# Course MA2C02, Hilary Term 2012
# Section 9: Vectors and Quaternions

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## Contents

# 9 Vectors and Quaternions

## 9.1 Vectors

*Vector quantities* are objects that have attributes of magnitude and direction. Many physical quantities, such as velocity, acceleration, force, electric field and magnetic field are examples of vector quantities. Displacements between points of space may also be represented using vectors.

Quantities that do not have a sense of direction associated with them are known as *scalar quantities*. Such physical quantities as temperature and energy are scalar quantities. Scalar quantities are usually represented by real numbers.

## 9.2 Displacement Vectors

Points of three-dimensional space may be represented, in a Cartesian coordinate system, by ordered triples $(x, y, z)$ of real numbers. Two ordered triples $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ of real numbers represent the same point of three-dimensional space if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$. The point whose Cartesian coordinates are given by the ordered triple $(0, 0, 0)$ is referred to as the *origin* of the Cartesian coordinate system.

It is usual to employ a Coordinate system such that the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are situated at a unit distance from the origin $(0, 0, 0)$, and so that the three lines that join the origin to these points are mutually perpendicular. Moreover it is customary to require that if the thumb of your right hand points in the direction from the origin to the point $(1, 0, 0)$, and if the first finger of that hand points in the direction from the origin to the point $(0, 1, 0)$, and if the second finger of that hand points in a direction perpendicular to the directions of the thumb and first finger, then that second finger points in the direction from the origin to the point $(0, 0, 1)$. (Thus if, at a point on the surface of the earth, away from the north and south pole, the point $(1, 0, 0)$ is located to the east of the origin, and the point $(0, 1, 0)$ is located to the north of the origin, then the point $(0, 0, 1)$ will be located above the origin.

Let $P_1$, $P_2$, $P_3$ and $P_4$ denote four points of three-dimensional space, represented in a Cartesian coordinate system by ordered triples as follows:

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2), \quad P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

The *displacement vector* $\overrightarrow{P_1, P_2}$ from the point $P_1$ to the point $P_2$ measures the distance and the direction in which one would have to travel in order to get from $P_1$ to $P_2$. This displacement vector may be represented by an ordered triple as follows:

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

The displacement vector $\overrightarrow{P_3P_4}$ is *equal* to the displacement vector $\overrightarrow{P_1P_2}$ if and only if

$$x_2 - x_1 = x_4 - x_3, \quad y_2 - y_1 = y_4 - y_3, \quad z_2 - z_1 = z_4 - z_3,$$

in which case we represent the fact that these two displacement vectors are equal by writing

$$\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}.$$

Geometrically, these two displacement vectors are equal if and only if $P_1$, $P_2$, $P_4$ and $P_3$ are the vertices of a parallelogram in three-dimensional space, in which case

$$x_3 - x_1 = x_4 - x_2, \quad y_3 - y_1 = y_4 - y_2, \quad z_3 - z_1 = z_4 - z_2,$$

and thus

$$\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$

These displacement vectors may be regarded as objects in their own right, and denoted by symbols of their own: we use a symbol such as $\vec{u}$ to denote the displacement vector $\overrightarrow{P_1P_2}$ from the point $P_1$ to the point $P_2$, and we write $\vec{u} = (u_x, u_y, u_z)$ where $u_x = x_2 - x_1$, $u_y = y_2 - y_1$ and $u_z = z_2 - z_1$.

## 9.3 The Parallelogram Law of Vector Addition

Let $P_1$, $P_2$, $P_3$ and $P_4$ denote four points of three-dimensional space, located such that $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$. Then (as we have seen) $\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}$ and the geometrical figure $P_1P_2P_4P_3$ is a parallelogram. Let

$$\vec{u} = \overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}, \quad \vec{v} = \overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$

Let

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2), \quad P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

Then $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$, where

$$u_x = x_2 - x_1 = x_4 - x_3, \quad u_y = y_2 - y_1 = y_4 - y_3, \quad u_z = z_2 - z_1 = z_4 - z_3,$$

$$v_x = x_3 - x_1 = x_4 - x_2, \quad v_y = y_3 - y_1 = y_4 - y_2, \quad v_z = z_3 - z_1 = z_4 - z_2,$$

Let $\vec{e} = \overrightarrow{P_1P_4}$. Then $\vec{e} = (e_x, e_y, e_z)$, where

$$e_x = x_4 - x_1 = u_x + v_x, \quad e_y = y_4 - y_1 = u_y + v_y, \quad e_z = z_4 - z_1 = u_z + v_z,$$

We say that the vector $\vec{e}$ is the *sum* of the vectors $\vec{u}$ and $\vec{v}$, and denote this fact by writing

$$\vec{e} = \vec{u} + \vec{v}.$$

This rule for addition of vectors is known as the *parallelogram rule*, due to its association with the geometry of parallelograms. Note that vectors are added, by adding together the corresponding components of the two vectors. For example,

$$(0, 3, 2) + (4, 8, -5) = (4, 11, -3).$$

Note that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

for all points $A$, $B$ and $C$ of space. Also

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

and

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in three-dimensional space. Thus addition of vectors satisfies the Commutative Law and the Associative Law.

The *zero vector* $\vec{0}$ is the vector $(0, 0, 0)$ that represents the displacement from any point in space to itself. The zero vector $\vec{0}$ has the property that

$$\vec{u} + \vec{0} = \vec{u}$$

for all vectors $\vec{u}$. Moroever, given any vector $\vec{u}$, there exists a vector, denoted by $-\vec{u}$, characterized by the property that

$$\vec{u} + (-\vec{u}) = \vec{0}.$$

If $\vec{u} = (u_x, u_y, u_z)$, then $-\vec{u} = (-u_x, -u_y, -u_z)$.

## 9.4 The Length of Vectors

Let $P_1$ and $P_2$ be points in space, and let $\vec{u}$ denote the displacement vector $\overrightarrow{P_1P_2}$ from the point $P_1$ to the point $P_2$. If $P_1 = (x_1, y_1, z_1)$ and $P_2 =$

$(x_2, y_2, z_2)$ then $\vec{u} = (u_x, u_y, u_z)$ where $u_x = x_2 - x_1$, $u_y = y_2 - y_1$ and $u_z = z_2 - z_1$.

The *length* (or *magnitude*) of the vector $\vec{u}$ is defined to be the distance from the point $P_1$ to the point $P_2$. This distance may be calculated using Pythagoras's Theorem. Let $Q = (x_2, y_2, z_1)$ and $R = (x_2, y_1, z_1)$. If the points $P_1$ and $P_2$ are distinct, and if $z_1 \neq z_2$, then the triangle $P_1QP_2$ is a right-angled triangle with hypotenuse $P_1P_2$, and it follows from Pythagoras's Theorem that

$$P_1P_2^2 = P_1Q^2 + QP_2^2 = P_1Q^2 + (z_2 - z_1)^2.$$

This identity also holds when $P_1 = P_2$, and when $z_1 = z_2$, and therefore holds wherever the points $P_1$ and $P_2$ are located. Similarly

$$P_1Q^2 = P_1R^2 + RQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(since $P_1RQ$ is a right-angled triangle with hypotenuse $P_1Q$ whenever the points $P_1$, $R$ and $Q$ are distinct), and therefore the length $|\vec{u}|$ of the displacement vector $\vec{u}$ from the point $P_1$ to the point $P_2$ satisfies the equation

$$|\vec{u}|^2 = P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = u_x^2 + u_y^2 + u_z^2.$$

In general we define the *length*, or *magnitude*, $|\vec{v}|$ of any vector quantity $\vec{v}$ by the formula

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

where $\vec{v} = (v_x, v_y, v_z)$. This ensures that the length of any displacement vector is equal to the distance between the two points that determine the displacement.

**Example** The vector $(3, 4, 12)$ is of length 13, since

$$3^2 + 4^2 + 12^2 = 5^2 + 12^2 = 13^2.$$

A vector whose length is equal to one is said to be a *unit vector*.

## 9.5 Scalar Multiples of Vectors

Let $\vec{v}$ be a vector, represented by the ordered triple $(v_x, v_y, v_z)$, and let $t$ be a real number. We define $t\vec{v}$ to be the vector represented by the ordered triple $(tv_x, tv_y, tv_z)$. Thus $t\vec{v}$ is the vector obtained on multiplying each of the components of $\vec{v}$ by the real number $t$.

Note that if $t > 0$ then $t\vec{v}$ is a vector, pointing in the same direction as $\vec{v}$, whose length is obtained on multiplying the length of $\vec{v}$ by the positive real number $t$.

Similarly if $t < 0$ then $t\vec{v}$ is a vector, pointing in the opposite direction to $\vec{v}$, whose length is obtained on multiplying the length of $\vec{v}$ by the positive real number $|t|$.

Note that

$$(s+t)\vec{u} = s\vec{u} + t\vec{u}, \quad t(\vec{u}+\vec{v}) = t\vec{u} + t\vec{v}, \text{ and } s(t\vec{u}) = (st)\vec{u},$$

for all vectors $\vec{u}$ and $\vec{v}$ and real numbers $s$ and $t$.

## 9.6 Linear Combinations of Vectors

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in three-dimensional space. A vector $\vec{v}$ is said to be a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there exist real numbers $t_1, t_2, \ldots, t_k$ such that

$$\vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k.$$

Let $O$, $P_1$ and $P_2$ be distinct points of three-dimensional space that are not colinear (i.e., that do not all lie on any one line in that space). The displacement vector $\overrightarrow{OP}$ of a point $P$ in three-dimensional space is a linear combination of the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ if and only if the point $P$ lies in the unique plane that contains the points $O$, $P_1$ and $P_2$.

## 9.7 Linear Dependence and Independence

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are said to be be *linearly dependent* if there exist real numbers $t_1, t_2, \ldots, t_k$, not all zero, such that

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}.$$

If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are not linearly dependent, then they are said to be *linearly independent*.

Note that if any of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is the zero vector, then those vectors are linearly dependent. Indeed if $\vec{v}_i = 0$ then these vectors satisfy a relation of the form

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{0}.$$

where $t_j = 0$ if $j \neq i$ and $t_i \neq 0$. We conclude that, in any list of linearly independent vectors, the vectors are all non-zero.

Also if any two of the vectors in the list $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are colinear, then these vectors are linearly dependent. For example, if $\vec{v}_1$ and $\vec{v}_2$ are colinear, then they satisfy a relation of the form $t_1\vec{v}_1 + t_2\vec{v}_2 = \vec{0}$, where $t_1$ and $t_2$ are not both zero. If we then set $t_i = 0$ when $i > 2$, then $\sum_{i=1}^{k} t_i\vec{v}_i = \vec{0}$.

If a vector $\vec{v}$ is expressible as a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_k$ then the vectors $\vec{v}_1, \ldots, \vec{v}_k, \vec{v}$ are linearly dependent. For there exist real numbers $s_1, \ldots, s_k$ such that

$$\vec{v} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k,$$

and then

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k - \vec{v} = 0.$$

**Theorem 9.1** *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in three-dimensional space which are linearly independent. Then, given any vector $\vec{s}$, there exist unique real numbers $p$, $q$ and $r$ such that*

$$\vec{s} = p\vec{u} + q\vec{v} + r\vec{w}.$$

**Proof** First we note that the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all non-zero, and no two of these vectors are colinear. Let $O$ denote the origin of a Cartesian coordinate system, and let $A$, $B$, $C$ and $P$ denote the points of three-dimensional space whose displacement vectors from the origin $O$ are $\vec{u}$, $\vec{v}$, $\vec{w}$ and $\vec{s}$ respectively. The points $O$, $A$, $B$ and $C$ are then all distinct, and there is a unique plane which contains the three points $O$, $A$ and $B$. This plane $OAB$ consists of all points whose displacement vector from the origin is expressible in the form $p\vec{u} + q\vec{w}$ for some real numbers $p$ and $q$. Now the vector $\vec{w}$ is not expressible as a linear combination of $\vec{u}$ and $\vec{v}$, and therefore the point $C$ does not belong to the plane $OAB$. Therefore the line parallel to $OC$ that passes through the point $P$ is not parallel to the plane $OAP$. This line therefore intersects the plane in a single point $Q$. Now the displacement vector of the point $Q$ from the origin is of the form $\vec{s} - r\vec{w}$ for some uniquely-determined real number $r$. But it is also expressible in the form $p\vec{u} + q\vec{v}$ for some uniquely-determined real numbers $p$ and $q$, because $Q$ lies in the plane $OAB$. Thus there exist real numbers $p$, $q$ and $r$ such that $\vec{s} - r\vec{w} = p\vec{u} + q\vec{v}$. But then

$$\vec{s} = p\vec{u} + q\vec{v} + r\vec{w}.$$

Moreover the point $Q$ and thus the real numbers $p$, $q$ and $r$ are uniquely determined by $\vec{s}$, as required. ∎

It follows from this theorem that no linearly independent list of vectors in three-dimensional space can contain more than three vectors, since were there a fourth vector in the list, then it would be expressible as a linear combination of the other three, and the vectors would not then be linearly independent.

## 9.8 The Scalar Product

Let $\vec{u}$ and $\vec{v}$ be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples $(u_1, u_2, u_3)$ and $(v_1, v_2, v_3)$ respectively. The *scalar product* of the vectors $\vec{u}$ and $\vec{v}$ is defined to be the real number $\vec{u} \,.\, \vec{v}$ defined by the formula

$$\vec{u} \,.\, \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

In particular,

$$\vec{u} \,.\, \vec{u} = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2,$$

for any vector $\vec{u}$, where $|\vec{u}|$ denotes the length of the vector $\vec{u}$.

Note that $\vec{u} \,.\, \vec{v} = \vec{v} \,.\, \vec{u}$ for all vectors $\vec{u}$ and $\vec{v}$. Also

$$(s\vec{u} + t\vec{v}) \,.\, \vec{w} = s\vec{u} \,.\, \vec{w} + t\vec{v} \,.\, \vec{w}, \quad \vec{u} \,.\, (s\vec{v} + t\vec{w}) = s\vec{u} \,.\, \vec{v} + t\vec{u} \,.\, \vec{w}$$

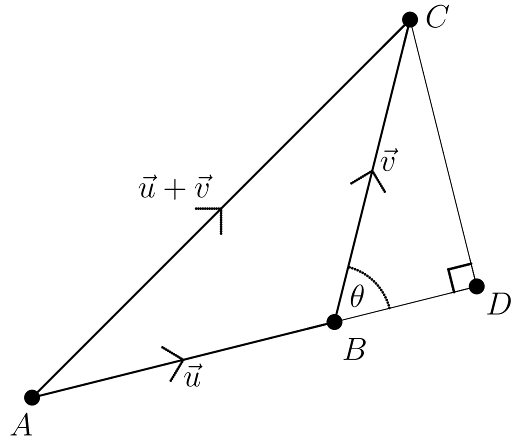
for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ and real numbers $s$ and $t$.

**Proposition 9.2** *Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in three-dimensional space. Then their scalar product $\vec{u} \,.\, \vec{v}$ is given by the formula*

$$\vec{u} \,.\, \vec{v} = |\vec{u}|\, |\vec{v}| \cos\theta,$$

*where $\theta$ denotes the angle between the vectors $\vec{u}$ and $\vec{v}$.*

**Proof** Suppose first that the angle $\theta$ between the vectors $\vec{u}$ and $\vec{v}$ is an acute angle, so that $0 < \theta < \frac{1}{2}\pi$. Let us consider a triangle $ABC$, where $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{BC} = \vec{v}$, and thus $\overrightarrow{AC} = \vec{u} + \vec{v}$. Let $ADC$ be the right-angled triangle constructed as depicted in the figure below, so that the line $AD$ extends $AB$ and the angle at $D$ is a right angle.




Then the lengths of the line segments $AB$, $BC$, $AC$, $BD$ and $CD$ may be expressed in terms of the lengths $|\vec{u}|$, $|\vec{v}|$ and $|\vec{u}+\vec{v}|$ of the displacement vectors $\vec{u}$, $\vec{v}$ and $\vec{u}+\vec{v}$ and the angle $\theta$ between the vectors $\vec{u}$ and $\vec{v}$ by means of the following equations:

$$AB = |\vec{u}|, \quad BC = |\vec{v}|, \quad AC = |\vec{u}+\vec{v}|,$$

$$BD = |\vec{v}|\,\cos\theta \quad \text{and} \quad DC = |\vec{v}|\,\sin\theta.$$

Then

$$AD = AB + BD = |\vec{u}| + |\vec{v}|\cos\theta.$$

The triangle $ADC$ is a right-angled triangle with hypotenuse $AC$. It follows from Pythagoras' Theorem that

$$\begin{aligned} |\vec{u}+\vec{v}|^2 &= AC^2 = AD^2 + DC^2 = (|\vec{u}| + |\vec{v}|\,\cos\theta)^2 + |\vec{v}|\,\sin^2\theta \\ &= |\vec{u}|^2 + 2|\vec{u}|\,|\vec{v}|\,\cos\theta + |\vec{v}|\,\cos^2\theta + |\vec{v}|\,\sin^2\theta \\ &= |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}|\,|\vec{v}|\,\cos\theta, \end{aligned}$$

because $\cos^2\theta + \sin^2\theta$.

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then

$$\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2, u_3+v_3),$$

and therefore

$$\begin{aligned} |\vec{u}+\vec{v}|^2 &= (u_1+v_1)^2 + (u_2+v_2)^2 + (u_3+v_3)^2 \\ &= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 + 2u_2v_2 + v_2^2 + u_3^2 + 2u_3v_3 + v_3^2 \\ &= |\vec{u}|^2 + |\vec{v}^2| + 2(u_1v_1 + u_2v_2 + u_3v_3) \\ &= |\vec{u}|^2 + |\vec{v}^2| + 2\vec{u}.\vec{v}. \end{aligned}$$

On comparing the expressions for $|\vec{u}+\vec{v}|^2$ given by the above equations, we see that $\vec{u}.\vec{v} = |\vec{u}|\,|\vec{v}|\,\cos\theta$ when $0 < \theta < \frac{1}{2}\pi$.

The identity $\vec{u}.\vec{v} = |\vec{u}|\,|\vec{v}|\,\cos\theta$ clearly holds when $\theta = 0$ and $\theta = \pi$. Pythagoras' Theorem ensures that it also holds when the angle $\theta$ is a right angle (so that $\theta = \frac{1}{2}\pi$. Suppose that $\frac{1}{2}\pi < \theta < \pi$, so that the angle $\theta$ is obtuse. Then the angle between the vectors $\vec{u}$ and $-\vec{v}$ is acute, and is equal to $\pi - \theta$. Moreover $\cos(\pi-\theta) = -\cos\theta$ for all angles $\theta$. It follows that

$$\vec{u}.\vec{v} = -\vec{u}.(-\vec{v}) = -|\vec{u}|\,|\vec{v}|\,\cos(\pi-\theta) = |\vec{u}|\,|\vec{v}|\,\cos\theta$$

when $\frac{1}{2}\pi < \theta < \pi$. We have therefore verified that the identity $\vec{u}.\vec{v} = |\vec{u}|\,|\vec{v}|\,\cos\theta$ holds for all non-zero vectors $\vec{u}$ and $\vec{v}$, as required. ∎

**Corollary 9.3** *Two non-zero vectors $\vec{u}$ and $\vec{v}$ in three-dimensional space are perpendicular if and only if $\vec{u}\,.\,\vec{v} = 0$.*

**Proof** It follows directly from Proposition 9.2 that $\vec{u}\,.\,\vec{v} = 0$ if and only if $\cos\theta = 0$, where $\theta$ denotes the angle between the vectors $\vec{u}$ and $\vec{v}$. This is the case if and only if the vectors $\vec{u}$ and $\vec{v}$ are perpendicular.

**Example** We can use the scalar product to calculate the angle $\theta$ between the vectors $(2,2,0)$ and $(0,3,3)$ in three-dimensional space. Let $\vec{u} = (2,2,0)$ and $\vec{v} = (3,3,0)$. Then $|\vec{u}|^2 = 2^2 + 2^2 = 8$ and $|\vec{v}|^2 = 3^2 + 3^2 = 18$. It follows that $(|\vec{u}|\,|\vec{v}|)^2 = 8 \times 18 = 144$, and thus $|\vec{u}|\,|\vec{v}| = 12$. Now $\vec{u}\,.\,\vec{v} = 6$. It follows that

$$6 = |\vec{u}|\,|\vec{v}|\,\cos\theta = 12\cos\theta.$$

Therefore $\cos\theta = \frac{1}{2}$, and thus $\theta = \frac{1}{3}\pi$.

We can use the scalar product to find the distance between points on a sphere. Now the Cartesian coordinates of a point $P$ on the unit sphere about the origin $O$ in three-dimensional space may be expressed in terms of angles $\theta$ and $\varphi$ as follows:

$$P = (\sin\theta\,\cos\varphi,\, \sin\theta\,\sin\varphi,\, \cos\theta).$$

The angle $\theta$ is that between the displacement vector $\overrightarrow{OP}$ and the vectical vector $(0,0,1)$. Thus the angle $\frac{1}{2}\pi - \theta$ represents the 'latitude' of the point $P$, when we regard the point $(0,0,1)$ as the 'north pole' of the sphere. The angle $\varphi$ measures the 'longitude' of the point $P$.

Now let $P_1$ and $P_2$ be points on the unit sphere, where

$$\begin{aligned} P_1 &= (\sin\theta_1\,\cos\varphi_1,\, \sin\theta_1\,\sin\varphi_1,\, \cos\theta_1), \\ P_2 &= (\sin\theta_2\,\cos\varphi_2,\, \sin\theta_2\,\sin\varphi_2,\, \cos\theta_2). \end{aligned}$$

We wish to find the angle $\psi$ between the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ of the points $P_1$ and $P_2$ from the origin. Now $|\overrightarrow{OP_1}| = 1$ and $|\overrightarrow{OP_2}| = 1$. On applying Proposition 9.2, we see that

$$\begin{aligned}
\cos\psi &= \overrightarrow{OP_1} \,.\, \overrightarrow{OP_2} \\
&= \sin\theta_1\,\sin\theta_2\,\cos\varphi_1\,\cos\varphi_2 + \sin\theta_1\,\sin\theta_2\,\sin\varphi_1\,\sin\varphi_2 \\
&\qquad + \cos\theta_1\,\cos\theta_2 \\
&= \sin\theta_1\,\sin\theta_2\,(\cos\varphi_1\,\cos\varphi_2 + \sin\varphi_1\,\sin\varphi_2) + \cos\theta_1\,\cos\theta_2 \\
&= \sin\theta_1\,\sin\theta_2\,\cos(\varphi_1 - \varphi_2) + \cos\theta_1\,\cos\theta_2.
\end{aligned}$$

## 9.9 The Vector Product

**Definition** Let $\vec{a}$ and $\vec{b}$ be vectors in three-dimensional space, with Cartesian components given by the formulae $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. The *vector product* $\vec{a} \times \vec{b}$ of the vectors $\vec{a}$ and $\vec{b}$ is the vector defined by the formula

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\; a_3b_1 - a_1b_3,\; a_1b_2 - a_2b_1).$$

Note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ for all vectors $\vec{a}$ and $\vec{b}$. Also $\vec{a} \times \vec{a} = \vec{0}$ for all vectors $\vec{a}$. It follows easily from the definition of the vector product that

$$(s\vec{a} + t\vec{b}) \times \vec{c} = s\vec{a} \times \vec{c} + t\vec{b} \times \vec{c}, \quad \vec{a} \times (s\vec{b} + t\vec{c}) = s\vec{a} \times \vec{b} + t\vec{a} \times \vec{c}$$

for all vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and real numbers $s$ and $t$.

**Proposition 9.4** *Let $\vec{a}$ and $\vec{b}$ be vectors in three-dimensional space $\mathbb{R}^3$. Then their vector product $\vec{a} \times \vec{b}$ is a vector of length $|\vec{a}|\,|\vec{b}|\,|\sin\theta|$, where $\theta$ denotes the angle between the vectors $\vec{a}$ and $\vec{b}$. Moreover the vector $\vec{a} \times \vec{b}$ is perpendicular to the vectors $\vec{a}$ and $\vec{b}$.*

**Proof** Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, and let $l$ denote the length $|\vec{a} \times \vec{b}|$ of the vector $\vec{a} \times \vec{b}$. Then

$$\begin{aligned}
l^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 \\
&\quad + a_3^2b_1^2 + a_1^2b_3^2 - 2a_3a_1b_3b_1 \\
&\quad + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 \\
&= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) \\
&\quad - 2a_2a_3b_2b_3 - 2a_3a_1b_3b_1 - 2a_1a_2b_1b_2
\end{aligned}$$

$$
\begin{aligned}
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - a_1^2 b_1^2 - a_2^2 b_2^2 - a_3^2 b_3^2 - 2a_2 b_2 a_3 b_3 - 2a_3 b_3 a_1 b_1 - 2a_1 b_1 a_2 b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \,.\, \vec{b})^2
\end{aligned}
$$

since

$$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2, \quad \vec{a} \,.\, \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

But $\vec{a} \,.\, \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta$ (Proposition 9.2). Therefore

$$l^2 = |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta) = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

(since $\sin^2\theta + \cos^2\theta = 1$ for all angles $\theta$) and thus $l = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$. Also

$$\vec{a} \,.\, (\vec{a} \times \vec{b}) = a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) = 0$$

and

$$\vec{b} \,.\, (\vec{a} \times \vec{b}) = b_1(a_2 b_3 - a_3 b_2) + b_2(a_3 b_1 - a_1 b_3) + b_3(a_1 b_2 - a_2 b_1) = 0$$

and therefore the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ (Corollary 9.3), as required. ∎

Using elementary geometry, and the formula for the length of the vector product $\vec{a} \times \vec{b}$ given by Proposition 9.4 it is not difficult to show that the length of this vector product is equal to the area of a parallelogram in three-dimensional space whose sides are represented, in length and direction, by the vectors $\vec{a}$ and $\vec{b}$.

**Remark** Let $\vec{a}$ and $\vec{b}$ be non-zero vectors that are not colinear (i.e., so that they do not point in the same direction, or in opposite directions). The direction of $\vec{a} \times \vec{b}$ may be determined, using the thumb and first two fingers of your right hand, as follows. Orient your right hand such that the thumb points in the direction of the vector $\vec{a}$ and the first finger points in the direction of the vector $\vec{b}$, and let your second finger point outwards from the palm of your hand so that it is perpendicular to both the thumb and the first finger. Then the second finger points in the direction of the vector product $\vec{a} \times \vec{b}$.

Indeed it is customary to describe points of three-dimensional space by Cartesian coordinates $(x, y, z)$ oriented so that if the positive $x$-axis and positive $y$-axis are pointed in the directions of the thumb and first finger respectively of your right hand, then the positive $z$-axis is pointed in the direction of the second finger of that hand, when the thumb and first two fingers are mutually perpendicular. For example, if the positive $x$-axis points towards the East, and the positive $y$-axis points towards the North, then the positive $z$-axis is chosen so that it points upwards. Moreover if $\vec{\imath} = (1, 0, 0)$ and $\vec{\jmath} = (0, 1, 0)$ then these vectors $\vec{\imath}$ and $\vec{\jmath}$ are unit vectors pointed in the direction of the positive $x$-axis and positive $y$-axis respectively, and $\vec{\imath} \times \vec{\jmath} = \vec{k}$, where $\vec{k} = (0, 0, 1)$, and the vector $\vec{k}$ points in the direction of the positive $z$-axis. Thus the 'right-hand' rule for determining the direction of the vector product $\vec{a} \times \vec{b}$ using the fingers of your right hand is valid when $\vec{a} = \vec{\imath}$ and $\vec{b} = \vec{\jmath}$.

If the directions of the vectors $\vec{a}$ and $\vec{b}$ are allowed to vary continuously, in such a way that these vectors never point either in the same direction or in opposite directions, then their vector product $\vec{a} \times \vec{b}$ will always be a non-zero vector, whose direction will vary continuously with the directions of $\vec{a}$ and $\vec{b}$. It follows from this that if the 'right-hand rule' for determining the direction of $\vec{a} \times \vec{b}$ applies when $\vec{a} = \vec{\imath}$ and $\vec{b} = \vec{\jmath}$, then it will also apply whatever the directions of $\vec{a}$ and $\vec{b}$, since, if your right hand is moved around in such a way that the thumb and first finger never point in the same direction, and if the second finger is always perpendicular to the thumb and first finger, then the direction of the second finger will vary continuously, and will therefore always point in the direction of the vector product of two vectors pointed in the direction of the thumb and first finger respectively.

**Example** We shall find the area of the parallelogram $OACB$ in three-dimensional space, where

$$O = (0, 0, 0), \quad A = (1, 2, 0), \qquad B = (-4, 2, -5), \quad C = (-3, 4, -5).$$

Note that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. Let $\vec{a} = \overrightarrow{OA} = (1, 2, 0)$ and $\vec{b} = \overrightarrow{OB} = (-4, 2, -5)$. Then $\vec{a} \times \vec{b} = (-10, 5, 10)$. Now $(-10, 5, 10) = 5(-2, 1, 2)$, and $|(-2, 1, 2)| = \sqrt{9} = 3$. It follows that

$$\operatorname{area} OACB = |\vec{a} \times \vec{b}| = 15.$$

Note also that the vector $(-2, 1, 2)$ is perpendicular to the parallelogram $OACB$.

**Example** We shall find the equation of the plane containing the points $A$, $B$ and $C$ where $A = (3, 4, 1)$, $B = (4, 6, 1)$ and $C = (3, 5, 3)$. Now if $\vec{u} = \overrightarrow{AB} = (1, 2, 0)$ and $\vec{v} = \overrightarrow{AC} = (0, 1, 2)$ then the vectors $\vec{u}$ and $\vec{v}$ are parallel to the plane. It follows that the vector $\vec{u} \times \vec{v}$ is perpendicular to this plane. Now

$\vec{u} \times \vec{v} = (4, -2, 1)$, and therefore the displacement vector between any two points of the plane must be perpendicular to the vector $(4, -2, 1)$. It follows that the function mapping the point $(x, y, z)$ to the quantity $4x - 2y + z$ must be constant throughout the plane. Thus the equation of the plane takes the form

$$4x - 2y + z = k,$$

for some constant $k$. We can calculate the value of $k$ by substituting for $x$, $y$ and $z$ the coordinates of any chosen point of the plane. On taking this chosen point to be the point $A$, we find that $k = 4 \times 3 - 2 \times 4 + 1 = 5$. Thus the equation of the plane is the following:

$$4x - 2y + z = 5.$$

(We can check our result by verifying that the coordinates of the points $A$, $B$ and $C$ do indeed satisfy this equation.)

## 9.10 Scalar Triple Products

Given three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in three-dimensional space, we can form the *scalar triple product* $\vec{u} \,.\, (\vec{v} \times \vec{w})$. This quantity can be expressed as the determinant of a $3 \times 3$ matrix whose rows contain the Cartesian components of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$. Indeed

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1),$$

and thus

$$\vec{u} \,.\, (\vec{v} \times \vec{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1).$$

The quantity on the right hand side of this equality defines the determinant of the $3 \times 3$ matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

We have therefore obtained the following result.

**Lemma 9.5** *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in three-dimensional space. Then*

$$\vec{u} \,.\, (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Using basic properties of determinants, or by direct calculation, one can easily obtain the identities

$$\begin{aligned}\vec{u} \,.\, (\vec{v} \times \vec{w}) &= \vec{v} \,.\, (\vec{w} \times \vec{u}) = \vec{w} \,.\, (\vec{u} \times \vec{v}) \\ &= -\vec{u} \,.\, (\vec{w} \times \vec{v}) = -\vec{v} \,.\, (\vec{u} \times \vec{w}) = -\vec{w} \,.\, (\vec{v} \times \vec{u})\end{aligned}$$

One can show that the absolute value of the scalar triple product $\vec{u}.(\vec{v}\times\vec{w})$ is the volume of the parallelepiped in three-dimensional space whose vertices are the points whose displacement vectors from some fixed point $O$ are $\vec{0}$, $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{u}+\vec{v}$, $\vec{u}+\vec{w}$, $\vec{v}+\vec{w}$ and $\vec{u}+\vec{v}+\vec{w}$. (A *parallelepiped* is a solid like a brick, but whereas the faces of a brick are rectangles, the faces of the parallelepiped are parallelograms.)

**Example** We shall find the volume of the parallelepiped in 3-dimensional space with vertices at $(0,0,0)$, $(1,2,0)$, $(-4,2,-5)$, $(0,1,1)$, $(-3,4,-5)$, $(1,3,1)$, $(-4,3,-4)$ and $(-3,5,-4)$. The volume of this parallelepiped is the absolute value of the scalar triple product $\vec{u} \,.\, (\vec{v} \times \vec{w})$, where

$$\vec{u} = (1,2,0), \quad \vec{v} = (-4,2,-5), \quad \vec{w} = (0,1,1).$$

Now

$$\begin{aligned}\vec{u} \,.\, (\vec{v} \times \vec{w}) &= (1,2,0) \,.\, (\,(-4,2,-5) \times (0,1,1)\,) \\ &= (1,2,0) \,.\, (7,4,-4) = 7 + 2 \times 4 = 15.\end{aligned}$$

Thus the volume of the paralellepiped is 15 units.

## 9.11 The Vector Triple Product Identity

**Proposition 9.6** *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in three-dimensional space. Then*

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \,.\, \vec{w})\,\vec{v} - (\vec{u} \,.\, \vec{v})\,\vec{w}.$$

**Proof** Let $\vec{q} = \vec{u} \times (\vec{v} \times \vec{w})$, and let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$, and $\vec{q} = (q_1, q_2, q_3)$. Then

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2,\; v_3 w_1 - v_1 w_3,\; v_1 w_2 - v_2 w_1).$$

and hence $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{q} = (q_1, q_2, q_3)$, where

$$\begin{aligned}q_1 &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= (u_2 w_2 + u_3 w_3) v_1 - (u_2 v_2 + u_3 v_3) w_1 \\ &= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_1 \\ &= (\vec{u} \,.\, \vec{w}) v_1 - (\vec{u} \,.\, \vec{v}) w_1\end{aligned}$$

Similarly

$$q_2 = (\vec{u} \,.\, \vec{w})v_2 - (\vec{u} \,.\, \vec{v})w_2$$

and

$$q_3 = (\vec{u} \,.\, \vec{w})v_2 - (\vec{u} \,.\, \vec{v})w_3$$

(In order to verify the formula for $q_2$ with an minimum of calculation, take the formulae above involving $q_1$, and cyclicly permute the subcripts 1, 2 and 3, replacing 1 by 2, 2 by 3, and 3 by 1. A further cyclic permutation of these subscripts yields the formula for $q_3$.) It follows that

$$\vec{q} = (\vec{u} \,.\, \vec{w})\,\vec{v} - (\vec{u} \,.\, \vec{v})\,\vec{w},$$

as required, since we have shown that the Cartesian components of the vectors on either side of this identity are equal. ∎

## 9.12 Orthonormal Triads of Unit Vectors

Let $\vec{u}$ and $\vec{v}$ be unit vectors (i.e., vectors of length one) that are perpendicular to each other, and let $\vec{w} = \vec{u} \times \vec{v}$. It follows immediately from Proposition 9.4 that $|\vec{w}| = |\vec{u}|\,|\vec{v}| = 1$, and that this unit vector $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \,.\, \vec{u} = \vec{v} \,.\, \vec{v} = \vec{w} \,.\, \vec{w} = 1$$

and

$$\vec{u} \,.\, \vec{v} = \vec{v} \,.\, \vec{w} = \vec{w} \,.\, \vec{u} = 0.$$

On applying the Vector Triple Product Identity (Proposition 9.6) we find that

$$\vec{v} \times \vec{w} = \vec{v} \times (\vec{u} \times \vec{v}) = (\vec{v} \,.\, \vec{v})\,\vec{u} - (\vec{v} \,.\, \vec{u})\,\vec{v} = \vec{u},$$

and

$$\vec{w} \times \vec{u} = -\vec{u} \times \vec{w} = -\vec{u} \times (\vec{u} \times \vec{v}) = -(\vec{u} \,.\, \vec{v})\,\vec{u} + (\vec{u} \,.\, \vec{u})\,\vec{v} = \vec{v},$$

Therefore

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} = \vec{w}, \quad \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} = \vec{u}, \quad \vec{w} \times \vec{u} = -\vec{u} \times \vec{w} = \vec{v},$$

Three unit vectors, such as the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ above, that are mutually perpendicular, are referred to as an *orthonormal triad* of vectors in three-dimensional space. The vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in any orthonormal triad are linearly independent. It follows directly from Theorem 9.1 that any vector in three-dimensional space may be expressed, uniquely, as a linear combination of the form

$$p\vec{u} + q\vec{v} + r\vec{w}.$$

Any Cartesian coordinate system on three-dimensional space determines an orthonormal triad $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$, where

$$\vec{\imath} = (1, 0, 0), \quad \vec{\jmath} = (0, 1, 0), \quad \vec{k} = (0, 0, 1).$$

The scalar and vector products of these vectors satisfy the same relations as the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ above. A vector represented in these Cartesian components by an ordered triple $(x, y, z)$ then satisfies the identity

$$(x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}.$$

## 9.13 Quaternions

A *quaternion* may be defined to be an expression of the form $w + xi + yj + zk$, where $w$, $x$, $y$ and $z$ are real numbers. There are operations of addition, subtraction and multiplication defined on the set $\mathbb{H}$ of quaternions. These are binary operations on that set.

Quaternions were introduced into mathematics in 1843 by William Rowan Hamilton (1805–1865).

The definitions of addition and subtraction are straightforward. The *sum* and *difference* of two quaternions $w + xi + yj + zk$ and $w' + x'i + y'j + z'k$ are given by the formulae

$$\begin{aligned} &(w + xi + yj + zk) + (w' + x'i + y'j + z'k) \\ &\qquad = (w + w') + (x + x')i + (y + y')j + (z + z')k; \\ &(w + xi + yj + zk) - (w' + x'i + y'j + z'k) \\ &\qquad = (w - w') + (x - x')i + (y - y')j + (z - z')k. \end{aligned}$$

If the quaternions $w + xi + yj + zk$ and $w' + x'i + y'j + z'k$ are denoted by $q$ and $q'$ respectively, then we may denote the sum and the difference of these quaternions by $q + q'$ and $q - q'$.

These operations of addition and subtraction of quaternions are binary operations on the set $\mathbb{H}$ of quaternions. It is easy to see that the operation of addition is commutative and associative, and that the *zero quaternion* $0 + 0i + 0j + 0k$ is an identity element for the operation of addition.

The operation of subtraction of quaternions is neither commutative nor associative. This results directly from the fact that the operation of subtraction on the set of real numbers is neither commutative nor associative.

Let $q$ be a quaternion. Then $q = w + xi + yj + zk$ for some real numbers $w$, $x$, $y$ and $z$, and there is a corresponding quaternion $-q$, with $-q =$

$(-w) + (-x)i + (-y)j + (-z)k$. Then $q + (-q) = (-q) + q = 0$, where 0 here denotes the zero quaternion $0 + 0i + 0j + 0k$. Thus, to every quaternion $q$ there corresponds a quaternion $-q$ that is the inverse of $q$ with respect to the operation of addition.

These properties of quaternions ensure that the quaternions constitute a group with respect to the operation of addition.

The definition of quaternion multiplication is somewhat more complicated than the definitions of addition and subtraction. The *product* of two quaternions $w + xi + yj + zk$ and $w' + x'i + y'j + z'k$ is given by the formula

$$\begin{aligned}
(w + xi + yj + zk) \times (w' + x'i + y'j + z'k) & \\
= \quad & (ww' - xx' - yy' - zz') + (wx' + xw' + yz' - zy')i \\
& +(wy' + yw' + zx' - xz')j + (wz' + zw' + xy' - yx')k.
\end{aligned}$$

We shall often denote the product $q \times q'$ of quaternions $q$ and $q'$ by $qq'$.

Given any real number $w$, let us denote the quaternion $w + 0i + 0j + 0k$ by $w$ itself. Let us also denote the quaternions $0 + 1i + 0j + 0k$, $0 + 0i + 1j + 0k$ and $0 + 0i + 0j + 1k$ by $i$, $j$ and $k$ respectively. It follows directly from the above formula defining multiplication of quaternions that

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$

where $i^2 = i \times i$, $ij = i \times j$ etc. It follows directly from these identities that

$$ijk = -1,$$

where $ijk = i \times (j \times k) = (i \times j) \times k$.

Let $q$ be a quaternion, given by the expression $w + xi + yj + zk$, where $w$, $x$, $y$ and $z$ are real numbers. One can easily verify that the quaternion $q$ can be formed from the seven quaternions $w$, $x$, $y$, $z$, $i$, $j$ and $k$ according to the formula

$$q = w + (x \times i) + (y \times j) + (z \times k).$$

The operation of multiplication on the set $\mathbb{H}$ of quaternions is not commutative. Indeed $i \times j = k$, but $j \times i = -k$.

One can however verify by a straightforward but somewhat tedious calculation that this operation of multiplication of quaternions is associative. Moreover the quaternion $1 + 0i + 0j + 0k$ is an identity element for this operation of multiplication.

A quaternion $w + xi + yj + zk$ is said to be *real* if $x = y = z = 0$. Such a quaternion may be identified with the real number $w$. In this way the set of real numbers may be regarded as a subset of the set of quaternions.

Although quaternion multiplication is not commutative, one can readily show that $a \times q = q \times a$ for all real numbers $a$ and for all quaternions $q$. Indeed if $q = w + xi + yj + zk$, where $w$, $x$, $y$ and $z$ are real numbers, then the rules of quaternion multiplication ensure that

$$a \times q = q \times a = (aw) + (ax)i + (ay)j + (az)k.$$

Let $q$ be a quaternion. Then $q = w + xi + yj + zk$ for some real numbers $w$, $x$, $y$ and $z$. We define the *conjugate* $\overline{q}$ of $q$ to be the quaternion $\overline{q} = w - xi - yj - zk$. The definition of quaternion multiplication may then be used to show that

$$q \times \overline{q} = \overline{q} \times q = w^2 + x^2 + y^2 + z^2.$$

We define the *modulus* $|q|$ of the quaternion $q$ by the formula

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Then $q\overline{q} = \overline{q}q = |q|^2$ for all quaternions $q$. Moreover $|q| = 0$ if and only if $q = 0$.

If $q$ and $r$ are quaternions, and if $\overline{q}$ and $\overline{r}$ denote the conjugates of $q$ and $r$ respectively, then the conjugate $\overline{q \times r}$ of the product $q \times r$ is given by the formula $\overline{q \times r} = \overline{r} \times \overline{q}$.

If $q$ is a non-zero quaternion, and if the quaternion $q^{-1}$ is defined by the formula $q^{-1} = |q|^{-2}\overline{q}$, then $qq^{-1} = q^{-1}q = 1$. We conclude therefore that, given any non-zero quaternion $q$, there exists a quaternion $q^{-1}$ that is the inverse of $q$ with respect to the property of multiplication.

These properties of quaternions ensure that the non-zero quaternions constitute a group with respect to the operation of multiplication.

## 9.14 Quaternions and Vectors

Let $q$ be a quaternion. We can write

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

where $q_0$, $q_1$, $q_2$ and $q_3$ are real numbers. We can then write

$$q = q_0 + \vec{q}$$

where

$$\vec{q} = q_1 i + q_2 j + q_3 k.$$

Following Hamilton, we can refer to $q_0$ as the *scalar part* of the quaternion $q$, and we can refer to $\vec{q}$ as the *vector part* of the quaternion $q$. Moreover $\vec{q}$ may be identified with the vector $(q_1, q_2, q_3)$ in three-dimensional space whose components (with respect to some fixed orthonormal basis) are $q_1$, $q_2$ and $q_3$. Thus a quaternion may be regarded as, in some sense, a formal sum of a scalar and a vector.

In particular, we can regard vectors as a special type of quaternion: a quaternion $q_0 + q_1 i + q_2 j + q_3 k$ represents a vector $\vec{q}$ in three-dimensional space if and only if $q_0 = 0$. Thus vectors are identified with those quaternions whose scalar part is zero.

Now let $\vec{q}$ and $\vec{r}$ be vectors, with Cartesian components $(q_1, q_2, q_3)$ and $(r_1, r_2, r_3)$ respectively. If we consider $\vec{q}$ and $\vec{r}$ to be quaternions (with zero scalar part), and multiply them together in accordance with the rules of quaternion multiplication, we find that

$$\vec{q}\,\vec{r} = -(\vec{q} \,.\, \vec{r}) + (\vec{q} \wedge \vec{r}),$$

where $\vec{q} . \vec{r}$ denotes the *scalar product* of the vectors $\vec{q}$ and $\vec{r}$, and $\vec{q} \wedge \vec{r}$ denotes the *vector product* of these vectors. Thus the scalar part of the quaternion $\vec{q}\,\vec{r}$ is $-\vec{q} \,.\, \vec{r}$, and the vector part is $\vec{q} \wedge \vec{r}$.

Note that $\vec{q}\,\vec{r}$ is itself a vector if and only if the vectors $\vec{q}$ and $\vec{r}$ are orthogonal.

More generally, let $q$ and $r$ be quaternions with scalar parts $q_0$ and $r_0$ and with vector parts $\vec{q}$ and $\vec{r}$, so that

$$q = q_0 + \vec{q}, \quad r = r_0 + \vec{r}.$$

Then

$$qr = q_0 r_0 - \vec{q} \,.\, \vec{r} + q_0 \vec{r} + r_0 \vec{q} + \vec{q} \wedge \vec{r},$$

and thus the scalar part of the quaternion $qr$ is

$$q_0 r_0 - \vec{q} \,.\, \vec{r},$$

and the vector part of the quaternion $qr$ is

$$q_0 \vec{r} + r_0 \vec{q} + \vec{q} \wedge \vec{r}.$$

Now let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be an orthonormal triad of vectors in three dimensional space, with

$$|\vec{u}| = |\vec{v}| = |\vec{w}| = 1,$$

$$\begin{aligned} \vec{u} \wedge \vec{v} = -\vec{v} \wedge \vec{u} &= \vec{w}, \\ \vec{v} \wedge \vec{w} = -\vec{w} \wedge \vec{v} &= \vec{u}, \\ \vec{w} \wedge \vec{u} = -\vec{u} \wedge \vec{w} &= \vec{v}, \end{aligned}$$

If we multiply these with one another in accordance with the rules for quaternion multiplication, we find that

$$\vec{u}^2 = \vec{v}^2 = \vec{w}^2 = -1,$$

$$\begin{aligned} \vec{u}\,\vec{v} = -\vec{v}\,\vec{u} &= \vec{w}, \\ \vec{v}\,\vec{w} = -\vec{w}\,\vec{v} &= \vec{u}, \\ \vec{w}\,\vec{u} = -\vec{u}\,\vec{w} &= \vec{v}, \end{aligned}$$

(Note that the rules for multiplying $\vec{u}$, $\vec{v}$ and $\vec{w}$ with one another correspond to Hamilton's rules for multiplying the basic quaternions $i$, $j$ and $k$ with one another, whenever $\vec{u}$, $\vec{v}$ and $\vec{w}$ constitute a positively oriented basis of three-dimensional space.)

## 9.15 Quaternions and Rotations

Let us consider the effect of a rotation through an angle $\theta$ about an axis in three-dimensional space passing through the origin. Let $l$, $m$ and $n$ be the cosines of the angles between the axis of the rotation and the three coordinate axes. In Cartesian coordinates, the axis of rotation is then in the direction of the vector $(l, m, n)$, where $l^2 + m^2 + n^2 = 1$. The angle $\theta$ and the direction cosines $l$, $m$, $n$ of the axis of the rotation together determine a quaternion $q$, with

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk).$$

Let $\overline{q}$ be the conjugate of $q$, given by the formula

$$\overline{q} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(li + mj + nk).$$

Let $(x, y, z)$ and $(x', y', z')$ be the Cartesian coordinates of two points in three-dimensional space, and let $r$ and $r'$ be the quaternions $r$ and $r'$ be defined by

$$r = xi + jy + zk \text{ and } r' = x'i + y'j + z'k.$$

We shall show that if $r' = qr\overline{q}$ then a rotation about the axis $(l, m, n)$ through an angle $\theta$ will send the point $(x, y, z)$ to the point $(x', y', z')$. (The effect of a rotation through an angle $\theta$ in the opposite sense can be calculated by replacing $\theta$ by $-\theta$ in the definition of the quaternion $q$.)

In this way the algebra of quaternions may be used in areas of application such as computer-aided design and the programming of computer games, in order to calculate the results of rotations applied to points in three-dimensional space.

Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be an orthonormal basis of three-dimensional space with $\vec{w} = \vec{u} \wedge \vec{v}$ (as above), and with $\vec{u}$ directed along the axis of the rotation. Let $\theta$ be a real number, specifying the angle of rotation, and let $q$ be the quaternion

$$\begin{aligned} q &= \cos\frac{\theta}{2} + \left(\sin\frac{\theta}{2}\right)\vec{u} \\ &= \cos\frac{\theta}{2} + l\sin\frac{\theta}{2}\,i + m\sin\frac{\theta}{2}\,j + n\sin\frac{\theta}{2}\,k, \end{aligned}$$

where

$$\vec{u} = (l, m, n), \quad l^2 + m^2 + n^2 = 1.$$

Then

$$q^{-1} = \overline{q} = \cos\frac{\theta}{2} - \left(\sin\frac{\theta}{2}\right)\vec{u},$$

since

$$\begin{aligned} q\,\overline{q} &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\vec{u}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\,\vec{u}\right) \\ &= \cos^2\frac{\theta}{2} + \left(\sin^2\frac{\theta}{2}\right)\vec{u}\,.\,\vec{u} \\ &\qquad - \left(\sin^2\frac{\theta}{2}\right)\vec{u}\wedge\vec{u} \\ &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \\ &= 1. \end{aligned}$$

Also we find that

$$\begin{aligned} q^2 &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\vec{u}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\vec{u}\right) \\ &= \cos^2\frac{\theta}{2} - \left(\sin^2\frac{\theta}{2}\right)\vec{u}\,.\,\vec{u} \\ &\qquad + 2\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}\,\vec{u} + \left(\sin^2\frac{\theta}{2}\right)\vec{u}\wedge\vec{u} \\ &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}\,\vec{u} \\ &= \cos\theta + \sin\theta\,\vec{u}. \end{aligned}$$

Let us now calculate the quaternion products $q\vec{u}\,\overline{q}$, $q\vec{v}\,\overline{q}$ and $q\vec{w}\,\overline{q}$. We first note that

$$\vec{u}\,\overline{q} = \overline{q}\,\vec{u}, \quad \vec{v}\,\overline{q} = q\vec{v}, \quad \vec{w}\,\overline{q} = q\vec{w}.$$

Therefore

$$\begin{aligned} q\vec{u}\,\overline{q} &= q\overline{q}\,\vec{u} = \vec{u}, \\ q\vec{v}\,\overline{q} &= q^2\vec{v} = (\cos\theta + \sin\theta\,\vec{u})\vec{v} \\ &= \cos\theta\vec{v} + \sin\theta\vec{w}, \\ q\vec{w}\,\overline{q} &= q^2\vec{w} = (\cos\theta + \sin\theta\,\vec{u})\vec{w} \\ &= \cos\theta\vec{w} - \sin\theta\vec{v} \end{aligned}$$

Thus if we define $T\colon\mathbb{R}^3 \to \mathbb{R}^3$ to be the transformation that sends a vector $\vec{r}$ to $q\vec{r}\,\overline{q}$, then $T$ fixes the vector $\vec{u}$, rotates the vector $\vec{v}$ about the direction of $\vec{u}$ through an angle $\theta$ towards $\vec{w}$, and rotates $\vec{w}$ about the direction of $\vec{u}$ through an angle $\theta$ towards $-\vec{v}$. This transformation $T$ is therefore a rotation about the direction of $\vec{u}$ through an angle $\theta$.

## Problems

1. Find the cosine of the angle between the vectors $(1, 2, 2)$ and $(0, 3, 4)$.

2. Calculate the components $a$, $b$ and $c$ of some non-zero vector $(a, b, c)$ that is orthogonal to $(1, 2, 1)$ and $(2, 5, 0)$.

3. Calculate the components $a$, $b$, $c$ of a non-zero vector that is perpendicular to the plane in $\mathbb{R}^3$ that passes through the points $(1, 0, 0)$, $(2, 3, 1)$ and $(0, 2, 3)$.

4. The points

$$(0, 0, 0),\ (1, 1, 0),\ (2, 3, 1),\ (-1, 0, 2),$$
$$(3, 4, 1),\ (0, 1, 2),\ (1, 3, 3) \text{ and } (2, 4, 3)$$

are the vertices of a parallelepiped in 3-dimensional space, since

$$\begin{aligned} (3, 4, 1) &= (1, 1, 0) + (2, 3, 1), \\ (0, 1, 2) &= (1, 1, 0) + (-1, 0, 2), \\ (1, 3, 3) &= (2, 3, 1) + (-1, 0, 2), \\ (2, 4, 3) &= (1, 1, 0) + (2, 3, 1) + (-1, 0, 2). \end{aligned}$$

Find the volume of this parallelepiped.

5. Calculate the products $q \times r$ and $r \times q$ of the quaternions $q$ and $r$ in each of the following cases:

   (i) $q = 1 - i$ and $r = 2i - j - k$;

   (ii) $q = 2j + k$ and $r = j + 2k$;

   (iii) $q = 1 - i - j$ and $r = 2i + j - k$.

   (This requires you to express $q \times r$ and $r \times q$ in the form $w + xi + yj + zk$ for appropriate real numbers $w$, $x$, $y$ and $z$).

6. Let $q$ and $r$ be the quaternions defined by $q = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$ and $r = i - j$. Calculate $qr\overline{q}$ (where $\overline{q}$ denotes the conjugate of the quaternion $q$).