Module MA2C02: Hilary Term 2013.

Assignment III.

To be handed in by Wednesday 20th March, 2013.
Please include both name and student number on any work handed in.

1. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 7y = e^{2x}\cos 3x.$$

The general solution of this differential equation is of the form $y = y_P + y_C$, where y_P is a particular integral, and where y_C, the complementary function, satisfies the differential equation $y_C'' - 12y_C' + 7y_C = 0$. The auxiliary polynomial $s^2 - 12s + 7$ has roots $6 \pm \sqrt{29}$. Indeed the roots are given by the standard formula for the roots of a quadratic: $\frac{1}{2}(12 \pm \sqrt{144 - 4 \times 7})$, and thus expression evaluates to $6 \pm sqrt29$. Therefore

$$y_C = Ae^{(6+\sqrt{29})x} + Be^{(6-\sqrt{29})x}.$$

We look for a particular integral of the form

$$y_P = pe^{2x}\cos 3x + qe^{2x}\sin 3x.$$

Now

$$\begin{aligned} y_P' &= (2p + 3q)e^{2x}\cos 3x + (2q - 3p)e^{2x}\sin 3x, \\ y_P'' &= (-5p + 12q)e^{2x}\cos 3x + (-5q - 12p)e^{2x}\sin 3x, \end{aligned}$$

But then

$$y_P'' - 12y_P' + 7y_P = (-22p - 24q)e^{2x}\cos 3x + (-22q + 24p)\sin 3x.$$

Therefore $-22p - 24q = 1$ and $-22q + 24p = 0$. It follows that $q = \frac{12}{11}p$ and $-22p - \frac{288}{11}p = 1$. Therefore $p = -\frac{11}{530}$ and $q = -\frac{12}{530}$. Thus the general solution to the differential equation is

$$y = -\frac{11}{530}e^{2x}\cos 3x - \frac{12}{530}e^{2x}\sin 3x + Ae^{(6+\sqrt{29})x} + Be^{(6-\sqrt{29})x}.$$

2. *Express* $\sin 4\theta$ *and* $\cos 5\theta$ *by formulae involving* $\sin\theta$ *and* $\cos\theta$ *and their powers.*

Using basic properties of the exponential and trigonometric functions, considered as functions of a complex variable:

$$
\begin{aligned}
\cos 4\theta + i \sin 4\theta &= e^{4i\theta} = (e^{i\theta})^4 \\
&= (\cos\theta + i\sin\theta)^4 \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta \\
&\qquad + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta \\
&\qquad - 4i\cos\theta\sin^3\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
&\qquad + 4i(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)
\end{aligned}
$$

It follows that

$$\sin 4\theta = 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta).$$

Also

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Similarly

$$
\begin{aligned}
\cos 5\theta + i\sin 5\theta &= e^{5i\theta} = (e^{i\theta})^5 \\
&= (\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta \\
&\qquad + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\qquad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta,
\end{aligned}
$$

and therefore

$$\cos 5\theta == \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

3. (a) *What is the cosine of the angle between the vectors* $(1, 1, 2)$ *and* $(1, -1, -2)$*?*

Let $\mathbf{v} = (1, 1, 2)$ and $\mathbf{w} = (1, -1, -2)$. Then $\mathbf{v}.\mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos\theta$, where θ is the angle between these vectors. Now

$$|\mathbf{v}|^2 = |\mathbf{w}|^2 = 1^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6,$$

and

$$\mathbf{v}.\mathbf{w} = 1 \times 1 + 1 \times (-1) + 2 \times (-2) = -4.$$

It follows that $6 \cos\theta = \mathbf{v}.\mathbf{w} = -4$, and therefore $\cos\theta = -\frac{2}{3}$.

(b) *Find the components of a non-zero vector orthogonal to the vectors* $(2, 3, 4)$ *and* $(3, 4, 5)$.

One such vector is the vector product of these two vectors. Now

$$\begin{aligned}(2, 3, 4) \times (3, 4, 5) &= (3 \times 5 - 4 \times 4, 4 \times 3 - 2 \times 5, 2 \times 4 - 3 \times 3) \\ &= (15 - 16, 12 - 10, 8 - 9) \\ &= (-1, 2, -1).\end{aligned}$$

Thus $(-1, 2, -1)$ has the required property. To verify, note that the scalar product of $(-1, 2, -1)$ with the two given vectors is zero.

(c) *Find the equation of the plane in* $\mathbb{R}^3$ *passing through the points* $(3, 0, 7)$*,* $(5, 1, 6)$ *and* $(6, 3, 8)$.

Now $(5, 1, 6) - (3, 0, 7) = (2, 1, -1)$ and $(6, 3, 8) - (3, 0, 7) = (3, 3, 1)$ and

$$\begin{aligned}(2, 1, -1) \times (3, 3, 1) &= (1 \times 1 - (-1) \times 3, (-1) \times 3 - 2 \times 1, 2 \times 3 - 1 \times 3) \\ &= (1 - (-3), -3 - 2, 6 - 3) \\ &= (4, -5, 3).\end{aligned}$$

Thus $(4, -5, 3)$ is orthogonal (or perpendicular) to the vectors $(2, 1, -1)$ and $(3, 3, 1)$, and is thus perpendicular to the plane. The plane is thus specified by an equation of the form $4x - 5y + 3z = k$ for some constant k. Substituting in the coordinates of one of the points, we find that

$$k = 4 \times 3 - 5 \times 0 + 3 \times 7 = 12 + 21 = 33.$$

Thus the equation of the plane is

$$4x - 5y + 3z = 33.$$

This can be verified by checking that all coordinates of all three given points satisfy this equation.