# Course MA2C02: Hilary Term 2010.

# Worked solutions for Assignment III.

1. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = x^2 e^{2x}.$$

**Solution.** The complementary function $y_C$ satisfies the differential equation $y_C'' - 2y_C' + 10y_C = 0$. The associated auxiliary polynomial $s^2 - 2s + 10$ has roots $1 + 3i$ and $1 - 3i$. It follows that

$$y_C = e^x(A\cos 3x + B\sin 3x).$$

We look for a particular integral of the form

$$y_P = (ax^2 + bx + c)e^{2x},$$

where $a$, $b$ and $c$ are fixed coefficients. (The derivative of a function of this form is another function of this form, but with different coefficients in place of $a$, $b$ and $c$.) On differentiating, we find that

$$\begin{aligned} y_P' &= (2ax^2 + (2a + 2b)x + b + 2c)e^{2x})e^{2x}, \\ y_P'' &= (4ax^2 + (8a + 4b)x + 2a + 4b + 4c)e^{2x})e^{2x}, \end{aligned}$$

and therefore

$$y_P'' - 2y_P' + 10y_P = (10ax^2 + (4a + 10b)x + 2a + 2b + 10c)e^{2x}.$$

Values of $a$, $b$ and $c$ need to be chosen such that this expression is equal to $x^2e^{2x}$. Therefore we must choose $1 = \frac{1}{10}$, $b = -\frac{4}{100}$, $-\frac{12}{1000}$. The general solution $y$ of the differential equation is given by $y = y_P + y_C$. Therefore

$$y = \left(\frac{1}{10}x^2 - \frac{4}{100}x - \frac{12}{1000}\right)e^{2x} + e^x(A\cos 3x + B\sin 3x).$$

2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function with period 6 whose values on the interval $[0, 6]$ are defined as follows:*

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 3; \\ 12 - 2x & \text{if } 3 \le x \le 6. \end{cases}$$

*Express the function $f$ as a Fourier series of the form*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{3}.$$

**Solution.** The function $f$ is an even function with period 6. It therefore follows from standard formulae that

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{2}{3}\int_{0}^{3} f(x)\,dx = \frac{2}{3}\left[x^2\right]_0^3 = 6,$$

and

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{\pi n x}{3}\,dx = \frac{2}{3}\int_{0}^{3} f(x)\cos\frac{\pi n x}{3}\,dx = \frac{4}{3}\int_{0}^{3} x\cos\frac{\pi n x}{3}\,dx$$

for $n > 0$. Thus $a_n = \displaystyle\int_0^3 u\frac{dv}{dx}\,dx$, where

$$u = \frac{4x}{3}, \quad \frac{dv}{dx} = \cos\frac{\pi n x}{3},$$

$$\frac{du}{dx} = \frac{4}{3}, \quad v = \frac{3}{\pi n}\sin\frac{\pi n x}{3}.$$

It follows from the formula for Integation by Parts that

$$\begin{aligned} a_n &= [uv]_0^3 - \int_0^3 v\frac{du}{dx}\,dx \\ &= \left[\frac{4x}{3}\sin\frac{\pi n x}{3}\right]_0^3 - \frac{4}{\pi n}\int_0^3 \sin\frac{\pi n x}{3}\,dx \\ &= \frac{12}{\pi^2 n^2}\left[\cos\frac{\pi n x}{3}\right]_0^3 \\ &= \frac{12}{\pi^2 n^2}(1 - (-1)^n) \\ &= \begin{cases} -\dfrac{24}{\pi^2 n^2} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

(We have used here the results that $\sin \pi n = 0$ and $\cos \pi n = (-1)^n$ for all integers $n$.) Thus

$$f(x) = 3 - \frac{24}{\pi^2}\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2}\cos\frac{\pi n x}{3}.$$

3. *Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 4-periodic sequence with $z_0 = 1$, $z_1 = 2$, $z_2 = 3i$ and $z_3 = -1 - i$. Find values of $c_0$, $c_1$, $c_2$ and $c_3$ such that*

$$z_n = c_0 + c_1 i^n + c_2(-1)^n + c_3(-i)^n$$

*for all integers $n$.*

**Solution.** It follows from standard formulae that

$$z_n = c_0 + c_1 i^n + c_2(-1)^n + c_3(-i)^n$$

for all integers $n$, where

$$\begin{aligned} c_k &= \frac{1}{4}\left(z_0 + z_1 i^{-k} + z_2 i^{-2k} + z_3 i^{-3k}\right) \\ &= \frac{1}{4}\left(1 + 2 \times (-i)^k + 3i \times (-1)^k + (-1 - i) \times i^k\right). \end{aligned}$$

Thus

$$c_0 = \tfrac{1}{2} + \tfrac{1}{2}i, \quad c_1 = \tfrac{1}{2} - \tfrac{3}{2}i, \quad c_2 = i, \quad c_3 = 0.$$