Course MA2C02: Hilary Term 2011.

Assignment III.

To be handed in by Wednesday 9th March, 2011.
Please include both name and student number on any work handed in.

1. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = \cos 3x.$$

The general solution y to this differential equation is of the form $y_P + y_C$, where y_P is a particular integral, and where y_C is a complementary function satisfying the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0.$$

Now the auxiliary polynomial $s^2 - 2s + 4$ associated with this differential equation has roots $1 \pm i\sqrt{3}$. It follows that

$$y_C = e^x \left(A \cos \sqrt{3}x + B \sin \sqrt{3}x \right).$$

We look for a particular integral y_P of the form $y_P = p\cos 3x + q \sin 3x$. Now

$$\begin{aligned} y_P &= p\cos 3x + q\sin 3x \\ y_P' &= 3q\cos 3x - 3p\sin 3x \\ y_P'' &= -9p\cos 3x - 9q\sin 3x \end{aligned}$$

and therefore

$$\begin{aligned} y_P'' - 2y_P' + 4y_P &= (-9p - 6q + 4p)\cos 3x + (-9q + 6p + 4q)\sin 3x \\ &= (-5p - 6q)\cos 3x + (-5q + 6p)\sin 3x. \end{aligned}$$

Therefore $-5p - 6q = 1$ and $-5q + 6p = 0$. It follows that $q = \frac{6}{5}p$ and $5p + \frac{36}{5}p = -1$, and therefore $61p = (25 + 36)p = -5$. Thus $p = -\frac{5}{61}$ and $q = -\frac{6}{61}$, and thus $y_P = -\frac{5}{61}\cos 3x - \frac{6}{61}\sin 3x$. We conclude that the general solution y to the differential equation is of the form

$$y = -\frac{5}{61}\cos 3x - \frac{6}{61}\sin 3x + e^x \left(A\cos\sqrt{3}x + B\sin\sqrt{3}x \right),$$

where A and B are arbitrary real constants.

2. *Any function y of a real variable x that solves the differential equation*

$$\frac{d^4y}{dx^4} - 16y = 0$$

may be represented by a power series of the form

$$y = \sum_{n=0}^{+\infty} \frac{y_n}{n!} x^n,$$

where the coefficients $y_0, y_1, y_2, y_3, \ldots$ of this power series are real numbers.

Find values of these coefficients y_n for $n = 0, 1, 2, 3, 4, \ldots$ that yield a solution to the above differential equation with $y_0 = 1$ and $y_1 = 0$ and $y_2 = -4$ and $y_3 = 0$. Hence or otherwise, find the solution to this differential equation.

On differentiating, we find that

$$y' = \sum_{n=0}^{+\infty} \frac{y_{n+1}}{n!} x^n, \quad y'' = \sum_{n=0}^{+\infty} \frac{y_{n+2}}{n!} x^n, \quad y''' = \sum_{n=0}^{+\infty} \frac{y_{n+3}}{n!} x^n,$$

$$y'''' = \sum_{n=0}^{+\infty} \frac{y_{n+4}}{n!} x^n.$$

Indeed

$$y^{(k)} = \sum_{n=0}^{+\infty} \frac{y_{n+k}}{n!} x^n$$

for all positive integers k. Thus if the function y satisfies the differential equation then

$$\sum_{n=0}^{+\infty} \frac{y_{n+4} - 16y_n}{n!} x^n$$

for all real numbers x, and therefore $y_{n+4} - 16y_n = 0$ for all non-negative integers n. Thus $y_{n+4} = 2^4 y_n$ for all non-negative integers n. Given that $y_1 = y_3 = 0$, it follows that $y_n = 0$ whenever n is odd. Given that $y_0 = 1$, it follows that $y_n = 2^n$ whenever n is a multiple of 4. Also, given that $y_2 = -4 = -2^2$, it follows that $y_n = -2^n$ whenever $n \equiv 2 \mod 4$, i.e., when $n = 2, 6, 10, 14, \ldots$. It follows that

$$\begin{aligned} y &= \sum_{m=0}^{+\infty} \frac{y_{2m}}{(2m)!} x^{2m} = \sum_{m=0}^{+\infty} \frac{(-1)^m 2^{2m}}{(2m)!} x^{2m} = \sum_{m=0}^{+\infty} \frac{(-1)^m (2x)^{2m}}{(2m)!} \\ &= \cos 2x. \end{aligned}$$

3. *Let* $(z_n : n \in \mathbb{Z})$ *be the doubly-infinite 3-periodic sequence with* $z_0 = 1$, $z_1 = 2$, $z_2 = -2$. *Find values of* c_0, c_1, c_2 *such that*

$$z_n = c_0 + c_1\omega^n + c_2\omega^{2n}$$

for all integers n, *where* $i = \sqrt{-1}$ *and*

$$\omega = e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{1}{2}(-1+\sqrt{3}\,i).$$

The standard formulae for the discrete Fourier transform ensure that

$$\begin{aligned}
c_0 &= \frac{1}{3}(z_0 + z_1 + z_2) = \frac{1}{3} \\
c_1 &= \frac{1}{3}(z_0 + \omega^{-1}z_1 + \omega^{-2}z_2) = \frac{1}{3}(z_0 + \omega^2 z_1 + \omega z_2) \\
&= \frac{1}{3}\left(z_0 - \frac{1}{2}z_1 - \frac{1}{2}z_2 - \frac{\sqrt{3}}{2}z_1 i + \frac{\sqrt{3}}{2}z_2 i\right) \\
&= \frac{1}{3} - \frac{2}{\sqrt{3}}i \\
c_2 &= \frac{1}{3}(z_0 + \omega^{-2}z_1 + \omega^{-4}z_2) = \frac{1}{3}(z_0 + \omega z_1 + \omega^2 z_2) \\
&= \frac{1}{3}\left(z_0 - \frac{1}{2}z_1 - \frac{1}{2}z_2 + \frac{\sqrt{3}}{2}z_1 i - \frac{\sqrt{3}}{2}z_2 i\right) \\
&= \frac{1}{3} + \frac{2}{\sqrt{3}}i
\end{aligned}$$