

Course MA2C01: Michaelmas Term 2012.

Assignment II—Worked Solutions.

To be handed in by Wednesday 23st January, 2013.

Please include both name and student number on any work handed in.

1. Let \mathbb{R}^3 be the set of all ordered triples of numbers, and let \otimes be the binary operation on \mathbb{R}^3 defined such that

$$(x_1, y_1, z_1) \otimes (x_2, y_2, z_2) = (x_1x_2, x_1y_2 + y_1z_2, z_1z_2)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$. Prove that (\mathbb{R}^3, \otimes) is a monoid. What is the identity element of this monoid? Is the monoid (\mathbb{R}^3, \otimes) a group?

We check whether the operation \otimes on \mathbb{R}^3 is associative.

$$\begin{aligned} & ((x_1, y_1, z_1) \otimes (x_2, y_2, z_2)) \otimes (x_3, y_3, z_3) \\ &= (x_1x_2, x_1y_2 + y_1z_2, z_1z_2) \otimes (x_3, y_3, z_3) \\ &= (x_1x_2x_3, x_1x_2y_3 + (x_1y_2 + y_1z_2)z_3, z_1z_2z_3) \\ &= (x_1x_2x_3, x_1x_2y_3 + x_1y_2z_3 + y_1z_2z_3, z_1z_2z_3) \\ & (x_1, y_1, z_1) \otimes ((x_2, y_2, z_2) \otimes (x_3, y_3, z_3)) \\ &= (x_1, y_1, z_1) \otimes (x_2x_3, x_2y_3 + y_2z_3, z_2z_3) \\ &= (x_1x_2x_3, x_1(x_2y_3 + y_2z_3), z_1z_2z_3) \\ &= (x_1x_2x_3, x_1x_2y_3 + x_1y_2z_3, z_1z_2z_3) \\ &= ((x_1, y_1, z_1) \otimes (x_2, y_2, z_2)) \otimes (x_3, y_3, z_3). \end{aligned}$$

Thus the operation \otimes on \mathbb{R}^3 is associative.

An element (e, f, g) of \mathbb{R}^3 is an identity element for the binary operation \otimes if and only if

$$(e, f, g) \otimes (x, y, z) = (x, y, z) \otimes (e, f, g) = (x, y, z)$$

for all $(x, y, z) \in \mathbb{R}^3$, i.e., if and only if

$$(xe, xf + yg, zg) = (ex, ey + fz, gz) = (x, y, z)$$

for all $(x, y, z) \in \mathbb{R}^3$. By inspection, we see that $(1, 0, 1)$ is an identity element. Thus (\mathbb{R}^3, \otimes) is a monoid.

This monoid is not a group. Indeed $(0,0,0) \otimes (x,y,z) = (0,0,0)$ for all $(x,y,z) \in \mathbb{R}^3$, and therefore there is no element (x,y,z) of \mathbb{R}^3 for which $(0,0,0) \otimes (x,y,z) = (1,0,1)$. It follows that the element $(0,0,0)$ of \mathbb{R}^3 is not invertible. Because this monoid contains a non-invertible element, it cannot be a group.

2. (a) *Describe the formal language over the alphabet $\{0,1\}$ generated by the context-free grammar whose only non-terminal is $\langle S \rangle$, whose start symbol is $\langle S \rangle$ and whose productions are the following:*

$$\begin{aligned}\langle S \rangle &\rightarrow 0 \\ \langle S \rangle &\rightarrow 1\langle S \rangle 1\end{aligned}$$

(i.e., describe the structure of the binary strings generated by the grammar). Is this context-free grammar a regular grammar?

The language generated by this grammar consists of all strings of binary digits consisting of n 1's followed by 0 followed by n 1's, where $n \geq 0$. This grammar is not a regular grammar, because the production $\langle S \rangle \rightarrow 1\langle S \rangle 1$ does not conform to any of the types of productions permitted in a regular grammar.

- (b) *Give the specification of a finite state acceptor that accepts the language over the alphabet $\{0,1\}$ consisting of all words where the number of occurrences of the digit 0 within the word is a multiple of 3. (In particular you should specify the set of states, the starting state, the finishing states, and the transition table that determines the transition function of the finite state acceptor.)*

Start state: S

Finishing State: S

Transition table:

	0	1
S	A	S
A	B	A
B	S	B

(The machine is in state S when the number of zeros in the string input at the relevant stage is divisible by 3; it is in state A when the number of zeros is congruent to 1 modulo 3; it is in state B when the number of

zeros is congruent to 2 modulo 3. The empty word is in the language so state S can be both a starting and a finishing state. No input gives rise to a string that cannot be completed to a string where the number of zeros is divisible by 3, and therefore no errors arise on input, and thus there is no need for an error state.)

(c) *Devise a regular grammar to generate the language specified in (b). (In particular, you should specify the nonterminals, the start state and the productions of the grammar.)*

Non-terminals: $\langle S \rangle, \langle A \rangle, \langle B \rangle$.

Start state: $\langle S \rangle$.

Productions:

$$\begin{aligned}\langle S \rangle &\rightarrow 0\langle A \rangle \\ \langle S \rangle &\rightarrow 1\langle S \rangle \\ \langle A \rangle &\rightarrow 0\langle B \rangle \\ \langle A \rangle &\rightarrow 1\langle A \rangle \\ \langle B \rangle &\rightarrow 0\langle S \rangle \\ \langle B \rangle &\rightarrow 1\langle B \rangle \\ \langle S \rangle &\rightarrow \varepsilon\end{aligned}$$

3. (a) *For each of the following graphs, answer the following questions (giving brief justifications for your answers):*

- *Is the graph connected?*
- *Is the graph regular?*
- *Does the graph have an Eulerian trail?*
- *Does the graph have an Eulerian circuit?*
- *Does the graph have a Hamiltonian circuit?*
- *Is the graph a tree?*

(i) *The graph (V, E_1) , where $V = \{a, b, c, d, e\}$ and*

$$E_1 = \{ae, bc, bd, cd\};$$

The graph is not connected: there is no path from a to b , for example.

The graph is not regular: vertices a and e have degree 1, whereas vertices b, c and d have degree 2.

Any graph that has an Eulerian trail, Eulerian circuit or Hamilton circuit is necessarily connected. This graph is not connected. Therefore it cannot have an Eulerian trail, Eulerian circuit or Hamilton circuit.

Trees are connected. Therefore this graph cannot have a tree. (Moreover the graph has a circuit, and cannot be a tree for that reason.)

- (ii) The graph (V, E_2) , where $V = \{a, b, c, d, e\}$ and

$$E_2 = \{ae, bc, bd, be\};$$

This graph is connected. It is not regular: vertices a, c and d are of degree 1, vertex b is of degree 3 and vertex e is of degree 2. It is a tree because it is connected and has no circuits. Because it has no circuits, it cannot have either an Eulerian or a Hamiltonian circuit. There is no Eulerian trail: in order for such a trail to exist, the graph would have to have at most two vertices of odd degree; this graph has four vertices of odd degree.

- (iii) The graph (V, E_3) , where $V = \{a, b, c, d, e\}$ and

$$E_3 = \{ab, ad, bc, cd, de\}.$$

This graph is connected and has an Eulerian trail $dabcde$. It is not regular: vertex e has degree 1, vertex d has degree 3, and vertices a, b and c have degree 2. This graph has a circuit $dabcd$ and therefore cannot be a tree. No circuit can pass through the vertex e , because no circuit can traverse an edge more than once, and therefore it is not possible for a circuit to go from d out to e and back again. Therefore the graph cannot have a circuit traversing every edge or passing through every vertex. In particular it cannot have either an Eulerian or a Hamiltonian circuit.

- (iv) The graph (V, E_4) , where $V = \{a, b, c, d, e\}$ and

$$E_4 = \{ab, ad, bc, bd, be, cd, de\}.$$

This graph is connected: every vertex is adjacent to the vertex b . The graph is not regular: vertices a, c and e are of degree 2, whereas vertices b and d are of degree 4.

This graph has an Eulerian circuit $abcdbeda$. This circuit is an Eulerian trail. This graph does not have a Hamiltonian circuit.

Indeed if such a circuit were to exist, one could choose the start vertex and the first edge to be the vertex b and the edge be respectively. The

circuit would continue from e to d . It would then go from c to one or other of the vertices a or c . But if it went to a , it would then be blocked from reaching c , because vertex c could not be reached from a without passing again through either b or d , and a Hamiltonian circuit can pass through a vertex at most once before reaching the vertex at which it starts and ends. And if it went to c , it would then be blocked from reaching a . Thus it is not possible to construct a Hamiltonian circuit in this graph.

(b) *Give an example of an isomorphism from the graph (V, E_4) specified in (iv) above to the graph (V', E') where $V' = \{p, q, r, s, t\}$ and*

$$E' = \{pq, pr, ps, pt, qr, qs, qt\}.$$

The function $\varphi: V \rightarrow V'$ is an isomorphism where $\varphi(a) = r$, $\varphi(b) = p$, $\varphi(c) = s$, $\varphi(d) = q$ and $\varphi(e) = t$.

(An isomorphism sets up a one-to-one correspondence between vertices of the two graphs that determines a one-to-one correspondence between edges. It follows that corresponding vertices must be of the same degree. Vertices b and d of the first graph have degree 4. Vertices p and q are the two vertices of the second graph that also have degree 4. It follows that b must map to one of the two vertices p and q , and d must map to the other one. It is then possible to complete the specification of the isomorphism by mapping vertices a , c and e of the first graph to the remaining vertices of the second graph.)