Course MA2C01: Michaelmas Term 2010.

Assignment I.

To be handed in by Wednesday 17th November, 2010.
Please include both name and student number on any work handed in.

1. *Use the Principle of Mathematical Induction to prove that*

$$\sum_{i=1}^{n} \frac{2i^2 - 1}{i^4} \leq 4 - \frac{2n+1}{n^2}$$

for all positive integers n.

The inequality holds when $n = 1$ since $L.H.S = 1$ and $R.H.S = 4 - 3 = 1$ when $n = 1$.

Suppose that the inequality holds for $n = m$, so that

$$\sum_{i=1}^{m} \frac{2i^2 - 1}{i^4} \leq 4 - \frac{2m+1}{m^2}.$$

Then

$$\begin{aligned} \sum_{i=1}^{m+1} \frac{2i^2 - 1}{i^4} &= \sum_{i=1}^{m} \frac{2i^2 - 1}{i^4} + \frac{2(m+1)^2 - 1}{(m+1)^4} \\ &\leq 4 - \frac{2m+1}{m^2} + \frac{2(m+1)^2 - 1}{(m+1)^4}. \end{aligned}$$

Thus it suffices to prove that

$$\frac{2m+1}{m^2} - \frac{2(m+1)^2 - 1}{(m+1)^4}. \geq \frac{2(m+1)+1}{(m+1)^2}$$

for all positive integers m.

Now

$$\begin{aligned} \frac{2m+1}{m^2} &= \frac{(2m+1)(m^4 + 4m^3 + 6m^2 + 4m + 1)}{m^2(m+1)^4}, \\ &= \frac{2m^5 + 9m^4 + 16m^3 + 14m^2 + 6m + 1}{m^2(m+1)^4}, \\ \frac{2(m+1)^2 - 1}{(m+1)^4} &= \frac{2m^4 + 4m^3 + m^2}{m^2(m+1)^4}, \end{aligned}$$

$$
\begin{aligned}
\frac{2(m+1)+1}{(m+1)^2} &= \frac{m^2(m^2+2m+1)(2m+3)}{m^2(m+1)^4}, \\
&= \frac{2m^5+7m^4+8m^3+3m^2}{m^2(m+1)^4},
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{2m+1}{m^2} - \frac{2(m+1)^2-1}{(m+1)^4} &= \frac{2m^5+7m^4+12m^3+13m^2+6m+1}{m^2(m+1)^4}, \\
&\geq \frac{2(m+1)+1}{(m+1)^2}.
\end{aligned}
$$

We conclude that

$$
\sum_{i=1}^{m+1} \frac{2i^2-1}{i^4} \geq 4 - \frac{2(m+1)+1}{(m+1)^2}.
$$

We have thus shown that if the inequality holds for $n = m$ then it holds for $n = m+1$. It follows that the inequality must hold for all positive integers n, by the Principle of Mathematical Induction.

2. *Let A, B and C be sets. Prove that*

$$
A \cap (B \setminus C) = (A \cap B) \setminus C.
$$

Let $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. Thus $x \in B$ and $x \notin C$. Now $x \in A$ and $x \in B$, therefore $x \in A \cap B$. Moreover $x \notin C$ and thus $x \in (A \cap B) \setminus C$.

Let $x \in (A \cap B) \setminus C$. Then $x \in A \cap B$ and $x \notin C$, and so $x \in A$ and $x \in B$. But $x \in B$ and $x \notin C$, and therefore $x \in B \setminus C$. Also $x \in A$, and therefore $x \in A \cap (B \setminus C)$.

We conclude that $A \cap (B \setminus C) = (A \cap B) \setminus C$, since every element of one of these sets is an element of the other.

3. *Let $\sim$ be the relation on the set $\mathbb{R}$ of real numbers, where real numbers x and y satisfy $x \sim y$ if and only if $y^3 - x^3$ is an integer. Determine whether or not the relation $\sim$ is an equivalence relation, and whether or not this relation is a partial order. [Give appropriate short proofs and/or counterexamples to justify your answers.]*

Let $x, y, z \in \mathbb{R}$. Then $x^3 - x^3 = 0$ and $0 \in \mathbb{Z}$, and thus $x \sim x$ for all $x \in \mathbb{R}$. Thus the relation $\sim$ is reflexive.

If $x \sim y$ then $y^3 - x^3 \in \mathbb{Z}$. But then $x^3 - y^3 = -(y^3 - x^3) \in \mathbb{Z}$, and therefore $y \sim x$. Thus the relation $\sim$ is symmetric.

If $x \sim y$ and $y \sim z$ then $y^3 - x^3 \in \mathbb{Z}$ and $z^3 - y^3 \in \mathbb{Z}$ and therefore

$$z^3 - y^3 = (z^3 - y^3) + (y^3 - x^3) \in \mathbb{Z},$$

and hence $x \sim z$. Therefore the relation $\sim$ is transitive.

Note that $1 \sim 2$ and $2 \sim 1$ by $1 \neq 2$. Thus the relation is not anti-symmetric.

The relation $\sim$ is reflexive, symmetric and transitive, and is thus an equivalence relation. It is not anti-symmetric, and therefore it is not a partial order.

4. *Let $f\colon [0,2] \to [0,2]$ be the function defined so that*

$$f(x) = \begin{cases} 2x & \textit{if } 0 \leq x \leq 1; \\ 3 - x & \textit{if } 1 \leq x \leq 2. \end{cases}$$

Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

The function x is not injective. Indeed $f(\frac{1}{2}) = f(2) = 1$ but $\frac{1}{2} \neq 2$. The function is surjective, because $y = f(\frac{1}{2}y)$ for all $y \in [0,2]$.