Course MA2C02: Hilary Term 2012.

Assignment I.

To be handed in by Wednesday 11th April, 2012.
Please include both name and student number on any work handed in.
Assignments handed in after Wednesday 4th April 2012 should be returned to the School of Mathematics Office, 17/18 Westland Row.

1. *Find the cosine of the angle between the vectors* $(1, 2, 4)$ *and* $(2, 2, 6)$.

   Let $\mathbf{v} = (1, 2, 4)$ and $\mathbf{w} = (2, 2, 6)$. Then

$$\mathbf{v}.\mathbf{w} = 1 \times 2 + 2 \times 2 + 4 \times 6 = 30,$$

$$|\mathbf{v}|^2 = 1^2 + 2^2 + 4^2 = 21, \quad |\mathbf{w}|^2 = 2^2 + 2^2 + 6^2 = 44.$$

   Thus the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is given by the formula

$$\cos\theta = \frac{\mathbf{v}.\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{30}{\sqrt{924}} = \frac{15}{\sqrt{231}} \approx 0.9869.$$

2. *Find a non-zero vector that is orthogonal (i.e., perpendicular) to the vectors* $(1, 2, 6)$ *and* $(2, 3, 9)$.

$$\begin{aligned}(1, 2, 6) \times (2, 3, 9) &= (2 \times 9 - 6 \times 3,\ 6 \times 2 - 1 \times 9,\ 1 \times 3 - 2 \times 2) \\ &= (0, 3, -1).\end{aligned}$$

   The vector $(0, 3, -1)$ is orthogonal to $(1, 2, 6)$ and $(2, 3, 9)$.

3. *Find the equation of the plane that passes through the points* $(1, 1, 2)$, $(2, 3, 5)$ *and* $(4, -3, 7)$.

   Now

$$(2, 3, 5) - (1, 1, 2) = (1, 2, 3), \quad (4, -3, 7) - (1, 1, 2) = (3, -4, 5)$$

   and

$$\begin{aligned}(1, 2, 3) \times (3, -4, 5) &= (2 \times 5 - 3 \times (-4),\ 3 \times 3 - 1 \times 5, \\ &\qquad 1 \times (-4) - 2 \times 3) \\ &= (22, 4, -10).\end{aligned}$$

The plane is parallel to the displacement vectors $(1, 2, 3)$ and $(3, -4, 5)$, and is therefore orthogonal to the vector $(22, 4, -10)$. The equation of the plane is thus of the form

$$22x + 4y - 10z = k$$

for some constant k. Setting $x = 1$, $y = 1$ and $z = 2$, we find that $k = 22 + 4 - 20 = 6$. Thus the equation of the plane passing through the points $(1, 1, 2)$, $(2, 3, 5)$ and $(4, -3, 7)$ is

$$22x + 4y - 10z = 6.$$

(This may be checked by verifying that the coordinates of the three given points satisfy the above equation.)

4. *Calculate the quaternion products* qr *and* rq *where* $q = 2 + 3j + k$ *and* $r = 3 + i + 4k$.

$$\begin{aligned}
qr &= (2 + 3j + k)(3 + i + 4k) \\
&= 2(3 + i + 4k) + 3j(3 + i + 4k) + k(3 + i + 4k) \\
&= 6 + 2i + 8k + 9j + 3ji + 12jk + 3k + ki + 4k^2 \\
&= 6 + 2i + 8k + 9j - 3k + 12i + 3k + j - 4 \\
&= 2 + 14i + 10j + 8k, \\
rq &= (3 + i + 4k)(2 + 3j + k) \\
&= (3 + i + 4k)2 + (3 + i + 4k)3j + (3 + i + 4k)k \\
&= 6 + 2i + 8k + 9j + 3ij + 12kj + 3k + ik + 4k^2 \\
&= 6 + 2i + 8k + 9j + 3k - 12i + 3k - j - 4 \\
&= 2 - 10i + 8j + 14k.
\end{aligned}$$

5. *Find an integer* x *which satisfies* $x \equiv 7$ *(mod.* 13*),* $x \equiv 3$ *(mod.* 5*) and* $x \equiv 2$ *(mod.* 3*).*

$$5 \times 3 = 15, \quad 13 \times 3 = 39, \quad 13 \times 5 = 65.$$

$$15 \equiv 2 \text{ (mod. 13)}, 15 \equiv 0 \text{ (mod. 5)}, 15 \equiv 0 \text{ (mod. 3)},$$

$$39 \equiv 0 \text{ (mod. 13)}, 39 \equiv 4 \text{ (mod. 5)}, 39 \equiv 0 \text{ (mod. 3)},$$

$$65 \equiv 0 \text{ (mod. 13)}, 65 \equiv 0 \text{ (mod. 5)}, 65 \equiv 2 \text{ (mod. 3)}.$$

Now

$$7 \times 2 \equiv 1 \text{ (mod. 13)}, \text{ and therefore } 105 = 7 \times 15 \equiv 1 \text{ (mod. 13)},$$

$$4 \times 4 \equiv 1 \text{ (mod. 5)}, \text{ and therefore } 156 = 4 \times 39 \equiv 1 \text{ (mod. 5)},$$

$$2 \times 2 \equiv 1 \text{ (mod. 3)}, \text{ and therefore } 130 = 2 \times 65 \equiv 1 \text{ (mod. 3)}.$$

Thus

$$105 \equiv 1 \text{ (mod. 13)}, 105 \equiv 0 \text{ (mod. 5)}, 105 \equiv 0 \text{ (mod. 3)},$$

$$156 \equiv 0 \text{ (mod. 13)}, 156 \equiv 1 \text{ (mod. 5)}, 156 \equiv 0 \text{ (mod. 3)},$$

$$130 \equiv 0 \text{ (mod. 13)}, 130 \equiv 0 \text{ (mod. 5)}, 130 \equiv 1 \text{ (mod. 3)}.$$

Now

$$7 \times 105 + 3 \times 156 + 2 \times 130 = 735 + 468 + 260 = 1463.$$

It follows that

$$1463 \equiv 7 \text{ (mod. 13)}, 1463 \equiv 3 \text{ (mod. 5)}, 1463 \equiv 2 \text{ (mod. 3)}.$$

(Note that $13 \times 5 \times 3 = 195$. Thus if x satisfies the required congruences if and only if $x \equiv 1463$ (mod. 195). The following are possible positive values of x:

$$98, \;\; 293, \;\; 488, \;\; 683, \;\; 878, \;\; 1073, \;\; 1268, \;\; 1463, \ldots)$$

6. *For each positive integer n, determine the value of the unique integer x_n satisfying $0 \leq x_n < 29$ for which $384^n \equiv x_n$ (mod. 29).*

   (N.B., there will exist integers r and m such that $384^j \equiv 384^k$ (mod. 29) whenever $j \geq r$, $k \geq r$ and $j \equiv k$ (mod. m). This fact should enable you to devise a specification that yields the value of 384^n (mod. 29) for all positive integers n, no matter how large.)

   The following is a transcription of an interactive Python session:

```
>>> pmod = 1
>>> for n in range(1,20):
...     pmod = (pmod * 384) % 29
...     print "if n = ", n, "then 384 ** n % 29 = ", pmod
...
if n =  1 then 384 ** n % 29 =  7
```

```
if n =  2 then 384 ** n % 29 =  20
if n =  3 then 384 ** n % 29 =  24
if n =  4 then 384 ** n % 29 =  23
if n =  5 then 384 ** n % 29 =  16
if n =  6 then 384 ** n % 29 =  25
if n =  7 then 384 ** n % 29 =  1
if n =  8 then 384 ** n % 29 =  7
if n =  9 then 384 ** n % 29 =  20
if n =  10 then 384 ** n % 29 =  24
if n =  11 then 384 ** n % 29 =  23
if n =  12 then 384 ** n % 29 =  16
if n =  13 then 384 ** n % 29 =  25
if n =  14 then 384 ** n % 29 =  1
if n =  15 then 384 ** n % 29 =  7
if n =  16 then 384 ** n % 29 =  20
if n =  17 then 384 ** n % 29 =  24
if n =  18 then 384 ** n % 29 =  23
if n =  19 then 384 ** n % 29 =  16
>>>
```

Looking at the output, we see that, for $n \geq 0$, 384^n (mod. 29) is periodic, with period 7. Therefore

$$384^n \text{ (mod. 29)} = \begin{cases} 1 & \text{if } n \equiv 0 \text{ (mod. 7)}; \\ 7 & \text{if } n \equiv 1 \text{ (mod. 7)}; \\ 20 & \text{if } n \equiv 2 \text{ (mod. 7)}; \\ 24 & \text{if } n \equiv 3 \text{ (mod. 7)}; \\ 23 & \text{if } n \equiv 4 \text{ (mod. 7)}; \\ 16 & \text{if } n \equiv 5 \text{ (mod. 7)}; \\ 25 & \text{if } n \equiv 6 \text{ (mod. 7)}. \end{cases}$$