Course MA2C01: Michaelmas Term 2009.

Worked Solutions for Assignment I.

1. *For each positive integer n, let $n!$ denote the product $1 \times 2 \times \cdots \times n$ of the integers between 1 and n. Prove that $(3n)! \leq (27)^n (n!)^3$ for all positive integers n.*

Solution. Note that $(3n)! = 3! = 6$ and $(27)^n (n!)^3 = 27$ when $n = 1$. Therefore the result holds when $n = 1$. Suppose that the result holds for $n = m$, where m is some positive integer, so that $(3m)! \leq (27)^m (m!)^3$. Now

$$(3(m+1))! = (3m)!(3m+1)(3m+2)(3m+3)$$

and

$$(27)^{m+1}((m+1)!)^3 = 27(27)^m (m!)^3 (m+1)^3 = (27)^m (m!)^3 \times 27(m+1)^3.$$

Also

$$(3m+1)(3m+2)(3m+3) \leq (3m+3)^3 = 27(m+1)^3.$$

It follows that $(3(m+1))! \leq (27)^m (m!)^3 \times 27(m+1)^3 = (27)^{m+1}((m+1)!)^3$. Thus if the identity $(3n)! \leq (27)^n (n!)^3$ holds when $n = m$ then it also holds when $n = m + 1$. It follows from the Principle of Mathematical Induction that this inequality holds for all positive integers n.

2. *Let A and B be sets. Prove that*

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

Solution. First we show that $(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A)$. Let $x \in (A \cup B) \setminus (A \cap B)$. Then either $x \in A$ or else $x \in B$. But $x \notin A \cap B$. Now if $x \in A$ then $x \in A \setminus B$, because $x \notin A \cap B$. Similarly if $x \in B$ then $x \in B \setminus A$. Thus $x \in (A \setminus B) \cup (B \setminus A)$ in the case where $x \in A$ and also in the case where $x \in B$. Thus if $x \in (A \cup B) \setminus (A \cap B)$ then $x \in (A \setminus B) \cup (B \setminus A)$. This shows that $(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A)$.

Next we show that $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$. Let $x \in (A \setminus B) \cup (B \setminus A)$. Then either $x \in A \setminus B$ or $x \in x \in B \setminus A$. Then $x \notin A \cap B$. Also either $x \in A$ or $x \in B$, and therefore $x \in A \cup B$. Thus $x \in (A \cup B) \setminus (A \cap B)$. This shows that $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$.

Now each of the sets $(A \setminus B) \cup (B \setminus A)$ and $(A \cup B) \setminus (A \cap B)$. is contained in the other. Therefore these sets are the equal.

3. *Let Q be the relation on the set $\mathbb{N}$ of (strictly) positive integers, where strictly positive integers x and y satisfy xQy if and only if $x^2 - y^2 = 2^k$ for some non-negative integer k. Also let R be the relation on the set $\mathbb{N}$, where strictly positive integers x and y satisfy xRy if and only if $x^2/y^2 = 2^k$ for some non-negative integer k. For each of the relations Q and R on the set $\mathbb{N}$, determine whether or not that relation is (i) reflexive, (ii) symmetric, (iii) anti-symmetric, (iv) transitive (v), an equivalence relation, (vi) a partial order. [Briefly justify your answers.]*

Solution.

The relation Q is not reflexive. Indeed $x^2 - y^2 = 0$ when $x = y$, but $2^k > 0$ for all non-negative integers k. Thus no element of $\mathbb{N}$ is related to itself.

The relation Q is not symmetric. 5 is related to 3, since $5^2 - 3^2 = 25 - 9 = 16 = 2^4$, but 3 is not related to 5.

Let x and y be positive integers. Suppose that xQy and yQx. Then $x^2 > y^2$, because $2^k > 0$ for all non-negative integers k. Also $y^2 > x^2$. But the inequalities $x^2 > y^2$ and $y^2 > x^2$ cannot be satisfied simultaneously. Now because there are no positive integers x and y for which xQy, yQx and $x \neq y$, the relation Q is anti-symmetric.

The relation Q is not transitive. Indeed $5Q3$, because $5^2 - 3^2 = 2^4$, and $3Q1$, because $3^1 - 1^2 = 2^3$, but 5 is not related to 1, since $5^2 - 1^2 = 24$, and 24 is not a power of 2.

The relation Q is neither an equivalence relation nor a partial order because it is not reflexive, and because it is not transitive.

The relation R is reflexive. Indeed $x^2/y^2 = 1 = 2^0$ when $x = y$.

The relation R is not symmetric. Indeed $2R1$, since $2^2/1^2 = 2^2$, but $1 \not R\, 2$.

The relation R is anti-symmetric. Indeed suppose that x and y are positive integers and that xRy and yRx. Then $x^2/y^2 = 2^k$ for some non-negative integer k, and therefore $x^2/y^2 \geq 1$ and $x^2 \geq y^2$. Similarly $y^2 \geq x^2$. Thus if xRy and yRx then $x^2 = y^2$. But this implies that $x = y$, because x and y are both positive. Thus if xRy and yRx then $x = y$. This shows that the relation R is anti-symmetric.

The relation R is transitive. Let x, y and z be positive integers. Suppose that xRy and yRz. Then there exist non-negative integers k and l such that $x^2/y^2 = 2^k$ and $y^2/z^2 = 2^l$. (Note that there is no reason why k should necessarily be equal to l, and therefore different symbols must be used to denote these quantities.) Then $x^2/z^2 = 2^k 2^l = 2^{k+l}$, and $k + l$ is a non-negative integer, and therefore xRz.

The relation R is not an equivalence relation, because it is not symmetric. The relation R is a partial order because it is reflexive, anti-symmetric and transitive.

4. *Let* $f\colon [0, +\infty) \to (0, 1]$ *be the function from the set* $[0, +\infty)$ *to the set* $(0, 1]$ *defined such that*

$$f(x) = \frac{1}{1 + x^2}$$

for all $x \in [0, +\infty)$*, where*

$$[0, +\infty) = \{x \in \mathbb{R} : 0 \leq x < +\infty\}, \quad (0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}.$$

(Thus $[0, +\infty)$ *is the set consisting of all non-negative real numbers.) Determine whether or not the function* f *is injective, whether or not it is surjective, and whether or not it is invertible.*

Solution. The function f is invertible. Let x and y be real numbers. Then

$$y = 1/(1 + x^2) \iff 1 + x^2 = \frac{1}{y} \iff x^2 = \frac{1 - y}{y}.$$

Now if $y \in (0, 1]$ then $(1 - y)/y \geq 0$, and therefore there exists a unique non-negative real number x such that $x^2 = (1 - y)/y$. It follows that the function f is invertible, and

$$f^{-1}(y) = \sqrt{\frac{1 - y}{y}}$$

for all $y \in (0, 1]$. The function f is therefore both injective and surjective.