Course MA2C01: Michaelmas Term 2011.

Assignment II — Worked Solutions

1. (a) *Let A be the set $\mathbb{C} \times \mathbb{C}$ consisting of all ordered pairs (z, w), where z and w are complex numbers. Let $\times$ denote the binary operation on A defined by $(z, w) \times (u, v) = (zu - wv, zv + wu)$ for all complex numbers z, w, u and v. Prove that $(A, \times)$ is a monoid. What is its identity element? Prove that an element (z, w) of A is invertible if and only if $z^2 + w^2 \neq 0$.*

   First we show that the binary operation $*$ is associative. Let

$$(z_1, w_1), (z_2, w_2), (z_3, w_3) \in \mathbb{C}.$$

Then

$$\begin{aligned}
&((z_1, w_1) \times (z_2, w_2)) \times (z_3, w_3) \\
&\quad = (z_1 z_2 - w_1 v_1, z_1 w_2 + w_1 z_2) \times (z_3, w_3) \\
&\quad = (z_1 z_2 z_3 - w_1 w_2 z_3 - z_1 w_2 w_3 - w_1 z_2 w_3, \\
&\qquad\quad z_1 z_2 w_3 - w_1 w_2 w_3 + z_1 w_2 z_3 + w_1 z_2 z_3) \\
&(z_1, w_1) \times ((z_2, w_2)) \times (z_3, w_3)) \\
&\quad = (z_1, w_1) \times (z_2 z_3 - w_2 w_3, z_2 w_3 + w_2 z_3) \\
&\quad = (z_1 z_2 z_3 - z_1 w_2 w_3 - w_1 z_2 w_3 - w_1 w_2 z_3, \\
&\qquad\quad w_1 z_2 z_3 - w_1 w_2 w_3 + w_1 z_2 w_3 + w_1 w_2 z_3) \\
&\quad = ((z_1, w_1) \times (z_2, w_2)) \times (z_3, w_3).
\end{aligned}$$

Thus the binary operation $*$ is associative.

An element (e, f) of A is an identity element if and only if

$$(ze - wf, zf + we) = (z, w) = (ez - fw, ew + fz)$$

for all $(z, w) \in A$. We conclude that $(1, 0)$ is an identity element (and is the only identity element). The binary operation $\times$ on A is associative and has an identity element. It follows that $(A, \times)$ is a monoid.

An element (z, w) of A is invertible if and only if there exists some element (u, v) of A such that

$$\begin{aligned}
(zu - wv, zv + wu) + (z, w) \times (u, v) = (1, 0) &= (u, v) \times (z, w) \\
&= (uz - vw, vz + uw).
\end{aligned}$$

It follows that $(z, w) \in A$ is invertible if and only if there exists some element (u, v) of A such that

$$zu - wv = 1, \quad zv + wu = 0.$$

If these equations are satisfied, then

$$u = (zu - wv)u = zu^2 + zv^2 = z(u^2 + v^2)$$

and

$$v = (zu - wv)v = -w(u^2 + v^2).$$

But then

$$u^2 + v^2 = (z^2 + w^2)(u^2 + v^2)^2$$

and therefore

$$(z^2 + w^2)(u^2 + v^2) = 1.$$

From these equations, it follows that an element (z, w) of A is invertible in the monoid A if and only if $z^2 + w^2 \neq 0$, in which case

$$(z, w)^{-1} = \left(\frac{z}{z^2 + w^2}, -\frac{w}{z^2 + w^2} \right).$$

(b) *Let* $(\mathbb{C}, \times)$ *be the monoid consisting of the set of complex numbers with the usual operation of multiplication, and let* $f\colon \mathbb{C} \to A$ *be the function from* $\mathbb{C}$ *to* A *which sends the complex number* $x + iy$ *to the ordered pair* (x, y) *for all real numbers* x *and* y*. Is the function* f *a homomorphism from* $(\mathbb{C}, \times)$ *to* $(A, \times)$*? Is this function an isomorphism?*

The function f is a homomorphism. Indeed

$$\begin{aligned} f(x + iy) \times f(u + iv) &= (x, y) \times (u, v) = (xy - yv, xv + yu) \\ &= f((xu - yv) + i(xv + yu)) \\ &= f((x + iy)(u + iv)). \end{aligned}$$

However the function f is not surjective, because its range is the proper subset $\mathbb{R} \times \mathbb{R}$ of A. It follows that the function f is not an isomorphism.

2. (a) *Describe the formal language over the alphabet* $\{0, 1\}$ *generated by the context-free grammar whose only non-terminal is* $\langle S \rangle$*, whose start symbol is* $\langle S \rangle$ *and whose productions are the following:*

$$\begin{aligned} \langle S \rangle &\to 0 \\ \langle S \rangle &\to 00\langle S \rangle \\ \langle S \rangle &\to \langle S \rangle 11 \end{aligned}$$

Is this context-free grammar a regular grammar?

The language generated by this context-free grammar consists of all strings of binary digits in which an odd number of 0's is followed by an even number of 1's.

The grammar is not a regular grammar because the second and third productions in the specification are impermissible in a regular grammar.

(b) *Give the specification of a finite state acceptor for the language over the alphabet* $\{a, b, c\}$ *consisting of all finite strings, such as aabbc, aabbbc and aaabbbc, that consist of two or more occurrences of the character a, followed by two or more occurrences of the character b, followed by a single occurrence of the character c. You should in particular specify the starting state, the finishing state or states, and the transition table for this finite state acceptor.*

Starting state: S Finishing state S: Transition Table:

	a	b	c
S	A	E	E
A	B	E	E
B	B	C	E
C	E	D	E
D	E	D	F
F	E	E	E
E	E	E	E

(c) *Give the specification of a regular grammar to generate the language over the alphabet* $\{a, b, c\}$ *that was defined in (b).*

Non-terminals: $\langle S\rangle$, $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$, $\langle D\rangle$, $\langle F\rangle$.

Start symbol: $\langle S\rangle$.

Productions:

$$\begin{aligned}
\langle S\rangle &\rightarrow a\langle A\rangle \\
\langle A\rangle &\rightarrow a\langle B\rangle \\
\langle B\rangle &\rightarrow a\langle B\rangle \\
\langle B\rangle &\rightarrow b\langle C\rangle \\
\langle C\rangle &\rightarrow b\langle D\rangle \\
\langle D\rangle &\rightarrow b\langle D\rangle \\
\langle D\rangle &\rightarrow c\langle F\rangle \\
\langle F\rangle &\rightarrow \varepsilon
\end{aligned}$$

3. *Consider a graph G with vertices a, b, c, d and e, and edges a b, a d, b c, b d, c d, c e and d e. Let V denote the set of vertices of this graph, so that* $V = \{a, b, c, d, e\}$.

(a) *Draw a diagram showing the vertices and edges of this graph.*

(b) *Determine the degrees of each of the vertices of the graph.*

Vertices a and e have degree 2, vertices b and c have degree 3, vertex d has degree 4.

(c) *Is this graph regular? [If not, briefly explain why not.]*

No. The vertices do not all have the same degree.

(d) *Is this graph complete? [If not, briefly explain why not.]*

No. There is no edge joining a and e for example. In a complete graph every pair of distinct vertices determines an edge of the graph.

(e) *Is this graph connected? [If not, briefly explain why not.]*

Yes.

(f) *Does this graph have an Eulerian circuit? [If so, give an example. If not, briefly explain why not.]*

No. The vertices b and c have degree 3. If the graph had an Eulerian circuit, the degree of every vertex would be even.

(g) *Does this graph have an Eulerian trail that starts at some vertex of the graph, and ends at some other vertex? [If so, give an example. If not, briefly explain why not.]*

Yes. b, a, d, b, c, $d\,e\,c$.

(h) *Does this graph have a Hamiltonian circuit? [If so, give an example.]*

Yes. a, $b\,c$, e, d, a.

(i) *Is this graph a tree? [If not, briefly explain why not.]*

No. Trees do not have circuits. This graph has circuits, e.g., $a\,b\,d$.

(j) *Does this graph have a spanning tree? [If so, give an example. If not, briefly explain why not.]*

Yes. Connected graphs have spanning trees. One such has vertices a, b, c, d and e and edges $a\,b$, $b\,c$, $c\,d$ and $d\,e$.

(k) *Does there exist a function* $\theta: V \to V$ *from the set* V *of vertices of the graph to itself that is an isomorphism from the graph* G *to itself and that satisfies* $\theta(a) = e$. *[If so, give an example. If not, briefly explain why not.]*

Yes. One such is defined such that $\theta(a) = e$, $\theta(b) = c$, $\theta(c) = b$, $\theta(d) = d$, $\theta(e) = a$.

(l) *Does there exist a function* $\varphi: V \to V$ *from the set* V *of vertices of the graph to itself that is an isomorphism from the graph* G *to itself and that satisfies* $\varphi(c) = d$. *[If so, give an example. If not, briefly explain why not.]*

No. If such an isomorphism were to exist then $\varphi(v)$ would have the same degree as v for all vertices v of the graph. But the degrees of vertices c and d are 3 and 4 respectively.