Module MA2C03—Additional Notes

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Contents

1 Real-Analytic Functions **1**
1.1 The Definition of Real-Analyticity 1
1.2 Fundamental Properties of Real-Analytic Functions 3
1.3 Taylor Series expansions of some Standard Functions 5

1 Real-Analytic Functions

1.1 The Definition of Real-Analyticity

We consider functions $f: D \to \mathbb{R}$ that are defined on open subsets D of the set $\mathbb{R}$ and take values in the set of real numbers itself. An *open* subset D of $\mathbb{R}$ is a subset of $\mathbb{R}$ that completely surrounds all its elements. The concept of open set can be more formally defined as follows:—

Definition A subset D of the set $\mathbb{R}$ of real numbers is said to be *open* if, given any element s of D, there exists some strictly positive real number r that is small enough to ensure that D contains all real numbers x that satisfy the inequalities $s - r \leq x \leq s + r$.

The purpose of introducing this definition is to ensure that, if a function $f: D \to \mathbb{R}$ is defined on an open subset D of $\mathbb{R}$, and if s is an element of D, then there exists some strictly positive real number r that is small enough to ensure that $f(x)$ is defined for all real numbers x satisfying $s-r \leq x \leq s+r$. The value $f(x)$ of the function f is then defined for all real numbers x that lie sufficiently close to s. Thus if the function f is defined *at* s then it is defined *around* s.

Definition Let $f\colon D \to \mathbb{R}$ be a function defined over an open subset D of the set $\mathbb{R}$ of real numbers and taking values in $\mathbb{R}$. The function f is said to be *real-analytic* on D if, given any point s of D, there exists a positive real number r, and real numbers $c_0, c_1, c_2, c_3, \ldots$ such that

$$f(s+x) = \sum_{n=0}^{+\infty} c_n x^n$$

for all real numbers x satisfying $|x| \leq r$.

A real-valued function $f\colon D \to \mathbb{R}$ defined over an open subset D of $\mathbb{R}$ is thus real-analytic on D if, given any element s of D, the values of the function in a sufficiently small neighbourhood of s can be determined by means of a convergent power series.

Example Let $D = \{x \in \mathbb{R} : x \neq 0\}$, and let $f\colon D \to \mathbb{R}$ be defined such that $f(x) = \dfrac{1}{x}$ for all non-zero real numbers x. Then the function f is real analytic. Indeed, given any non-zero real number s, the function f can be expressed in a neighbourhood of s by means of the power series

$$\begin{aligned} f(x) = \frac{1}{s+x} &= \frac{1}{s} - \frac{1}{s^2}\,x + \frac{1}{s^3}\,x^2 - \frac{1}{s^4}\,x^3 + \frac{1}{s^5}\,x^4 - \frac{1}{s^6}\,x^5 + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{s^{n+1}} x^n \end{aligned}$$

which converges for all real numbers satisfying $|x| < |s|$.

Many of the standard functions of a real-variable are real-analytic functions wherever they are defined. Thus polynomial functions, exponential functions, logarithm functions and trigonometric functions such as the sine, cosine, tangent, cotangent, secant and cosecant functions are real-analytic functions wherever they are defined. Also sums, differences, products, quotients and compositions of real-analytic functions are real-analytic wherever they are defined.

However there exist functions that can be differentiated any number of times that are not real-analytic. A standard example of such a function is the function $s\colon \mathbb{R} \to \mathbb{R}$ defined such that

$$s(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It turns out that this function can be differentiated any number of times, and its derivatives of all orders are equal to zero. This function is *very flat* around $x = 0$. However the function cannot be expressed as the sum of a power series that converges around $x = 0$.

Thus, if one is, for example, using methods based on the use of power series to determine solutions of differential equations, such methods determine solutions of the differential equations that are real-analytic. However if a differential equation is expressed in terms of functions that are real-analytic then its solutions can be expected to be real-analytic. Therefore one can find all solutions of the differential equation by searching for the real-analytic solutions.

1.2 Fundamental Properties of Real-Analytic Functions

1. *Any function expressible as the sum of a convergent power series is real-analytic. In particular any polynomial function is real analytic.*

2. *Sums, differences, products, quotients and compositions of real analytic functions are real-analytic wherever they are defined.*

3. *The exponential, logarithm, sine, cosine, tangent, cotangent, secant and cosecant functions are real-analytic.*

4. *The inverse function of a real-analytic function is real-analytic wherever it is defined.*

5. *Convergent power series can be differentiated and integrated term by term.*

   Thus if

   $$y = f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots,$$

   then

   $$\frac{dy}{dx} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

   and

   $$\int y\,dx = C + c_0 x + \frac{1}{2}c_1 x^2 + \frac{1}{3}c_2 x^3 + \frac{1}{4}c_3 x^4 + \frac{1}{5}c_4 x^5 + \cdots,$$

   where C is some constant of integration.

6. *Let $f\colon D \to \mathbb{R}$ is a real-analytic function defined over some open subset D of $\mathbb{R}$, let s be an element of D, let r be a positive real number chosen such that $s + x \in D$ for all real numbers x satisfying $|x| \leq r$, and let $c_0, c_1, c_2, c_3 \ldots$ are real constants such that*

$$f(s+x) = \sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

for all real numbers x satisfying $|x| \leq r$. Then

$$c_n = \frac{f^{(n)}(s)}{n!}$$

for all non-negative integers n, where $f^{(n)}(s)$ denotes the nth derivative of $f(x)$ at $x = s$.

Indeed

$$\begin{aligned} \frac{d^k}{dx^k}\Big(f(s+x)\Big) &= \frac{d^k}{dx^k}\left(\sum_{n=0}^{+\infty} c_n x^n\right) = \sum_{n=0}^{+\infty} c_n \frac{d^k}{dx^k}\,(x^n) \\ &= k!c_k + \sum_{n=k+1}^{+\infty} \frac{n!c_n}{(n-k)!} x^{n-k}. \end{aligned}$$

On setting $x = 0$, we find that

$$f^{(k)}(s) = \frac{d^k}{dx^k}\Big(f(s+x)\Big)\bigg|_{x=0} = k!c_k.$$

Therefore $c_k = \dfrac{f^{(k)}(s)}{k!}$.

7. *If r is a positive real number, and if $c_0, c_1, c_2, c_3 \ldots$ are real constants with the property that*

$$0 = \sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

for all real numbers x satisfying $-r \leq x \leq r$, then $c_n = 0$ for all non-negative integers n.

8. *Let $f\colon D \to \mathbb{R}$ be a real-analytic function defined over some open subset D of $\mathbb{R}$, and let s be an element of D. Then*

$$\begin{aligned} f(s+x) &= f(s) + f'(s)x + \frac{f''(s)}{2}\,x^2 + \frac{f'''(s)}{6}\,x^3 + \frac{f^{(4)}(s)}{24}\,x^4 + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{f^{(n)}(s)}{n!}\,x^n, \end{aligned}$$

for all values of x sufficiently close to zero, where where f denotes the function in question, and $f^{(n)}(a)$ denotes the nth derivative of $f(x)$ at $x = a$.

1.3 Taylor Series expansions of some Standard Functions

The following Taylor Series expansions of the corresponding functions are well-known:—

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ &= \sum_{n=0}^{+\infty} x^n, \quad (-1 < x < 1), \\ (1+x)^s &= 1 + sx + \frac{s(s-1)}{2}x^2 + \frac{s(s-1)(s-2)}{6}x^3 \\ &\qquad + \frac{s(s-1)(s-2)(s-3)}{24}x^4 + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{s(s-1)\cdots(s-n+1)}{n!}x^n \quad (-1 < x < 1), \\ \log(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^{n-1}x^n}{n} \quad (-1 < x < 1), \\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R}), \\ \sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad (x \in \mathbb{R}), \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad (x \in \mathbb{R}).
\end{aligned}
$$