy and ySz. Then $x^2 + y^2$ and $y^2 + z^2$ are both divisible by 2. Adding, we find that $x^2 + 2y^2 + z^2$ is divisible by 2. On subtracting $2y^2$, which is itself divisible by 2, we find that $x^2 + z^2$ is divisible by 2, and thus xSz.

(Alternatively, to prove transitivity, one may observe that if xSy and ySz then x and y are either both odd or else both even, and similarly y and z are either both odd or both even. Thus if x is even then z is even and xSz, or if x is odd then z is odd and xSz.)

The relation S on $\mathbb{Z}$ is an equivalence relation, because it is reflexive, symmetric and transitive. It is not a partial order, because it is not anti-symmetric.

3. *Let $f\colon [0,3] \to [0,9]$ be the function defined so that $f(x) = x^3 - 3x^2 + 3x$ for all $x \in [0,3]$. Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers. (Note that $[0,3]$ denotes the set of all* real numbers *between* 0 *and* 3 *inclusive.)*

   On differentiating the function f, we find that $f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$. Because the derivative of f is greater than or equal to zero, and only equals zero when $x = 1$, the function $f\colon [0,3] \to [0,9]$ is injective. Indeed if $u, v \in [0,3]$ satisfy $u \neq v$, then either $u < v$, in which case $f(u) < f(v)$ and thus $f(u) \neq f(v)$, or else $u > v$, in which case $f(u) > f(v)$ and thus $f(u) \neq f(v)$. Now $f(0) = 0$ and $f(3) = 9$. It then follows from the continuity of the function f that $f(x)$ takes on all values between 0 and 9 and x increases from 0 to 3. Therefore the function f is surjective.

   In fact $f(x) = (x-1)^3 + 1$. Thus if $u, v \in [0,3]$ satisfy $f(u) = f(v)$ then $(u-1)^3 = (v-1)^3$, and therefore $u - 1 = v - 1$ and $u = v$. This gives an alternative proof that the function is injective. Also, given $y \in [0,9]$ there exists some $w \in [-1,2]$ such that $w^3 = y - 1$. Then $w + 1 \in [0,3]$ and $f(w+1) = w^3 + 1 = y$. This gives an alternative proof that the function is surjective.

4. *Let $\mathbb{R}$ be the set of all real numbers, and let $*$ be the binary operation on $\mathbb{R}$ defined such that*

$$x * y = 3xy + 2x + 2y + \frac{2}{3}.$$

for all $x, y \in \mathbb{R}$. Prove that $(\mathbb{R}^2, *)$ is a monoid. What is the identity element of this monoid? Determine which elements of the monoid are invertible. Is the monoid $(\mathbb{R}, *)$ a group?

Let $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned}
(x * y) * z &= 3(3xy + 2x + 2y + \tfrac{2}{3})z + 2(3xy + 2x + 2y + \tfrac{2}{3}) + 2z + \tfrac{2}{3} \\
&= 9xyz + 6xz + 6yz + 2z + 6xy + 4x + 4y + \tfrac{4}{3} + 2z + \tfrac{2}{3} \\
&= 9xyz + 6(xy + xz + yz) + 4(x + y + z) + 2 \\
x * (y * z) &= 3x(3yz + 2y + 2z + \tfrac{2}{3}) + 2x + 2(3yz + 2y + 2z + \tfrac{2}{3}) + \tfrac{2}{3} \\
&= 9xyz + 6xy + 6xz + 2x + 2x + 6yz + 4y + 4z + \tfrac{4}{3} + \tfrac{2}{3} \\
&= 9xyz + 6(xy + xz + yz) + 4(x + y + z) + 2 \\
&= (x * y) * z
\end{aligned}$$

Thus the binary operation $*$ on $\mathbb{R}$ is associative.

A real number e is an identity element for $*$ if and only if $x*e = e*x = x$ for all real numbers x. Thus e is an identity element if and only if $3xe + 2x + 2e + \frac{2}{3} = x$ for all real numbers x. Now

$$\begin{aligned}
3xe + 2x + 2e + \tfrac{2}{3} &= x \\
\iff 3xe + x + \tfrac{2}{3}(3e + 1) &= 0 \\
\iff (3e + 1)(x + \tfrac{2}{3}) &= 0
\end{aligned}$$

It follows that $-\frac{1}{3}$ is an identity element for $*$ on $\mathbb{R}$.

Real numbers x and y are inverses of one another if and only if $3xy + 2x + 2y + \frac{2}{3} = -\frac{1}{3}$. This is the case if and only if $(3x+2)y = -2x-1$. This equation can be solved to give y in terms of x if and only if $3x + 2 \neq 0$. It follows that the real number x is invertible in the monoid $(\mathbb{R}, *)$ if and only if $x \neq -\frac{2}{3}$. The monoid is not a group, because not every element of the monoid is invertible.