Module MA2C03—Additional Notes

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Contents

2 Solving Ordinary Differential Equations by the Method of Power Series

2.1 Summary and Essential Points of the Discussion

Subsection 2.2 summarizes basic facts concerning Taylor series expansions of the exponential, sine and cosine functions.

Subsection 2.3 covers the differentiation of power series term by term. It is shown that if $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ then $\frac{dy}{dx} = \sum_{n=0}^{+\infty} \frac{y_{n+1}}{n!} x^n$. The coefficient of x^n in the power series is expressed in the form $\frac{y_n}{n!}$, where y_n is some real constant. The reason for expressing the coefficient of x^n in this moderately complicated fashion is that it leads to a memorable formula for the derivatives of the power series, so that, to write down the power series for the kth derivative of y with respect to x, we have merely to replace y_n by y_{n+k} in the formula expressing y as a power series in x.

Subsection 2.4 shows that if there exists some strictly positive real number r such that $\sum_{n=0}^{+\infty} c_n x^n$ converges for all real numbers x satisfying $-r \leq x \leq r$ then the coefficients c_n are given by the formula $c_n = \frac{f^{(n)}(0)}{n!}$, where $f(x) = \sum_{n=0}^{+\infty} c_n x^n$ (Proposition A). It follows that if the power series converges to the zero function then its coefficients $c_0, c_1, c_2, c_3, \ldots$ must all be zero (Corollary B). This result is used frequently in the examples that follow.

Subsection 2.5 describes the use of the method of power series to solve the differential equation

$$\frac{d^6y}{dx^6} - 64y = 0.$$

A key step is to use the fact that if $\sum_{n=0}^{+\infty} c_n x^n = 0$ for all (sufficiently small) values of the independent variable x, then the coefficients c_n of the power series must satisfy $c_n = 0$ for all non-negative integers n. This result is Corollary B.

Subsection 2.6 applies the same method to solve the differential equation

$$\frac{d^3y}{dx^3} + 343y = 0.$$

Subsection 2.7 and following subsections provide a discussion of the differential equation $\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, solving this equation by the method of power series in the case where the roots of the auxiliary polynomial s^2+bs+0 are real and distinct.

First it is shown that if there exists a solution y of the differential equation $\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ that can be represented in the form $y=\sum_{n=0}^{+\infty}\frac{y_n}{n!}x^n$, then $y_{n+2}+by_{n+1}+cy_n=0$ for all non-negative integers n. The cases when $b^2>4c$, $b^2=4c$ and $b^2<4c$ are then investigated separately, and the standard solutions are obtained in those cases.

In Subsection 2.11 the method employed in previous subsections is generalized to show that if y is a solution of the differential equation

$$c_k\frac{d^ky}{dx^k}+c_{k-1}\frac{d^{k-1}y}{dx^{k-1}}+\cdots c_1\frac{dy}{dx}+c_0y=0,$$

where $c_k\neq 0$, and if $y=\sum_{n=0}^{+\infty}\frac{y_n}{n!}x^n$, then

$$c_ky_{n+k}+c_{k-1}y_{n+k-1}+\cdots+c_1y_{n+1}+c_0y_n=0$$

for all non-negative integers n. It follows that $y_k,y_{k+1},y_{k+2},\ldots$ may be determined recursively from the values of $y_0,y_1,\ldots,y_{k-1}$.

Subsection 2.12 contains a brief discussion of solutions of linear homogeneous differential equations with variable coefficients, using the method of power series. The method is used to find the solution to the differential equation

$$\frac{dy}{dx}-2xy=0.$$

by representing the solution to that equation in the form $y=\sum_{n=0}^{+\infty}a_nx^n$.

2.2 Taylor Series expansions of Exponential and Trigonometric Functions

The following Taylor Series expansions of the exponential, sine and cosine functions are as follows:—

$$e^x \quad = \quad 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+\cdots$$

$$
\begin{aligned}
&= \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R}), \\
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad (x \in \mathbb{R}), \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad (x \in \mathbb{R}).
\end{aligned}
$$

This power series expansions are Taylor series of the form

$$
\begin{aligned}
f(a+x) &= f(a) + f'(a)x + \frac{f''(a)}{2}\,x^2 + \frac{f'''(a)}{6}\,x^3 + \frac{f^{(4)}(a)}{24}\,x^4 + \cdots \\
&= \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}\,x^n,
\end{aligned}
$$

where f denotes the function in question, and $f^{(n)}(a)$ denotes the nth derivative of $f(x)$ at $x = a$. One can calulate the value of $f(x)$ to whatever accuracy is desired by adding sufficiently many terms of the infinite series.

The power series expansions for e^x, $\sin x$ and $\cos x$ given above are particularly relevant when obtaining solutions of ordinary differential equations with constant coefficients. Note that

$$
\begin{aligned}
e^{cx} &= 1 + cx + \frac{c^2x^2}{2} + \frac{c^3x^3}{6} + \frac{c^4x^4}{24} + \frac{c^5x^5}{120} + \frac{c^6x^6}{720} + \frac{c^7x^7}{5040} + \cdots \\
&= \sum_{n=0}^{+\infty} \frac{c^n x^n}{n!}, \\
\sin cx &= cx - \frac{c^3x^3}{6} + \frac{c^5x^5}{120} - \frac{c^7x^7}{5040} + \cdots \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k c^{2k+1} x^{2k+1}}{(2k+1)!}, \\
\cos cx &= 1 - \frac{c^2x^2}{2} + \frac{c^4x^4}{24} - \frac{c^6x^6}{720} + \cdots \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k c^{2k} x^{2k}}{(2k)!}
\end{aligned}
$$

for all real numbers x.

2.3 Differentiating Power Series Term by Term

Let

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \\ &= \sum_{n=0}^{+\infty} a_n x^n, \end{aligned}$$

where the coefficients $a_0, a_1, a_2, a_3, \ldots$ are real numbers. The derivative of y with respect to x may then be found by differentiating term by term. We find that

$$\begin{aligned} \frac{dy}{dx} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \\ &= \sum_{k=1}^{+\infty} k a_k x^{k-1}. \end{aligned}$$

Now we can substitute $k = n+1$ in the above identity to obtain a summation over values of n satisfying $n \geq 0$. We find that

$$\frac{dy}{dx} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n.$$

It is convenient to express a_n in terms of y_n, where $y_n = n! a_n$. Then $y_0, y_1, y_2, y_3, \ldots$ are real constants that determine the power series. We find that

$$a_n = \frac{y_n}{n!} \quad \text{and} \quad (n+1)a_{n+1} = (n+1) \times \frac{y_{n+1}}{(n+1)!} = \frac{y_{n+1}}{n!}.$$

We conclude that if $y = \displaystyle\sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ then

$$\frac{dy}{dx} = \sum_{n=0}^{+\infty} \frac{y_{n+1}}{n!} x^n, \quad \frac{d^2y}{dx^2} = \sum_{n=0}^{+\infty} \frac{y_{n+2}}{n!} x^n, \quad \frac{d^3y}{dx^2} = \sum_{n=0}^{+\infty} \frac{y_{n+3}}{n!} x^n. \quad \text{etc.}$$

In general we find that if $y = \displaystyle\sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ then $\displaystyle\frac{d^ky}{dx^k} = \sum_{n=0}^{+\infty} \frac{y_{n+k}}{n!} x^n$.

The above discussion implicitly assumes that it is permissible to differentiate power series term by term. This is a result that is proved in courses whose purpose is to develop rigorous proofs of the main theorems of differential and integral calculus.

2.4 Uniqueness of Power Series Representations

Proposition A. *Let* $\sum_{n=0}^{+\infty} c_n x^n$ *be a power series whose coefficients*

$$c_0, c_1, c_2, c_3, \ldots$$

are real numbers. Suppose that there exists some positive integer r *with the property that the power series converges whenever* $-r \leq x \leq r$. *Let*

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

for all real numbers x *satisfying* $-r \leq x \leq r$. *Then*

$$c_n = \frac{f^{(n)}(0)}{n!}$$

for all non-negative integers n, *where* $f^{(n)}(0)$ *denotes the* n*th derivative of the* $f(x)$ *at* $x = 0$.

Proof. On calculating the kth derivative of x^n we find that

$$\frac{d^k}{dx^k}\Bigl(x^n\Bigr) = \begin{cases} 0 & \text{if } n < k; \\ k! & \text{if } n = k; \\ \dfrac{n! x^{n-k}}{(n-k)!} & \text{if } n > k. \end{cases}$$

(This result may be proved by induction on k.)

Power series may be differentiated term by term. Therefore,

$$f^{(k)}(x) = \frac{d^k}{dx^k}\Bigl(f(x)\Bigr) = k! c_k + \sum_{n=k+1}^{+\infty} \frac{n! c_n}{(n-k)!} x^{n-k}.$$

On setting $x = 0$, we find that $f^{(k)}(0) = k! c_k$. The result follows. Q. E. D.

Corollary B. *Let* $\sum_{n=0}^{+\infty} c_n x^n$ *be a power series whose coefficients*

$$c_0, c_1, c_2, c_3, \ldots$$

are real numbers. Suppose that there exists a strictly positive real number r *such that*

$$\sum_{n=0}^{+\infty} c_n x^n = 0$$

for all real numbers x *satisfying* $-r \leq x \leq r$. *Then* $c_n = 0$ *for all non-negative integers* n.

Proof. This result follows immediately from Proposition A, because the infinite series converges to the zero function when $-r \leq x \leq r$. Q. E. D.

Corollary C. *Let* $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{+\infty} b_n x^n$ *for all values of* x *sufficients close to zero. Suppose that* $f(x) = g(x)$ *for all values of* x *sufficiently close to zero. Then* $a_n = b_n$ *for all non-negative integers* n.

Proof. Note that

$$f(x) - g(x) = \sum_{n=0}^{+\infty} (a_n - b_n) x^n.$$

Thus if $f(x) = g(x)$ for all values of x sufficiently close to zero then it follows from Corollary B that $a_n - b_n = 0$ for all non-negative integers n. The result follows. Q. E. D.

2.5 Solving the Differential Equation $\dfrac{d^6 y}{dx^6} - 64y = 0$ using the Method of Power Series

Consider the differential equation

$$\frac{d^6 y}{dx^6} - 64y = 0.$$

Note that $24 = 2^6$. We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined. Then $\dfrac{d^6 y}{dx^6} = \sum_{n=0}^{+\infty} \dfrac{y_{n+6}}{n!} x^n$. It follows that

$$\sum_{n=0}^{+\infty} \frac{y_{n+6} - 64 y_n}{n!} x^n = \frac{d^6 y}{dx^6} - 64y = 0.$$

Now the above identity is an identity of the form $\sum_{n=0}^{+\infty} c_n x^n = 0$, where $c_0, c_1, c_2, \ldots$ are real constants. The left hand side must be zero *for all values of the independent variable* x. It follows from this that $c_n = 0$ for all non-negative integers n. This result is a consequence of Corollary B.

Corollary B thus ensures that $y_{n+6} = 64 y_n = 2^6 y_n$ for all non-negative integers n. Thus the values of y_n for all non-negative integers n are determined by the values of y_0, y_1, y_2, y_3, y_4 and y_5.

Suppose that $y_0 = 1$, $y_1 = 2$, $y_2 = 4$, $y_3 = 8$, $y_4 = 16$ and $y_5 = 32$. Then the recursion relation $y_{n+6} = 2^6 y_n$ ensures that $y_n = 2^n$ for all non-negative integers n. It then follows that

$$y = \sum_{n=0}^{+\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{+\infty} \frac{(2x)^n}{n!} = e^{2x}.$$

Next suppose that $y_0 = 1$, $y_1 = -2$, $y_2 = 4$, $y_3 = -8$, $y_4 = 16$ and $y_5 = -32$. Then the recursion relation $y_{n+6} = 2^6 y_n = (-2)^6 y_n$ ensures that $y_n = (-2)^n$ for all non-negative integers n. It then follows that

$$y = \sum_{n=0}^{+\infty} \frac{(-2)^n}{n!} x^n = \frac{(-2x)^n}{n!} = e^{-2x}.$$

Next suppose that $y_0 = 1$, $y_1 = 0$, $y_2 = 0$, $y_3 = 8$, $y_4 = 0$ and $y_5 = 0$. Then

$$y_n = \begin{cases} 2^n & \text{if } n \text{ is divisible by 3;} \\ 0 & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

It would be possible with a bit of work to express the solution to the differential equation in terms of exponential and trigonometrical functions.

2.6 Solving the Differential Equation $\dfrac{d^3y}{dx^3} + 343y = 0$ using the Method of Power Series

Consider the differential equation

$$\frac{d^3y}{dx^3} + 343y = 0.$$

Note that $343 = 7^3$. We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined.

Now $\frac{d^3y}{dx^3} = \sum_{n=0}^{+\infty} \frac{y_{n+3}}{n!} x^n$. It follows that

$$\sum_{n=0}^{+\infty} \frac{y_{n+3} + 343y_n}{n!} x^n = \frac{d^3y}{dx^3} + 343y = 0.$$

Now because the power series on the left hand side of the above identity converges to zero for all values of the independent variable x, its coefficients must all be equal to zero. Therefore $y_{n+3} + 343y_n = 0$ for all non-negative integers n. It follows that $y_{n+3} = (-7)^3 y_n$ for all non-negative integers n.

The values of y_n for all non-negative integers n will be determined by the values of y_0, y_1 and y_2.

Suppose that $y_0 = 1$, $y_1 = -7$ and $y_2 = 49$. Then $y_n = (-7)^n$ for all non-negative integers n. In this case we find that

$$y = \sum_{n=0}^{+\infty} \frac{(-7x)^n}{n!} = e^{-7x}.$$

Next suppose that $y_0 = 5$, $y_1 = 0$ and $y_2 = 0$. Then $y_n = 5 \times (-7)^n$ whenever n is divisible by 3, and $y_n = 0$ whenever n is not divisible by 3. (With some work, one could then find an expression for y as a function of x expressed in terms of exponential and trigonometrical functions.)

2.7 The Difference Equation that determines the Solution of the Differential Equation $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$.

Consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined.

Now

$$\frac{dy}{dx} = \sum_{n=0}^{+\infty} \frac{y_{n+1}}{n!} x^n \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{n=0}^{+\infty} \frac{y_{n+2}}{n!} x^n.$$

It follows that

$$\sum_{n=0}^{+\infty} \frac{y_{n+2} + by_{n+1} + cy_n}{n!} x^n = \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Because the power series on the left hand side of the above identity converges to the zero function, the coefficient of x^n must be equal to zero for all non-negative integers n. Therefore

$$y_{n+2} + by_{n+1} + cy_n = 0 \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Note that $y_{n+2} = -by_{n+1} - cy_n$ for all non-negative integers n. Thus if y_0 and y_1 are given, the values of $y_2, y_3, y_3, \ldots$ could be successively computed (e.g., by means of a computer program utilizing an appropriate recursive function).

We consider solutions to the above difference equation for the coefficients y_n that grow as some power of n. Suppose that $y_n = As^n$ for all non-negative integers n, where A and s are numerical constants. Then

$$y_{n+2} + by_{n+1} + cy_n = (s^2 + bs + c)y_n.$$

It follows that the infinite sequence $y_0, y_1, y_2, y_3 \ldots$ satisfies the required difference equation if and only if s is a root of the auxiliary polynomial s^2+bs+c.

2.8 Solution of the Differential Equation $\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy = 0$ when $b^2 > 4c$.

Consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined. We have shown that

$$y_{n+2} + by_{n+1} + cy_n = 0 \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Moreover if $y_n = As^n$ for all non-negative integers n, where A and s are numerical constants, then the infinite sequence $y_0, y_1, y_2, y_3 \ldots$ satisfies the above difference equation if and only if $s^2 + bs + c = 0$.

Now suppose that $b^2 > 4c$. Then the auxiliary polynomial $s^2 + bs + c$ has two distinct real roots u and v. Let A and B be real constants, and let

$$y_n = Au^n + Bv^n$$

for all non-negative integers n. Then $y_{n+2} + by_{n+1} + cy_n = 0$ for all n. Let

$$y = \sum_{n=0}^{\infty} \frac{Au^n + Bv^n}{n!} x^n = A \sum_{n=0}^{\infty} \frac{(ux)^n}{n!} + B \sum_{n=0}^{\infty} \frac{(vx)^n}{n!} = Ae^{ux} + Be^{vx}.$$

Then y is a solution of the differential equation $y'' + by' + c$. Moreover the constants A and B determine the values of y_0 and y_1, and therefore determine the values of $y_2, y_3, y_4, \ldots$. We have thus found the general solution of the differential equation when $b^2 > 4c$.

2.9 Solution of the Differential Equation $\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy = 0$ when $b^2 = 4c$.

Consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined. We have shown that

$$y_{n+2} + by_{n+1} + cy_n = 0 \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Moreover if $y_n = As^n$ for all non-negative integers n, where A and s are numerical constants, then the infinite sequence $y_0, y_1, y_2, y_3 \ldots$ satisfies the above difference equation if and only if $s^2 + bs + c = 0$.

Suppose that $b^2 = 4c$. Let $r = -\frac{1}{2}b$. Then $s^2 + bs + c = (s - r)^2$. The difference equation $y_{n+2} + by_{n+1} + cy_n = 0$ is satisfied when $y_n = Ar^n$ for all non-negative integers n. Also

$$(n + 2)r^{n+2} + b(n + 1)r^{n+1} + cnr^n = n(r^2 + br + c)r^n + (2r + b)r^{n+1} = 0.$$

It follows that the difference equation $y_{n+2} + by_{n+1} + cy_n = 0$ is satisfied when $y_n = Bnr^n$. The sum of any two solutions to the difference equation also solves the difference equation. Thus the solutions to the difference equation $y_{n+2} + by_{n+1} + cy_n = 0$ are of the form $y_n = Ar^n + Bnr^n$ in the case where $b^2 = 4c$. Let

$$y = \sum_{n=0}^{\infty} \frac{Ar^n + Bnr^n}{n!} x^n = A \sum_{n=0}^{\infty} \frac{(rx)^n}{n!} + B \sum_{n=1}^{\infty} \frac{(rx)^n}{(n-1)!} = (A + Brx)e^{rx}.$$

Then y is a solution of the differential equation $y''+by'+cy = 0$. Moreover the constants A and B determine the values of y_0 and y_1, and therefore determine the values of $y_2, y_3, y_4, \ldots$. We have thus found the general solution of the differential equation when $b^2 = 4c$.

2.10 Solution of the Differential Equation $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ when $b^2 < 4c$.

Consider the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$ where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined. We have shown that

$$y_{n+2} + b y_{n+1} + c y_n = 0 \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Moreover if $y_n = A s^n$ for all non-negative integers n, where A and s are numerical constants, then the infinite sequence $y_0, y_1, y_2, y_3 \ldots$ satisfies the above difference equation if and only if $s^2 + bs + c = 0$.

Suppose that the auxiliary polynomial $s^2 + bs + c$ has no real roots. Then it has two complex roots of the form $p \pm iq$, where p and q are real numbers and $i = \sqrt{-1}$. Let C and D be *complex* constants, let $u_n = (p + iq)^n$ and $v_n = (p - iq)^n$, and let

$$w_n = C u_n + D u_n = C(p + iq)^n + D(p - iq)^n$$

for all non-negative integers n. Then

$$u_{n+2} + b u_{n+1} + c u_n = 0 \text{ and } v_{n+2} + b v_{n+1} + c v_n = 0,$$

and therefore

$$w_{n+2} + b w_{n+1} + c w_n = 0$$

for all non-negative integers n. Now the numbers w_n are real numbers for all n if and only if $D = \overline{C}$ (i.e., if and only if D is the complex conjugate of C). In this case we may write $C = A + iB$ and $D = A - iB$, where A and B are real constants.

Thus let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{(A + iB)(p + iq)^n + (A - iB)(p - iq)^n}{n!} x^n \\ &= (A + iB) \sum_{n=0}^{\infty} \frac{((p + iq)x)^n}{n!} + (A - iB) \sum_{n=0}^{\infty} \frac{((p - iq)x)^n}{n!} \\ &= (A + iB) e^{(p+iq)x} + (A - iB) e^{(p-iq)x}, \end{aligned}$$

where $e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$ for all complex numbers z. Then y is a solution of the differential equation $y'' + by' + cy$ when the roots of the auxiliary polynomial $s^2 + bs + c$ are $p \pm iq$. Standard properties of the exponential function, applicable when the argument is a complex number, ensure that

$$\begin{aligned} e^{(p+iq)x} &= e^{px} e^{iqx} = e^{px}(\cos qx + i \sin qx), \\ e^{(p-iq)x} &= e^{px} e^{iqx} = e^{px}(\cos qx - i \sin qx). \end{aligned}$$

It follows that

$$\begin{aligned} y &= e^{px}\left((A+iB)(\cos qx + i\sin qx) + (A-iB)(\cos qx - i\sin qx)\right) \\ &= e^{px}(2A\cos qx - 2B\sin qx). \end{aligned}$$

The constants A and B determine the coefficients y_0 and y_1 in the representation of y as a power series. We have thus found the general solution to the differential equation $y'' + by' + c$ in the case where $b^2 < 4c$.

2.11 Solution of Homogeneous Linear Ordinary Differential Equations with Constant Coefficients

Let y be a solution of a differential equation of the form

$$c_k\frac{d^ky}{dx^k} + c_{k-1}\frac{d^{k-1}y}{dx^{k-1}} + \cdots c_1\frac{dy}{dx} + c_0y = 0,$$

where $c_k \neq 0$. We suppose that the solution y can be represented by a power series of the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!}x^n$ Then

$$\sum_{n=0}^{\infty} \frac{c_ky_{n+k} + c_{k-1}y_{n+k-1} + \cdots + c_1y_{n+1} + c_0y_n}{n!} = 0.$$

It follows from Corollary B that

$$c_ky_{n+k} + c_{k-1}y_{n+k-1} + \cdots + c_1y_{n+1} + c_0y_n = 0$$

for all non-negative integers n. But then

$$y_{n+k} = -\frac{1}{c_k}\left(c_{k-1}y_{n+k-1} + \cdots + c_1y_{n+1} + c_0y_n\right)$$

for all non-negative integers k. Thus, given the values of $y_0, y_1, y_2, \ldots, y_{k-1}$, one can recursively determine the values of $y_k, y_{k+1}, y_{k+2}, y_{k+3}, \ldots$, and it would be a straightforward exercise to program a computer to carry out that task. The values of the coefficients y_n would then determine the solution of the differential equation.

2.12 Solution of Homogeneous Linear Ordinary Differential Equations with Variable Coefficients

In seeking solutions of homogeneous linear differential equations with constant coefficients, we have represented the desired solution y as the sum of a power series expressed in the form $y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n$. This representation is convenient for finding solutions of homogeneous differential equations with constant coefficients. It is less convenient when those coefficients are variable. In that case it may be simpler to look for solutions of the form $y = \sum_{n=0}^{+\infty} a_n x^n$, where $a_0, a_1, a_2, \ldots$ are the coefficients to be determined.

Consider, for example, the differential equation

$$\frac{dy}{dx} - 2xy = 0.$$

We look for a solution of the form $y = \sum_{n=0}^{+\infty} a_n x^n$. On substituting this expression for y in the differential equation, we find that

$$\sum_{n=0}^{+\infty} \left(n a_n x^{n-1} - 2 a_n x^{n+1}\right) = 0.$$

Now

$$\sum_{n=0}^{+\infty} n a_n x^{n-1} = \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k \quad \text{and} \quad \sum_{n=0}^{+\infty} 2 a_n x^{n+1} = \sum_{k=1}^{+\infty} 2 a_{k-1} x^k.$$

Therefore

$$a_1 + \sum_{k=1}^{+\infty} \left((k+1) a_{k+1} - 2 a_{k-1}\right) x^k = 0.$$

It then follows that $a_1 = 0$ and $(k+1)a_{k+1} = 2a_{k-1}$ whenever $k \geq 1$. Applying these identities for even values of k, we find that $y_3 = 0$, $y_5 = 0$, $y_7 = 0$, etc., and applying these identities when $k = 2j + 1$ for some non-negative integer j, we find that $(j + 1)a_{2j+2} = a_{2j}$ for all non-negative integers j. It follows that $a_{2j} = \frac{a_0}{j!}$ for all non-negative integers j, and thus

$$y = \sum_{j=0}^{+\infty} \frac{a_0 x^{2j}}{j!} = a_0 e^{x^2}.$$

One may readily verify that this function does indeed satisfy the differential equation.