Course MA2C01: Michaelmas Term 2012.

Assignment I—Worked Solutions.

To be handed in by Wednesday 31st October, 2012.
Please include both name and student number on any work handed in.

1. *Use the Principle of Mathematical Induction to prove that*

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

for all positive integers n and real numbers x satisfying $x \neq 1$.

When $n = 1$ the left hand side equals $1 + x$ and the right hand side equals $\dfrac{1 - x^2}{1 - x}$. Now $1 - x^2 = (1 + x)(1 - x)$. Thus the right hand side equals $1 + x$, and thus equals the left hand side when $n = 1$. The result therefore holds when $n = 1$. (Alternatively one could not that the identity holds when $n = 0$, and use this as the basis for the induction proof.)

Suppose that

$$\sum_{i=0}^{k} x^i = \frac{1 - x^{k+1}}{1 - x}$$

for some positive (or non-negative) integer k. Then

$$\begin{aligned}
\sum_{i=0}^{k+1} x^i &= \sum_{i=0}^{k} x^i + x^{k+1} \\
&= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \\
&= \frac{1 - x^{k+1}}{1 - x} + \frac{x^{k+1}(1 - x)}{1 - x} \\
&= \frac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x} \\
&= \frac{1 - x^{k+2}}{1 - x}.
\end{aligned}$$

Thus if the required identity holds when $n = k$ then it holds when $n = k + 1$. The identity therefore holds for all positive integers n by the Principle of Mathematical Induction.

2. *Let A, B and C be sets. Prove that*

$$(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.$$

We show that each element of $(A \setminus C) \cup (B \setminus C)$ is an element of $(A \cup B) \setminus C$, and that each element of $(A \cup B) \setminus C$ is an element of $(A \setminus C) \cup (B \setminus C)$.

Let $x \in (A \setminus C) \cup (B \setminus C)$. Then either $x \in A \setminus C$ or $x \in B \setminus C$. It follows that either $x \in A$ or $x \in B$, and therefore $x \in A \cup B$. But, in both cases just considered, $x \notin C$. Therefore $x \in (A \cup B) \setminus C$. This proves that

$$(A \setminus C) \cup (B \setminus C) \subset (A \cup B) \setminus C.$$

Now let $x \in (A \cup B) \setminus C$. Then $x \in A \cup B$ and $x \notin C$. But then either $x \in A$ and $x \in B$. If $x \in A$ then $x \in A \setminus C$, because $x \notin C$. Similarly, if $x \in B$ then $x \in B \setminus C$, because $x \notin C$. Thus $x \in (A \setminus C) \cup (B \setminus C)$. This proves that

$$(A \cup B) \setminus C \subset (A \setminus C) \cup (B \setminus C).$$

Therefore

$$(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.$$

3. *Let Q be the relation on the set $\mathbb{R}^*$ of non-zero real numbers, where non-zero real numbers x and y satisfy xQy if and only if $\dfrac{x^2}{y^2}$ is an rational number. Determine*

   (i) *whether or not the relation Q is* reflexive*,*

   (ii) *whether or not the relation Q is* symmetric*,*

   (iii) *whether or not the relation Q is* anti-symmetric*,*

   (iv) *whether or not the relation Q is* transitive*,*

   (v) *whether or not the relation Q is a* equivalence relation*,*

   (vi) *whether or not the relation Q is a* partial order.

   Note that a rational number is a number that can be expressed as a fraction n/d whose numerator n and denominator d are integers. Not all real numbers are rational numbers: both $\sqrt{2}$ and π are examples of real numbers that are not rational numbers.

   [Justify your answers with short proofs and/or counterexamples.]

$\frac{x^2}{y^2} = 1$ whenever $x = y$, and the number 1 is a rational number. It follows that xQx for all real numbers x. Thus the relation Q is *reflexive.*

If $x, y \in \mathbb{R}^*$ and xQy then $\frac{x^2}{y^2} = q$ for some non-zero rational number q. But then $\frac{y^2}{x^2} = \frac{1}{q}$, and $1/q$ is also a rational number. It follows that yQx. Thus the relation Q is *symmetric.*

Note that $1Q2$ and $2Q1$, but $1 \neq 2$. This counter-example shows that the relation Q is not *anti-symmetric.*

If $x, y, z \in \mathbb{R}^*$, xQy and yQz then $\frac{x^2}{y^2} = q_1$ and $\frac{y^2}{z^2} = q_2$ for some rational numbers q_1 and q_2. But then $\frac{x^2}{z^2} = \frac{x^2}{y^2}\frac{y^2}{z^2} = q_1q_2$, and q_1q_2 is a rational number. This shows that the relation Q is *transitive.*

The relation Q is an *equivalence relation* because it is reflexive, symmetric and transitive. Because this relation is not anti-symmetric, it is not a *partial order.*

4. *Let* $f\colon [0,4] \to [0,10]$ *be the function defined so that*

$$f(x) = \begin{cases} x^3 & \textit{if } 0 \leq x \leq 2; \\ x + 6 & \textit{if } 2 < x \leq 4. \end{cases}$$

Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

Consider the behaviour of this function as x increases from 0 to 4. The value of the function increases continuously from 0 to 8 as x increases from 0 to 2. It then increases continuously from 8 to 10 as x increases from 2 to 4.

Let $u, v \in [0, 10]$ satisfy $u \leq v$ and $f(u) = f(v)$. If $f(u) \leq 8$ then $u, v \in [0, 2]$ and $u = v = \sqrt[3]{f(u)}$. if $f(u) > 8$ then $u, v \in (2, 4]$, and $u = v = f(u) - 6$. It follows that if $u, v \in [0, 4]$ satisfy $f(u) = f(v)$ then $u = v$. Thus the function f is injective.

Let $y \in [0, 10]$. If $0 \leq y \leq 8$ then $y = f(x)$, where $x = \sqrt[3]{y}$. If $8 < y \leq 10$ then $y = f(x)$, where $x = y - 6$. Therefore each element y of $[0, 10]$ is in the range of the function. Thus the function is surjective.