Course MA2C03: Michaelmas Term 2013.

Assignment I—Worked Solutions.

1. *Use the Principle of Mathematical Induction to prove that*

$$\sum_{k=1}^{n} 7^k k = \frac{7}{36}\Big((6n-1)7^n + 1\Big)$$

for all positive integers n.

The required equality holds when $n = 1$, since both sides are then equal to 7. Suppose that the equality holds when $n = m$ for some natural number m, so that

$$\sum_{k=1}^{m} 7^k k = \frac{7}{36}\Big((6m-1)7^m + 1\Big).$$

Then

$$\begin{aligned}
\sum_{k=1}^{m+1} 7^k k &= \sum_{k=1}^{m} 7^k k + 7^{m+1}(m+1) \\
&= \frac{7}{36}\Big((6m-1)7^m + 1\Big) + 7^{m+1}(m+1) \\
&= \frac{7}{36}\Big((6m-1)7^m + 1 + 36(m+1)7^m\Big) \\
&= \frac{7}{36}\Big((42m+35)7^m + 1\Big) = \frac{7}{36}\Big((6m+5)7^{m+1} + 1\Big) \\
&= \frac{7}{36}\Big((6(m+1)-1)5^{m+1} + 1\Big).
\end{aligned}$$

and thus the equality holds when $n = m + 1$. It follows from the Principle of Mathematical Induction that the equality holds for all natural numbers n.

2. *Let* A *and* B *be sets. Prove that*

$$(A \cup B) \setminus (A \setminus B) = B.$$

We prove that every element of the set on the left hand side is an element of the set on the right hand side, and vice versa. Let $x \in (A \cup B) \setminus (A \setminus B)$. Then $x \in A \cup B$ and $x \notin A \setminus B$. Now $x \notin A \setminus B$

implies that either $x \notin A$ or else $x \in A \cap B$. Thus $x \in A$ and $x \notin A \setminus B$ together imply that $x \in A \cap B$ and thus $x \in B$. Thus if $x \in A \cup B$ and $x \notin A \setminus B$ then $x \in B$. We have thus shown that $(A \cup B) \setminus (A \setminus B)$ is a subset of B.

Now let $x \in B$. Then $x \in A \cup B$ and $x \notin A \setminus B$, and thus $x \in (A \cup B) \setminus (A \setminus B)$. We have thus shown that B is a subset of $(A \cup B) \setminus (A \setminus B)$. Therefore $(A \cup B) \setminus (A \setminus B) = B$, as required.

3. *Let S be the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xSy if and only if $x^3 - x \geq y^3 - y$. Determine*

   (i) whether or not the relation S is reflexive*,*

   (ii) whether or not the relation S is symmetric*,*

   (iii) whether or not the relation S is anti-symmetric*,*

   (iv) whether or not the relation S is transitive*,*

   (v) whether or not the relation S is a equivalence relation*,*

   (vi) whether or not the relation S is a partial order*.*

   [Justify your answers with short proofs and/or counterexamples.]

   If integers x and y satisfy $x = y$ then $x^3 - x = y^3 - y$, and therefore xSy. Thus xSx for all integers x. We conclude that the relation S on the set $\mathbb{Z}$ of integers is reflexive.

   The relation S on $\mathbb{Z}$ is not symmetric. Indeed if $x = 3$ and $y = 2$ then $x^3 - x = 24$ and $y^3 - y = 6$. Thus xSy, but $y \not S x$.

   The relation S is not anti-symmetric. Note that $x^3 - x = 0$ when $x = 0$ and $x = 1$ (and also when $x = -1$). It follows that $0S1$ and $1S0$ but $0 \neq 1$.

   The relation S is transitive. Indeed let x, y and z be integers satisfying xSy and ySz. Then $x^3 - x \geq y^3 - y \geq z^3 - z$, and therefore $x^3 - x \geq z^3 - z$, and thus xSz.

   The relation S on $\mathbb{Z}$ is not an equivalence relation because it is not symmetric.

   The relation S on $\mathbb{Z}$ is not a partial order relation because it is not anti-symmetric.

4. *Let* $f\colon [1,4] \to [0,6]$ *be the function defined so that* $f(x) = x^2 - 4x + 4$ *for all* $x \in [1,4]$*. Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers. (Note that* $[1,4]$ *denotes the set of all* real numbers *between* 1 *and* 4 *inclusive, and therefore includes fractions such as* $\frac{3}{2}$ *and irrational numbers like* $\sqrt{2}$ *and* π*.)*

   We consider the behaviour of the function f on the interval $[1,4]$. Now $f'(x) = 2x - 4$ (where $f'(x)$ denotes the derivative of the function f at x for all $x \in [1,4]$. It follows that $f'(x) < 0$ when $1 \leq x < 2$ and $f'(x) > 0$ when $2 < x \leq 4$. Thus the function f is strictly decreasing on the interval $[1,2]$ and is strictly increasing on the function $[2,4]$. Also $f(1) = 1$, $f(2) = 0$ and $f(4) = 4$. The function f therefore will not be injective, and indeed $f(1) = f(3) = 1$.

   Now the range of the function is the interval $[0,4]$, since f maps the interval $[1,2]$ onto the whole of the interval $[0,1]$, and maps the interval $[2,4]$ onto the whole of the interval $[0,4]$. Therefore there does not exist any $x \in [1,4]$ satisfying $f(x) = 5$, though 5 is an element of the codomain $[0,6]$ of the function. Therefore the function $f\colon [1,4] \to [0,6]$ is not surjective.