Course MA2C02, Hilary Term 2010
Section 2: Trigonometric Identities, Complex Exponentials and Periodic Sequences

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Contents

2 Trigonometric Identities, Complex Exponentials and Periodic Sequences

2.1 Basic Trigonometric Identities

An anticlockwise rotation about the origin through an angle of θ radians sends a point (x, y) of the plane to the point (x', y'), where

$$\left\{ \begin{array}{rcl} x' & = & x\cos\theta - y\sin\theta \\ y' & = & x\sin\theta + y\cos\theta \end{array} \right. \tag{9}$$

(This follows easily from the fact that such a rotation takes the point $(1, 0)$ to the point $(\cos\theta, \sin\theta)$ and takes the point $(0, 1)$ to the point $(-\sin\theta, \cos\theta)$.) An anticlockwise rotation about the origin through an angle of ϕ radians then sends the point (x', y') of the plane to the point (x'', y''), where

$$\left\{ \begin{array}{rcl} x'' & = & x'\cos\phi - y'\sin\phi \\ y'' & = & x'\sin\phi + y'\cos\phi \end{array} \right. \tag{10}$$

Now an anticlockwise rotation about the origin through an angle of $\theta + \phi$ radians sends the point (x, y), of the plane to the point (x'', y''), and thus

$$\left\{ \begin{array}{rcl} x'' & = & x\cos(\theta+\phi) - y\sin(\theta+\phi) \\ y'' & = & x\sin(\theta+\phi) + y\cos(\theta+\phi) \end{array} \right. \tag{11}$$

But if we substitute the expressions for x' and y' in terms of x, y and θ provided by equation (9) into equation (10), we find that

$$\left\{ \begin{array}{rcl} x'' & = & x(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) - y(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) \\ y'' & = & x(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) + y(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) \end{array} \right. \tag{12}$$

On comparing equations (11) and (12) we see that

$$\cos(\theta+\phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi, \tag{13}$$

and

$$\sin(\theta+\phi) = \sin\theta\,\cos\phi + \cos\theta\,\sin\phi. \tag{14}$$

On replacing ϕ by $-\phi$, and noting that $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, we find that

$$\cos(\theta-\phi) = \cos\theta\,\cos\phi + \sin\theta\,\sin\phi, \tag{15}$$

and

$$\sin(\theta-\phi) = \sin\theta\,\cos\phi - \cos\theta\,\sin\phi. \tag{16}$$

If we add equations (13) and (15) we find that

$$\cos\theta\,\cos\phi = \tfrac{1}{2}(\cos(\theta+\phi) + \cos(\theta-\phi)). \tag{17}$$

If we subtract equation (13) from equation (15) we find that

$$\sin\theta\,\sin\phi = \tfrac{1}{2}(\cos(\theta-\phi) - \cos(\theta+\phi)). \tag{18}$$

And if we add equations (14) and (16) we find that

$$\sin\theta\,\cos\phi = \tfrac{1}{2}(\sin(\theta+\phi) + \sin(\theta-\phi)). \tag{19}$$

If we substitute $\phi = \theta$ in equations (13) and (14), and use the identity $\cos^2\theta + \sin^2\theta = 1$, we find that

$$\sin 2\theta = 2\sin\theta\,\cos\theta \tag{20}$$

and

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \tag{21}$$

It then follows from equation (21) that

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta) \tag{22}$$

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta). \tag{23}$$

Remark Equations (9) and (10) may be written in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Also equation (11) may be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and therefore

$$\begin{aligned} \cos(\theta+\phi) &= \cos\theta\,\cos\phi - \sin\theta\,\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\,\cos\phi + \cos\theta\,\sin\phi. \end{aligned}$$

This provides an alternative derivation of equations (13) and (14).

2.2 Basic Trigonometric Integrals

On differentiating the sine and cosine function, we find that

$$\begin{aligned} \frac{d}{dx}\sin kx &= k\cos kx & (24)\\ \frac{d}{dx}\cos kx &= -k\sin kx. & (25) \end{aligned}$$

for all real numbers k.

It follows that

$$\begin{aligned} \int \sin kx &= -\frac{1}{k}\cos kx + C & (26)\\ \int \cos kx &= \frac{1}{k}\sin kx + C, & (27) \end{aligned}$$

for all non-zero real numbers k, where C is a constant of integration.

Theorem 2.1 *Let m and n be positive integers. Then*

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx\,dx &= 0, & (28)\\ \int_{-\pi}^{\pi} \sin nx\,dx &= 0, & (29)\\ \int_{-\pi}^{\pi} \cos mx\,\cos nx\,dx &= \begin{cases} \pi & \textit{if } m = n,\\ 0 & \textit{if } m \neq n, \end{cases} & (30)\\ \int_{-\pi}^{\pi} \sin mx\,\sin nx\,dx &= \begin{cases} \pi & \textit{if } m = n,\\ 0 & \textit{if } m \neq n, \end{cases} & (31)\\ \int_{-\pi}^{\pi} \sin mx\,\cos nx\,dx &= 0. & (32) \end{aligned}$$

Proof First we note that

$$\int_{-\pi}^{\pi} \cos nx\,dx = \left[\frac{1}{n}\sin nx\right]_{-\pi}^{\pi} = \frac{1}{n}\left(\sin n\pi - \sin(-n\pi)\right) = 0$$

and

$$\int_{-\pi}^{\pi} \sin nx\,dx = \left[-\frac{1}{n}\cos nx\right]_{-\pi}^{\pi} = -\frac{1}{n}\left(\cos n\pi - \cos(-n\pi)\right) = 0$$

for all non-zero integers n, since $\cos n\pi = \cos(-n\pi) = (-1)^n$ and $\sin n\pi = \sin(-n\pi) = 0$ for all integers n.

Let m and n be positive integers. It follows from equations (17) and (18) that

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = \tfrac{1}{2} \int_{-\pi}^{\pi} (\cos((m-n)x) + \cos((m+n)x)) \, dx.$$

and

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \tfrac{1}{2} \int_{-\pi}^{\pi} (\cos((m-n)x) - \cos((m+n)x)) \, dx$$

But

$$\int_{-\pi}^{\pi} \cos((m+n)x) \, dx = 0$$

(since $m+n$ is a positive integer, and is thus non-zero). Also

$$\int_{-\pi}^{\pi} \cos((m-n)x) \, dx = 0 \text{ if } m \neq n,$$

and

$$\int_{-\pi}^{\pi} \cos((m-n)x) \, dx = 2\pi \text{ if } m = n$$

(since $\cos((m-n)x) = 1$ when $m = n$). It follows that

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx &= \int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \tfrac{1}{2} \int_{-\pi}^{\pi} \cos((m-n)x) \, dx \\ &= \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases} \end{aligned}$$

Using equation (19), we see also that

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, dx = \tfrac{1}{2} \int_{-\pi}^{\pi} (\sin((m+n)x) + \sin((m-n)x)) \, dx = 0$$

for all positive integers m and n. (Note that $\sin((m-n)x) = 0$ in the case when $m = n$). ∎

2.3 Basic Properties of Complex Numbers

We shall extend the definition of the exponential function so as to define a value of e^z for any complex number z. First we note some basic properties of complex numbers.

A *complex number* is a number that may be represented in the form $x+iy$, where x and y are real numbers, and where $i^2 = -1$. The real numbers x

and y are referred to as the *real* and *imaginary* parts of the complex number $x + iy$, and the symbol i is often denoted by $\sqrt{-1}$. One adds or subtracts complex numbers by adding or subtracting their real parts, and adding or subtracting their imaginary parts. Thus

$$(x+iy)+(u+iv) = (x+u)+i(y+v). \qquad (x+iy)-(u+iv) = (x-u)+i(y-v).$$

Multiplication of complex numbers is defined such that

$$(x + iy) \times (u + iv) = (xu - yv) + i(xv + uy).$$

The reciprocal $(x + yi)^{-1}$ of a non-zero complex number $x + iy$ is given by the formula

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}.$$

Complex numbers may be represented by points of the plane (through the Argand diagram). A complex number $x + iy$ represents, and is represented by, the point of the plane whose Cartesian coordinates are (x, y). One often therefore refers to the set of all complex numbers as the *complex plane*. This complex plane is pictured as a flat plane, containing lines, circles etc., and distances and angles are defined in accordance with the usual principles of plane geometry and trigonometry.

The *modulus* of a complex number $x + iy$ is defined to be the quantity $\sqrt{x^2 + y^2}$: it represents the distance of the corresponding point (x, y) of the complex plane from the origin $(0, 0)$. The modulus of a complex number z is denoted by $|z|$.

Let z and w be complex numbers. Then z lies on a circle of radius $|z|$ centred at 0, and the point $z + w$ lies on a circle of radius $|w|$ centred at z. But this circle of radius $|w|$ centred at z is contained within the disk bounded by a circle of radius $|z| + |w|$ centred at the origin, and therefore $|z + w| \leq |z| + |w|$. This basic inequality is essentially a restatement of the basic geometric result that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. Indeed the complex numbers 0, z and $z + w$ represent the vertices of a triangle in the complex plane whose sides are of length $|z|$, $|w|$ and $|z+w|$. The inequality is therefore often referred to as the *Triangle Inequality*.

Let z and w be complex numbers, and let $z = x + iy$ and $w = u + iv$. Then $zw = (xu - yv) + i(xv + yu)$ and therefore

$$\begin{aligned} |zw|^2 &= (xu - yv)^2 + (xv + yu)^2 \\ &= (x^2u^2 + y^2v^2 - 2xyuv) + (x^2v^2 + y^2u^2 + 2xyuv) \\ &= (x^2 + y^2)(u^2 + v^2) = |z|^2|w|^2. \end{aligned}$$

It follows that $|zw| = |z|\,|w|$ for all complex numbers z and w. A straightforward proof by induction on n then shows that $|z^n| = |z|^n$ for all complex numbers z and non-negative integers n.

2.4 Complex Numbers and Trigonometrical Identities

Let θ and φ be real numbers, and let

$$z = \cos\theta + i\sin\theta, \quad w = \cos\varphi + i\sin\varphi,$$

where $i = \sqrt{-1}$. Then

$$\begin{aligned} zw &= (\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi) + i(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi) \\ &= \cos(\theta+\varphi) + i\sin(\theta+\varphi). \end{aligned}$$

2.5 The Exponential of a Complex Number

Let z be a complex number, and, for each non-negative integer m, let

$$p_m(z) = \sum_{n=0}^{m} \frac{z^n}{n!}.$$

Then $p_0(z), p_1(z), p_2(z), \ldots$ is an infinite sequence of complex numbers. Moreover one can show that, as the integer m increases without limit, the value of the complex number $p_m(z)$ approaches a limiting value $p_\infty(m)$, so that, given any strictly positive real number ε (no matter how small), there exists some positive integer M such that $|p_m(z) - p_\infty(z)| < \varepsilon$ whenever $m \geq M$. (The quantity $|p_m(z) - p_\infty(z)|$ measures the distance in the complex plane from $p_m(z)$ to $p_\infty(z)$, and thus quantifies the error that results on approximating the quantity $p_\infty(z)$ by $p_m(z)$. The size of this error can be made as small as we please, provided that we choose a value of m that is sufficiently large.) This limiting value $\exp p_\infty(z)$ is said to be the *limit* $\lim\limits_{m\to+\infty} p_m(z)$ of $p_m(z)$ as m tends to $+\infty$. The exponential e^z of the complex number z is defined to be the value of this limit. Thus

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!} = p_\infty(z) = \lim_{m\to+\infty} p_m(z) = \lim_{m\to+\infty} \left(\sum_{n=0}^{m} \frac{z^n}{n!} \right).$$

We may also write

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots,$$

The exponential e^z of the complex number z is also denoted by $\exp z$. The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$, mapping the set of complex numbers to itself, which sends each complex number z to its exponents e^z.

2.6 Euler's Formula

Theorem 2.2 (Euler's Formula)

$$e^{i\theta} = \cos\theta + i\,\sin\theta$$

for all real numbers θ.

Proof Let us take the real and imaginary parts of the infinite series that defines $e^{i\theta}$. Now $i^2 = -1, i^3 = -i$ and $i^4 = 1$, and therefore

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} = C(\theta) + iS(\theta),$$

where

$$\begin{aligned} C(\theta) &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \frac{\theta^{12}}{12!} - \cdots \\ S(\theta) &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \frac{\theta^{13}}{13!} - \cdots. \end{aligned}$$

However the infinite series that define these functions $C(\theta)$ and $S(\theta)$ are the Taylor series for the trigonometric functions $\cos\theta$ and $\sin\theta$. Thus $C(\theta) = \cos\theta$ and $S(\theta) = \sin\theta$ for all real numbers θ, and therefore $e^{i\theta} = \cos\theta + i\sin\theta$, as required. ∎

Note that if we set $\theta = \pi$ in Euler's formula we obtain the identity

$$e^{i\pi} + 1 = 0.$$

The following identities follow directly from Euler's formula.

Corollary 2.3

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

for all real numbers θ.

It is customary to define the values $\cos z$ and $\sin z$ of the cosine and sine functions at any complex number z by the formulae

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

Corollary 2.3 ensures that the cosine and sine functions defined for complex values of the argument in this fashion agree with the standard functions for real values of the argument defined through trigonometry.

2.7 Multiplication of Complex Exponentials

Let z and w be complex numbers. Then

$$e^z\, e^w = \left(\sum_{j=0}^{\infty} \frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty} \frac{w^k}{k!}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{z^j w^k}{j!k!}.$$

Thus the value of the product $e^z\, e^w$ is equal to the value of the infinite double sum that is obtained on adding together the quantities $z^j w^k/(j!k!)$ for all ordered pairs (j,k) of non-negative integers. A fundamental result in the theory of infinite series ensures that, in this case, the value of this infinite double sum is independent of the order of summation, and that, in particular, we can evaluate this double sum by first adding together, for each non-negative integer n, the values of the quantities $z^j w^k/(j!k!)$ for all ordered pairs (j,k) of negative numbers with $j+k=n$, and then adding together the resultant quantities for all non-negative values of the integer n. Thus

$$e^z e^w = \sum_{n=0}^{\infty}\left(\sum_{\substack{(j,k)\\ j+k=n}} \frac{z^j w^k}{j!k!}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{j=0}^{n}\frac{n!}{j!(n-j)!} z^j w^{n-j}\right).$$

(Here we have used the fact that if $j+k=n$ then $k=n-j$.) Now the quantity $\dfrac{n!}{j!(n-j)!}$ is the binomial coefficient $\dbinom{n}{j}$. It follows from the Binomial Theorem that

$$\sum_{j=0}^{n}\frac{n!}{j!(n-j)!} z^j w^{n-j} = (z+w)^n.$$

If we substitute this identity in the formula for the product $e^z e^w$, we find that

$$e^z e^w = \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!} = e^{z+w}.$$

We have thus obtained the following result.

Theorem 2.4

$$e^z e^w = e^{z+w}$$

for all complex numbers z and w.

On combining the results of Theorem 2.4 and Euler's Formula (Theorem 2.2), we obtain the following identity for the value of the exponential of a complex number.

Corollary 2.5

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

for all complex numbers $x + iy$.

2.8 Complex Roots of Unity

Lemma 2.6 *Let ω be a complex number satisfying the equation $\omega^n = 1$ for some positive integer n. Then*

$$\omega = e^{\frac{2\pi mi}{n}} = \cos \frac{2\pi m}{n} + i \sin \frac{2\pi m}{n}$$

for some integer m.

Proof The modulus $|\omega|$ of ω is a positive real number satisfying the equation $|\omega|^n = |\omega^n| = 1$. It follows that $\omega = e^{i\theta} = \cos\theta + i\sin\theta$ for some real number θ. Now

$$(e^{i\theta})^2 = e^{i\theta}e^{i\theta} = e^{2i\theta}, \quad (e^{i\theta})^3 = e^{2i\theta}e^{i\theta} = e^{3i\theta}, \text{ etc.,}$$

and a straightforward proof by induction on r shows that

$$(e^{i\theta})^r = e^{ri\theta} = \cos r\theta + i \sin r\theta$$

for all positive integers r. Now $\omega^n = 1$. It follows that

$$1 = (e^{i\theta})^n = e^{ni\theta} = \cos n\theta + i \sin n\theta,$$

and thus $\cos n\theta = 1$ and $\sin n\theta = 0$. But these conditions are satisfied if and only if $n\theta = 2\pi m$ for some integer m, in which case $\omega = e^{2\pi mi/n}$, as required. ∎

We see that, for any positive integer n, there exist exactly n complex numbers ω satisfying $\omega^n = 1$. These are of the form $e^{2\pi mi/n}$ for $m = 0, 1, \ldots, n-1$. They lie on the unit circle in the complex plane (i.e., the circle of radius 1 centred on 0 in the complex plane) and are the vertices of a regular n-sided polygon in that plane.

2.9 Representation of Periodic Sequences

Definition A *doubly-infinite* sequence $(z_n : n \in \mathbb{Z})$ of complex numbers associates to every integer n a corresponding complex number z_n.

Definition We say that doubly-infinite sequence $(z_n : n \in \mathbb{Z})$ of complex numbers is *m-periodic* if $z_{n+m} = z_n$ for all integers n.

Lemma 2.7 *Let m be a positive integer, and let $\omega_m = e^{2\pi i/m}$. Then the value of $\sum_{k=0}^{m-1} \omega_m^{kn}$ is determined, for any integer n, as follows:*

$$\sum_{k=0}^{m-1} \omega_m^{kn} = \begin{cases} m & \text{if } n \text{ is divisible by } m; \\ 0 & \text{if } n \text{ is not divisible by } m. \end{cases}$$

Proof The complex number ω_m has the property that $\omega_m^m = 1$. Also

$$(1 - z)(1 + z + z^2 + \cdots + z^{m-1}) = 1 - z^m$$

for any complex number z. It follows that

$$(1 - \omega_m^n) \sum_{k=0}^{m-1} \omega_m^{kn} = 1 - \omega_m^{mn} = 0$$

for all integers n, and therefore

$$\sum_{k=0}^{m-1} \omega_m^{kn} = 0 \quad \text{provided that} \quad \omega_m^n \neq 1.$$

Now $\omega_m^n = 1$ if and only if the integer n is divisible by m. We can therefore conclude that

$$\sum_{k=0}^{m-1} \omega_m^{kn} = \begin{cases} m & \text{if } n \text{ is divisible by } m, \\ 0 & \text{if } n \text{ is not divisible by } m, \end{cases}$$

as required. ∎

Theorem 2.8 *Let $(z_n : n \in \mathbb{Z})$ be a doubly-infinite sequence of complex numbers which is m-periodic . Then*

$$z_n = \sum_{k=0}^{m-1} c_k \omega_m^{kn},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} z_j \omega_m^{-kj}.$$

Proof It follows from the definition of the numbers c_k that

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} z_j \omega_m^{-kj} \omega_m^{kn} = \frac{1}{m} \sum_{j=0}^{m-1} \left(z_j \sum_{k=0}^{m-1} \omega_m^{(n-j)k} \right),$$

for all integers n. Now it follows from Lemma 2.7 that

$$\sum_{k=0}^{m-1} \omega_m^{(n-j)k} = 0$$

unless $n - j$ is divisible by m, in which case

$$\sum_{k=0}^{m-1} \omega_m^{(n-j)k} = m.$$

Moreover, given any integer n, there is a unique integer r between 0 and $m - 1$ for which $n - r$ is divisible by m. It follows that

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = z_r \quad \text{where } 0 \leq r < m \text{ and } r \equiv n \pmod{m}.$$

Moreover $z_r = z_n$, because the sequence $(z_n : n \in \mathbb{Z})$ is m-periodic. Thus

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = z_n$$

for all integers n, as required. ∎

Example Let $(z_n : n \in \mathbb{Z})$ be an 3-periodic sequence with $z_0 = 2$, $z_1 = 4$, $z_2 = 5$. Let $\omega = \omega_3 = e^{2\pi i/3}$. It follows from Theorem 2.8 that

$$z_n = c_0 + c_1 \omega^n + c_2 \omega^{2n}$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{3} \left(z_0 + z_1 \omega^{-k} + z_2 \omega^{-2k} \right).$$

for $k = 0, 1, 2$. Now $\omega^{-1} = \omega^2$ and $\omega^{-2} = \omega$, because $\omega^3 = 1$. Therefore

$$c_k = \frac{1}{3} \left(z_0 + z_1 \omega^{2k} + z_2 \omega^k \right),$$

and thus

$$\begin{aligned} c_0 &= \tfrac{1}{3}(2+4+5) = \tfrac{11}{3}, \\ c_1 &= \tfrac{1}{3}(2+4\omega^2+5\omega), \\ c_2 &= \tfrac{1}{3}(2+4\omega+5\omega^2). \end{aligned}$$

Now

$$\begin{aligned} \omega &= \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3} = \tfrac{1}{2}(-1+\sqrt{3}\,i), \\ \omega^2 &= \cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3} = \tfrac{1}{2}(-1-\sqrt{3}\,i). \end{aligned}$$

It follows that

$$c_1 = \tfrac{1}{6}(-5+\sqrt{3}\,i), \quad c_2 = \tfrac{1}{6}(-5-\sqrt{3}\,i).$$

Example Let $(z_n : n \in \mathbb{Z})$ be an 4-periodic sequence with $z_0 = 2$, $z_1 = 4$, $z_2 = 5$, $z_3 = 1$. Now if ω_4 is defined as in the statement of Theorem 2.8 then $\omega_4 = e^{2\pi i/4} = i$. It follows from Theorem 2.8 that

$$z_n = c_0 + c_1 i^n + c_2(-1)^n + c_3(-i)^n$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$\begin{aligned} c_k &= \frac{1}{4}\left(z_0 + z_1 i^{-k} + z_2 i^{-2k} + z_3 i^{-3k}\right) \\ &= \frac{1}{4}\left(2 + 4\times(-i)^k + 5\times(-1)^k + i^k\right). \end{aligned}$$

Thus

$$c_0 = 3, \quad c_1 = -\tfrac{3}{4} - \tfrac{3}{4}i, \quad c_2 = \tfrac{1}{2}, \quad c_3 = -\tfrac{3}{4} + \tfrac{3}{4}i.$$

2.10 Periodic Sequences of Real Numbers

Theorem 2.9 *Let $(x_n : n \in \mathbb{Z})$ be a doubly-infinite sequence of real numbers which is m-periodic . Then*

$$x_n = \sum_{k=0}^{m-1}\left(p_k \cos\frac{2\pi kn}{m} + q_k \sin\frac{2\pi kn}{m}\right),$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$p_k = \frac{1}{m}\sum_{j=0}^{m-1} x_j \cos\frac{2\pi kj}{m}, \quad q_k = \frac{1}{m}\sum_{j=0}^{m-1} x_j \sin\frac{2\pi kj}{m}.$$

Proof It follows from Theorem 2.8 that

$$x_n = \sum_{k=0}^{m-1} c_k \omega_m^{kn},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \omega_m^{-kj}.$$

Now

$$\begin{aligned} \omega_m^n &= \cos \frac{2n\pi}{m} + i \sin \frac{2n\pi}{m} \\ \omega_m^{-n} &= \cos \frac{2n\pi}{m} - i \sin \frac{2n\pi}{m} \end{aligned}$$

for all integers n. Now $c_k = p_k - q_k i$ for $k = 0, 1, \ldots, m-1$, where

$$p_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \cos \frac{2\pi kj}{m}, \quad q_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \sin \frac{2\pi kj}{m}.$$

(Note that p_k and q_k are real numbers for all k. It follows that

$$x_n = \mathrm{Re}\left(\sum_{k=0}^{m-1} c_k \omega_m^{kn} \right) = \sum_{k=0}^{m-1} \left(p_k \cos \frac{2\pi kn}{m} + q_k \sin \frac{2\pi kn}{m} \right),$$

where $\mathrm{Re}\left(\sum_{k=0}^{m-1} c_k \omega_m^{kn} \right)$ denotes the real part of $\sum_{k=0}^{m-1} c_k \omega_m^{kn}$. ∎

Problems

1. Let $(z_n : n \in \mathbb{Z})$ be the doubly-infinite 3-periodic sequence with $z_0 = 1$, $z_1 = 2$ and $z_2 = 6$. Find values of a_0, a_1 and a_2 such that

   $$z_n = a_0 + a_1 \omega^n + a_2 \omega^{2n}$$

   for all integers n, where $\omega = e^{2\pi i k/3}$. (Note that $\omega = \frac{1}{2}(-1 + \sqrt{3}\, i)$, $\omega^2 = e^{-2\pi i k/3} = \frac{1}{2}(-1 - \sqrt{3}\, i)$ and thus $\omega^3 = 1$ and $\omega + \omega^2 = -1$.)