# Course MA2C03: Michaelmas Term 2013.

# Assignment II—Worked Solutions.

1. *Let $\mathbb{R}^2$ be the set of all ordered pairs of numbers, and let $\otimes$ be the binary operation on $\mathbb{R}^2$ defined such that*

$$(a, b) \otimes (c, d) = (a + c, ac + b + d)$$

*for all $(a, b), (c, d) \in \mathbb{R}^2$. Prove that $(\mathbb{R}^2, \otimes)$ is a monoid. What is the identity element of this monoid? Is the monoid $(\mathbb{R}^2, \otimes)$ a group?*

Let $(a, b), (c, d), (e, f) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&((a, b) \otimes (c, d)) \otimes (e, f) \\
&\qquad = (a + c, ac + b + d) \otimes (e, f) \\
&\qquad = (a + c + e, (a + c)e + (ac + b + d) + f) \\
&\qquad = (a + c + e, ae + ce + ac + b + d + f) \\
&(a, b) \otimes ((c, d) \otimes (e, f)) \\
&\qquad = (a, b) \otimes (c + e, ce + d + f) \\
&\qquad = (a + c + e, a(c + e) + b + (ce + d + f)) \\
&\qquad = (a + c + e, ac + ae + ce + b + d + f) \\
&\qquad = ((a, b) \otimes (c, d)) \otimes (e, f)
\end{aligned}$$

Therefore the binary operation $\otimes$ on $\mathbb{R}^2$ is associative.

An element $(e, f)$ of $\mathbb{R}^2$ is an identity element if and only if $(a, b) \otimes (e, f) = (e, f) \otimes (a, b) = (a, b)$ for all $(a, b) \in \mathbb{R}^2$. This is the case if and only if $a + e = a$ and $ae + b + f = b$ for all real numbers $a$ and $b$. These identities are clearly satisfied if $e = 0$ and $f = 0$. It follows that $(0, 0)$ is an identity element for the associative binary operation $\otimes$ on $\mathbb{R}^2$, and therefore $(\mathbb{R}^2, \otimes)$ is a monoid.

An element $(c, d)$ of $\mathbb{R}^2$ is the inverse of $(a, b)$ if and only if

$$(a, b) \otimes (c, d) = (c, d) \otimes (a, b) = (0, 0).$$

This is the case if and only if $a + c = 0$ and $ac + b + d = 0$. These equations are satisfied if and only if $c = -a$ and $d = a^2 - b$. It follows that each element $(a, b)$ of $\mathbb{R}^2$ is invertible, and $(a, b)^{-1} = (-a, a^2 - b)$. It follows that $(\mathbb{R}^2, \otimes)$ is a group.

2. (a) *Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose only non-terminal is $\langle S \rangle$, whose start symbol is $\langle S \rangle$ and whose productions are the following:*

$$\begin{aligned} \langle S \rangle &\rightarrow 1 \\ \langle S \rangle &\rightarrow 0\langle S \rangle \end{aligned}$$

*(i.e., describe the structure of the binary strings generated by the grammar). Is this context-free grammar a regular grammar?*

The language consists of all strings

$$1,\ 01,\ 001,\ 0001, \ldots$$

of binary digits made up of $n$ occurrences of the digit 0 followed by a single digit 1 for some $n \geq 0$.

(b) *Give the specification of a finite state acceptor that accepts the language over the alphabet $\{a, b, c\}$ consisting of all words, including the empty word, where the string aba never occurs as a substring. (In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.)*

States: S, A, B, E.

Start state: A.

Finishing states: S, A, B.

Transition table:

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| S | A | S | S |
| A | A | B | S |
| B | E | S | S |
| E | E | E | E |

*Comments:* the state E is an error state that traps errors. An error occurs if and only if the letter $a$ is entered at a time when the previous characters entered make up a string concluding with the substring $ab$. This happens if and only if the finite state machine is in state B. The machine enters state B when the character $b$ is entered at a time where the previous character entered was a letter $a$. The machine is constructed so that whenever the character $a$ is entered either the machine enters the error state or else it enters state A. Because one can halt the machine with a valid string any time up to the point that an error has occurred, the states S, A and B will all be finishing states.

(c) *Devise a regular grammar to generate the language specified in (b). (In particular, you should specify the nonterminals, the start symbol and the productions of the grammar.)*

Nonterminals: $\langle S\rangle$, $\langle A\rangle$, $\langle B\rangle$.
Start symbol: $\langle S\rangle$.
Productions:

$$\begin{array}{lcl}
\langle S\rangle & \to & a\langle A\rangle \\
\langle S\rangle & \to & b\langle S\rangle \\
\langle S\rangle & \to & c\langle S\rangle \\
\langle A\rangle & \to & a\langle A\rangle \\
\langle A\rangle & \to & b\langle B\rangle \\
\langle A\rangle & \to & c\langle S\rangle \\
\langle B\rangle & \to & b\langle S\rangle \\
\langle B\rangle & \to & c\langle S\rangle \\
\langle S\rangle & \to & \varepsilon \\
\langle A\rangle & \to & \varepsilon \\
\langle B\rangle & \to & \varepsilon
\end{array}$$

3. (a) *Let $(V, E)$ be the graph with vertices $a$, $b$, $c$, $d$, $e$ and edges $a\,b$, $a\,e$, $b\,c$, $b\,e$, $c\,d$, $d\,e$.*
   *(i) Is the graph complete?*
   *(ii) Is the graph connected?*
   *(iii) Is the graph regular?*
   *(iv) Does the graph have an Eulerian trail?*
   *(v) Does the graph have an Eulerian circuit?*
   *(vi) Does the graph have a Hamiltonian circuit?*
   *(vii) Is the graph a tree?*

The graph is not complete, because not every pair distinct vertices determines the endpoints of an edge of the graph. For example, there is no edge with endpoints $a$ and $c$.

The graph is connected. Indeed there is a walk $a\,b\,c\,d\,e$ in the graph that visits all vertices and therefore determines within itself a walk from any vertex to any other.

The graph is not regular, because vertices do not all have the same degree. Vertices $b$ and $e$ are of degree 3, whereas vertices $a$, $c$ and $d$ are of degree 2.

The graph has an Eulerian trail $b\,a\,e\,d\,c\,b\,e$ from the vertex $b$ to the vertex $e$. (Any Eulerian trail that is not a circuit must start and end at the vertices that are of odd degree. Such a trail must therefore either start at $b$ and end at $e$, or else start at $e$ and end at $b$.)

The graph does not have any Eulerian circuit. A theorem states that if a graph possesses an Eulerian circuit then all vertices must be of even degree. However the vertices $b$ and $e$ are of odd degree.

The graph possesses Hamiltonian circuits. One such is $a\,b\,c\,d\,e\,a$.

The graph is not a tree because it has circuits. One circuit is the Hamiltonian circuit specified above. Another is $a\,b\,e\,a$.

(b) *Give an example of an isomorphism from the graph* $(V, E)$ *specified in (a) to the graph* $(V_1, E_1)$ *with vertices* $p$, $q$, $r$, $s$, $t$ *and edges and edges* $p\,q$, $p\,s$, $q\,t$, $r\,s$, $r\,t$, $s\,t$.

There are two such isomorphisms. One is $\varphi\colon V \to V_1$ and the other is $\psi\colon V \to V_1$, where

$$\varphi(a) = r, \quad \varphi(b) = s, \quad \varphi(c) = p, \quad \varphi(d) = q, \quad \varphi(e) = t,$$

and

$$\varphi(a) = r, \quad \varphi(b) = t, \quad \varphi(c) = q, \quad \varphi(d) = p, \quad \varphi(e) = s.$$

Note that the vertices $b$ and $e$ of the graph $(V, E)$ are of degree 3 and therefore must map under any isomorphism to vertices of $(V_1, E_1)$ that are also of degree 3. Thus either $b$ and $e$ must map to $s$ and $t$ respectively, or else $b$ and $e$ must map to $t$ and $s$ respectively. The vertex $a$ is the only vertex of $(V, E)$ that is adjacent to both vertices $b$ and $e$ of degree 3, and must therefore map to the unique vertex $r$ of $(V_1, E_1)$ that is adjacent to both $s$ and $t$.