Course MA2C02: Hilary Term 2011.

Assignment IV—Worked Solutions

1. *Let* $f\colon [0,1] \to \mathbb{R}$ *be the real-valued function on the interval* $[0,1]$ *defined as follows:*
$$f(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq \frac{1}{4}; \\ \frac{4}{3}(1-x) & \text{if } \frac{1}{4} \leq x \leq 1. \end{cases}$$
This function may be represented as a Fourier series of the form
$$f(x) = \sum_{n=1}^{+\infty} b_n \sin n\pi x,$$
where
$$\begin{aligned} b_n &= 2\int_0^1 f(x)\sin n\pi x\,dx \\ &= 2\int_0^{\frac{1}{4}} f(x)\sin n\pi x\,dx + 2\int_{\frac{1}{4}}^1 f(x)\sin n\pi x\,dx. \end{aligned}$$
Find the values of the coefficients b_n *for* $n = 1,2,3,4,\ldots$. *(Note that*
$$\sin\frac{1}{4}\pi = \cos\frac{1}{4}\pi = \frac{1}{\sqrt{2}},$$
$\sin\theta = \cos(\frac{1}{2}\pi - \theta)$ *for all real numbers* θ, $\sin\frac{1}{2}\pi = -\sin\frac{3}{2}\pi = 1$, *and*
$$\sin\frac{1}{4}\pi = \sin\frac{3}{4}\pi = -\sin\frac{5}{4}\pi = -\sin\frac{7}{4}\pi, \text{ etc.)}$$

$$\begin{aligned} b_n &= 2\int_0^{\frac{1}{4}} f(x)\sin n\pi x\,dx + 2\int_{\frac{1}{4}}^1 f(x)\sin n\pi x\,dx \\ &= 8\int_0^{\frac{1}{4}} x\sin n\pi x\,dx + \frac{8}{3}\int_{\frac{1}{4}}^1 (1-x)\sin n\pi x\,dx. \end{aligned}$$

Now
$$\int_0^{\frac{1}{4}} x\sin n\pi x\,dx = \int_0^{\frac{1}{4}} u\frac{dv}{dx}\,dx = [uv]_0^{\frac{1}{4}} - \int_0^{\frac{1}{4}} v\frac{du}{dx}\,dx,$$

where

$$u = x; \quad v = -\frac{1}{n\pi}\cos n\pi x;$$
$$\frac{du}{dx} = 1; \qquad v = \sin n\pi x.$$

It follows that

$$\begin{aligned}
\int_0^{\frac{1}{4}} x\sin n\pi x\,dx &= \left[-\frac{1}{n\pi}x\,\cos n\pi x\right]_0^{\frac{1}{4}} + \frac{1}{n\pi}\int_0^{\frac{1}{4}}\cos n\pi x\,dx \\
&= -\frac{1}{4n\pi}\,\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^{\frac{1}{4}} \\
&= -\frac{1}{4n\pi}\,\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}.
\end{aligned}$$

Similarly

$$\int_{\frac{1}{4}}^1 (1-x)\sin n\pi x\,dx. = \int_{\frac{1}{4}}^1 u\frac{dv}{dx}\,dx = [uv]_{\frac{1}{4}}^1 - \int_{\frac{1}{4}}^1 v\frac{du}{dx}\,dx,$$

where

$$u = 1-x; \quad v = -\frac{1}{n\pi}\cos n\pi x;$$
$$\frac{du}{dx} = -1; \qquad v = \sin n\pi x.$$

It follows that

$$\begin{aligned}
\int_{\frac{1}{4}}^1 (1-x)\sin n\pi x\,dx. &= \left[-\frac{1}{n\pi}(1-x)\,\cos n\pi x\right]_{\frac{1}{4}}^1 \\
&\quad - \frac{1}{n\pi}\int_{\frac{1}{4}}^1 \cos n\pi x\,dx \\
&= \frac{3}{4n\pi}\,\cos\frac{n\pi}{4} - \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_{\frac{1}{4}}^1 \\
&= \frac{3}{4n\pi}\,\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}.
\end{aligned}$$

(Here we have used the fact that $\sin n\pi = 0$ for all integers n.) It follows that

$$\begin{aligned}
b_n &= 8\left(-\frac{1}{4n\pi}\,\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right) \\
&\quad + \frac{8}{3}\left(\frac{3}{4n\pi}\,\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right) \\
&= \frac{32}{3n^2\pi^2}\sin\frac{n\pi}{4}
\end{aligned}$$

Thus

$$b_n = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{8} \\ \dfrac{32}{3\sqrt{2}n^2\pi^2} & \text{if } n \equiv 1 \pmod{8} \\ \dfrac{32}{3n^2\pi^2} & \text{if } n \equiv 2 \pmod{8} \\ \dfrac{32}{3\sqrt{2}n^2\pi^2} & \text{if } n \equiv 3 \pmod{8} \\ 0 & \text{if } n \equiv 4 \pmod{8} \\ -\dfrac{32}{3\sqrt{2}n^2\pi^2} & \text{if } n \equiv 5 \pmod{8} \\ -\dfrac{32}{3n^2\pi^2} & \text{if } n \equiv 6 \pmod{8} \\ -\dfrac{32}{3\sqrt{2}n^2\pi^2} & \text{if } n \equiv 7 \pmod{8} \end{cases}$$

2. *Find the lengths of the vectors* $(3, 6, 6)$ *and* $(4, 4, 7)$ *and also the cosine of the angle between them.*

Let $\mathbf{u} = (3, 3, 6)$ and $\mathbf{v} = (4, 4, 7)$. Then

$$|\mathbf{u}|^2 = 3^2 + 6^2 + 6^2 = 9 + 36 + 26 = 81$$

and

$$|\mathbf{v}|^2 = 4^2 + 4^2 + 7^2 = 16 + 16 + 49 = 81.$$

It follows that $|\mathbf{u}| = |\mathbf{v}| = \sqrt{81} = 9$. Thus the vectors $(3, 6, 6)$ and $(4, 4, 7)$ are of length 9.

Calculating the scalar product, we find that

$$\mathbf{u}.\mathbf{v} = (3, 6, 6).(4, 4, 7) = 3 \times 4 + 6 \times 4 + 6 \times 7 = 12 + 24 + 42 = 78.$$

But $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Therefore

$$\cos\theta = \frac{78}{81} = \frac{26}{27} \quad (= 0.9629629629629629\ldots).$$

3. *Calculate the components of a non-zero vector* (a, b, c) *in* $\mathbb{R}^3$ *that is orthogonal to the vectors* $(1, 1, 2)$ *and* $(2, 3, 2)$.

The vector product of $(1,1,2)$ and $(2,3,2)$ satisfies the requirement. Now

$$(1,1,2)\times(2,3,2) = (1\times2-2\times3, 2\times2-1\times2, 1\times3-1\times2) = (-4,2,1).$$

The vector $(-4,2,1)$ is orthogonal to both $(1,1,2)$ and $(2,3,2)$. (To check: the scalar product of $(-4,2,1)$ with each of $(1,1,2)$ and $(2,3,2)$ is zero.)

4. *Let the quaternions q and r be defined as follows:*

$$q = 1 - i - 2k, \quad r = j - 3k.$$

Calculate the quaternion products qr and rq. [Hamilton's basic formulae for quaternion multiplication state that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.]$$

$$\begin{aligned}
(1-i-2k)(j-3k) &= j - 3k - i(j-3k) - 2k(j-3k) \\
&= j - 3k - ij + 3ik - 2kj + 6k^2 \\
&= j - 3k - k - 3j + 2i - 6 \\
&= -6 + 2i - 2j - 4k, \\
(j-3k)(1-i-2k) &= j - 3k - (j-3k)i - 2(j-3k)k \\
&= j - 3k - ji + 3ki - 2jk + 6k^2 \\
&= j - 3k + k + 3j - 2i - 6 \\
&= -6 - 2i + 4j - 2k.
\end{aligned}$$