Course MA2C01: Michaelmas Term 2009.

Worked Solutions for Assignment II.

1. *Let c be a fixed positive integer, and let $\otimes$ denote the binary operation on the set $\mathbb{Z}$ of integers defined by the formula*

$$x \otimes y = xy + c(x+y) + c^2 - c$$

for all integers x, y and z.

(a) *Is $(\mathbb{Z}, \otimes)$ a semigroup? [Justify your answer.]*

Solution. $(\mathbb{Z}, \otimes)$ is a semigroup if and only if the binary operation $\otimes$ is associative. Now

$$\begin{aligned}
(x \otimes y) \otimes z &= (xy + c(x+y) + c^2 - c) \otimes z \\
&= xyz + cxz + cyz + c^2 z - cz + cxy + c^2 x + c^2 y \\
&\quad + c^3 - c^2 + cz + c^2 - c \\
&= xyz + c(xz + yz + xy) + c^2(x+y+z) + c^3 - c, \\
x \otimes (y \otimes z) &= x \otimes (yz + c(y+z) + c^2 - c) \\
&= xyz + cxy + cxz + c^2 x - cx + cx + cyz + c^2 y \\
&\quad + c^2 z + c^3 - c^2 + c^2 - c \\
&= xyz + c(xy + xz + yz) + c^2(x+y+z) + c^3 - c \\
&= (x \otimes y) \otimes z
\end{aligned}$$

Thus the operation $\otimes$ on $\mathbb{Z}$ is associative, and therefore $(\mathbb{Z}, \otimes)$ is a semigroup.

(b) *Is $(\mathbb{Z}, \otimes)$ a monoid? If so, what is its identity element?*

Solution. The semigroup $(\mathbb{Z}, \otimes)$ is a monoid if and only if it has an identity element e. If so, this identity element must satisfy

$$xe + cx + ce + c^2 - c = x.$$

for all $x \in \mathbb{Z}$. But then

$$(e + c - 1)(x + c) = 0$$

for all $x \in \mathbb{Z}$. Examination of this formula shows that there is an identity element e, and moreover $e = 1 - c$. Thus $(\mathbb{Z}, \otimes)$ is a monoid with identity element $1 - c$.

(c) *Which of the elements of* $\mathbb{Z}$ *are invertible? Is* $(\mathbb{Z}, \otimes)$ *a group?*

An integer x is invertible in this monoid if and only if there exists some integer y such that $x \otimes y = 1 - c$. Now

$$\begin{aligned} x \otimes y &= 1 - c \\ \iff xy + c(x + y) + c^2 - c &= 1 - c \\ \iff (x + c)(y + c) &= 1 \end{aligned}$$

Thus x is invertible if and only if $c \neq -c$ and $1/(x+c)$ is an integer. It follows that the invertible elements of the monoid are $1-c$ and $-1-c$. The monoid $(\mathbb{Z}, \otimes)$ is not a group since it has elements that are not invertible.

2. *Construct a regular grammar that generates the language* L *over the alphabet* $\{0, 1\}$*, where*

$$L = \{1, 1000, 1000000, 1000000000, \ldots\},$$

so that a string of binary digits belongs to L *if and only if it consists of the digit* 1 *followed by a string of* $3n$ *zeroes, for some non-negative integer* n*. You should specify your formal grammar in Backus-Naur form.*

Solution. Non terminals: $\langle S\rangle$, $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$.
Start symbol: $\langle S\rangle$.
Productions:

$$\begin{aligned} \langle S\rangle &\rightarrow 1\langle A\rangle \\ \langle A\rangle &\rightarrow 0\langle B\rangle | \varepsilon \\ \langle B\rangle &\rightarrow 0\langle C\rangle \\ \langle C\rangle &\rightarrow 0\langle A\rangle \end{aligned}$$

3. *Answer the following questions concerning the graph with vertices a, b, c, d, e and f pictured above. [Justify all your answers.]*

(a) *Is the graph complete?*

Solution. Not complete. There is no edge from a to f.

(b) *Is the graph regular?*

Solution. Regular. All vertices are of degree 3.

(c) *Is the graph connected?*

Solution. Connected. All vertices may be joined to a by a path of length at most 2.

(d) *Does the graph have an Eulerian circuit?*

Solution. No. Were a Eulerian circuit to exist, the degrees of all vertices would need to be even. This is not the case.

(e) Does the graph have a Hamiltonian circuit?

Solution. Yes. $a\,b\,e\,d\,f\,c\,a$ is one such circuit.

(f) *Give an example of a spanning tree for the graph, specifying the vertices and edges of the spanning tree.*

Solution. One such spanning tree has vertices a, b, c, d, e and f and edges $a\,b$, $b\,c$, $b\,e$, $e\,d$, $e\,f$. (There are many others. Note that any spanning tree is connected, has all six vertices, and has five edges.)

(g) *Given an example of an isomorphism between the graph pictured above and that pictured below. (You should specify the isomorphism as a function between the sets* $\{a, b, c, d, e, f\}$ *and* $\{u, v, w, x, y, z\}$ *of vertices of the two graphs.)*

Solution.
One such isomorphism

$$\varphi\colon \{a, b, c, d, e, f\} \to \{u, v, w, x, y, z\}$$

is defined so that

$$\varphi(a) = x, \quad \varphi(b) = y, \quad \varphi(c) = w, \quad \varphi(d) = u, \quad \varphi(e) = z, \quad \varphi(f) = v.$$

(Any such isomorphism must send edges to edges and thus must send triangles to triangles. Thus the vertices of the triangle $a\,b\,c$ must either be mapped to the vertices of the triangle $x\,y\,w$, in some order which in this case is arbitrary, or else to the vertices of the triangle $u\,z\,w$. Moreover d is adjacent to a, and therefore $\varphi(d)$ must be adjacent to $\varphi(a)$, and similarly for e and f.)