Course MA2C01: Michaelmas Term 2010.

Assignment II.

To be handed in by Wednesday 26th January, 2011.
Please include both name and student number on any work handed in.

1. (a) Let $*$ denote the binary operation on the set $\mathbb{R}^3$ of ordered triples of real numbers defined such that

$$(a_1, a_2, a_3) * (b_1, b_2, b_3)$$
$$= \Bigl(a_1b_3 + a_2b_2 + a_3b_1,\ \ a_1b_1 + a_2b_3 + a_3b_2,\ \ a_1b_2 + a_2b_1 + a_3b_3,\Bigr).$$

Prove that $\mathbb{R}^3, *)$ is a monoid. Is this monoid a group? [Justify your answers.]

(b) Let $f\colon \mathbb{R}^3 \to \mathbb{C}$ be the function defined such that

$$f(a_1, a_2, a_3) = a_3 - \frac{1}{2}(a_1 + a_2) + \frac{\sqrt{3}}{2}(a_1 - a_2)i$$

for all $a_1, a_2, a_3 \in \mathbb{R}$, where $i^2 = -1$. Prove that f is a homomorphism between the monoids $(\mathbb{R}^3, *)$ and $(\mathbb{C}, \times)$, where $\times$ denotes the standard multiplication operation on the set $\mathbb{C}$ of complex numbers.

2. (a) Describe the language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose non-terminals are $\langle S \rangle$ and $\langle A \rangle$, whose start symbol is $\langle S \rangle$ and whose prodcutions are

$$\langle S \rangle \to \langle S \rangle \langle A \rangle, \quad \langle S \rangle \to 1, \quad \langle A \rangle \to 01.$$

Is the context-free grammar a regular grammar?

(b) Let L be the language over the alphabet $\{0, 1\}$ consisting of those finite strings of binary digits in which neither 010 nor 101 occurs as a substring. Give the description of a finite state acceptor for the language L, specifying the starting state, the finishing state or states, and the transition table for this finite state acceptor.

(c) Construct a regular context-free grammar that generates the langle L described in (b).

3. Answer the following questions concerning the graph G with vertices V and edges E, where

$$\begin{aligned} V &= \{a,\ b,\ c,\ d\ \ e\} \\ E &= \{a\,b,\ \ a\,c,\ \ b\,c,\ \ c\,d,\ \ c\,e,\ \ d\,e\}. \end{aligned}$$

[Briefly justify all your answers.]

(a) Is the graph complete?

(b) Is the graph regular?

(c) Is the graph connected?

(d) Does the graph have an Eulerian circuit?

(e) Does the graph have a Hamiltonian circuit?

(f) Give an example of a spanning tree for the graph, specifying the vertices and edges of the spanning tree.

(g) Given an example of an isomorphism $\varphi\colon V \to V$ from the given graph to itself which satisfies $\varphi(a) = d$. [You should specify the isomorphism as a function from the set $\{a, b, c, d, e, f\}$ to itself.]

4. (a) Let graphs G_1 and G_2 be trees, where there are no vertices or edges that are common to both G_1 and G_2. Let v_1 be a vertex of G_1, let v_2 be a vertex of G_2, and let G be the graph consisting of the vertices and edges of G_1, the vertices and edges of G_2 and the edge $v_1\,v_2$. Explain why the graph G is a tree.

(b) Let the graphs G, G_1 and G_2 be as in (a), let w_1 be a vertex of G_1 distinct from v_1, let w_2 be a vertex of G_2 distinct from v_2, and let G' be the graph formed from the graph G by adding an extra edge $w_1\,w_2$. Is the graph G' a tree? [Justify your answer.]