Course MA2C01: Michaelmas Term 2011.
Assignment I — Worked Solutions

1. *Use the Principle of Mathematical Induction to prove that*

$$\sum_{i=1}^{n} \frac{1}{i^3} \leq \frac{3}{2} - \frac{1}{2n^2}$$

for all positive integers n.

$$\sum_{i=1}^{n} \frac{1}{i^3} = 1 \quad \text{and} \quad \frac{3}{2} - \frac{1}{2n^2} = 1$$

when $n = 1$. The inequality therefore holds when $n = 1$.

Suppose that the inequality holds when $n = m$ so that

$$\sum_{i=1}^{m} \frac{1}{i^3} \leq \frac{3}{2} - \frac{1}{2m^2}$$

Then

$$\sum_{i=1}^{m+1} \frac{1}{i^3} = \sum_{i=1}^{m} \frac{1}{i^3} + \frac{1}{(m+1)^3} \leq \frac{3}{2} - \frac{1}{2m^2} + \frac{1}{(m+1)^3}.$$

It thus suffices to show that

$$\frac{1}{2m^2} - \frac{1}{(m+1)^3} \geq \frac{1}{2(m+1)^2}$$

for all positive integers m. Now

$$\begin{aligned} &\frac{1}{2m^2} - \frac{1}{(m+1)^3} - \frac{1}{2(m+1)^2} \\ &\qquad = \frac{(m+1)^3 - 2m^2 - m^2(m+1)}{2m^2(m+1)^3} \\ &\qquad = \frac{(m^3 + 3m^2 + 3m + 1) - 2m^2 - m^3 - m^2)}{2m^2(m+1)^3} \\ &\qquad = \frac{3m+1}{2m^2(m+1)^3}. \end{aligned}$$

It follows that

$$\frac{1}{2m^2} - \frac{1}{(m+1)^3} - \frac{1}{2(m+1)^2} \geq 0$$

for all positive integers m, and therefore

$$\sum_{i=1}^{m+1} \frac{1}{i^3} \leq \frac{3}{2} - \frac{1}{2(m+1)^2}.$$

The required result therefore follows by the Principle of Mathematical Induction.

2. *Let A, B and C be sets. Prove that*

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Let $x \in (A \setminus B) \cup (B \setminus A)$. We show that $x \in (A \cup B) \setminus (A \cap B)$. Now either both $x \in A$ and $x \notin B$, or else both $x \in B$ and $x \notin A$. If $x \in A$ and $x \notin B$ then $x \in A \cup B$ and $x \notin A \cap B$, and therefore $x \in (A \cup B) \setminus (A \cap B)$. Similarly if $x \in B$ and $x \notin A$ then $x \in A \cup B$ and $x \notin A \cap B$, and therefore $x \in (A \cup B) \setminus (A \cap B)$. Thus if $x \in (A \setminus B) \cup (B \setminus A)$ then $x \in (A \cup B) \setminus (A \cap B)$. This proves that

$$(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B).$$

Now let $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in A \cup B$, and therefore either $x \in A$ or $x \in B$. But $x \notin (A \cap B)$, and therefore x cannot belong to both of the sets A and B. Therefore x must belong to exactly one of the sets A and B, and therefore either $x \in A \setminus B$ or else $x \in B \setminus A$. It follows that $x \in (A \setminus B) \cup (B \setminus A)$. This proves that

$$(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A).$$

We conclude from the above results that

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B),$$

as required.

3. *Let Q be the relation on the set $\mathbb{R}$ of real numbers, where real numbers x and y satisfy xQy if and only if e^{x-y} is an integer. Determine*

(i) whether or not the relation Q is reflexive*,*

(ii) whether or not the relation Q is symmetric*,*

(iii) whether or not the relation Q is anti-symmetric*,*

(iv) whether or not the relation Q is transitive*,*

(v) whether or not the relation Q is a equivalence relation*,*

(vi) whether or not the relation Q is a partial order*.*

[Justify your answers with short proofs and/or counterexamples.]

(i) $e^{x-y} = e^0 = 1$ when $x = y$, and therefore $e^{x-y} \in \mathbb{Z}$ when $x = y$. It follows that xQx for all real numbers x, and thus the relation Q is *reflexive*.

(ii) Let x and y be real numbers. Then

$$e^{y-x} = e^{-(x-y)} = \frac{1}{e^{x-y}}.$$

Now if $x = \log 2$ and $y = 0$ then $e^{x-y} = 2$ and $e^{y-x} = \frac{1}{2}$. Now $2 \in \mathbb{Z}$ and $\frac{1}{2} \notin \mathbb{Z}$. It follows that xQy and $y \not{Q}\, x$ when $x = \log 2$ and $y = 0$. Thus the relation Q is not symmetric.

(iii) Let x and y be real numbers. Suppose that xQy and yQx. Then e^{x-y} is an integer, and $1/e^{x-y}$ is also an integer. Now the only integers whose reciprocals are also integers are the integers 1 and -1. Also $e^{x-y} > 0$ for all real numbers x and y. It follows that xQy and yQx if and only if $e^{x-y} = 1$, in which case $x - y = 0$ and therefore $x = y$. This shows that the relation Q is *anti-symmetric*.

(iv) Let x, y and z be real numbers. Suppose that xQy and yQz. Then e^{x-y} and e^{y-z} are integers. Now

$$e^{x-z} = e^{(x-y)+(y-z)} = e^{x-y}e^{y-z}.$$

It follows that e^{x-z} is a product of integers, and is thus itself an integer. Therefore xQz. This shows that the relation Q is *transitive*.

(v) The relation Q is not an equivalence relation because it is not symmetric.

(vi) The relation Q is a partial order because it is reflexive, anti-symmetric and transitive.

4. *Let* $f\colon [0,3] \to [0,4]$ *be the function defined so that*

$$f(x) = -x^2 + 2x + 3.$$

Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

We consider the behaviour of the function f on the interval $[0,3]$. Computing some of its values, we find that $f(0) = 3$, $f(1) = 4$, $f(2) = 3$ and $f(3) = 0$. These computations suffice to demonstrate that the function f is not injective, since $f(0) = f(2) = 3$. Also it follows from the continuity of the function f that $f(x)$ takes on all values between 4 and 0 as x increases from 1 to 3. Therefore the function f is surjective.

We conclude therefore that the function f is surjective, but it is not injective.