Course MA2C01: Michaelmas Term 2009.

Assignment I.

To be handed in by Wednesday 18th November, 2009.
Please include both name and student number on any work handed in.

1. For each positive integer n, let $n!$ denote the product $1 \times 2 \times \cdots \times n$ of the integers between 1 and n. Prove that $(3n)! \leq (27)^n (n!)^3$ for all positive integers n.

2. Let A and B be sets. Prove that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

3. Let Q be the relation on the set $\mathbb{N}$ of (strictly) positive integers, where strictly positive integers x and y satisfy xQy if and only if $x^2 - y^2 = 2^k$ for some non-negative integer k. Also let R be the relation on the set $\mathbb{N}$, where strictly positive integers x and y satisfy xRy if and only if $x^2/y^2 = 2^k$ for some non-negative integer k. For each of the relations Q and R on the set $\mathbb{N}$, determine whether or not that relation is (i) reflexive, (ii) symmetric, (iii) anti-symmetric, (iv) transitive (v), an equivalence relation, (vi) a partial order. [Briefly justify your answers.]

4. Let $f\colon [0, +\infty) \to (0, 1]$ be the function from the set $[0, +\infty)$ to the set $(0, 1]$ defined such that

$$f(x) = \frac{1}{1 + x^2}$$

for all $x \in [0, +\infty)$, where

$$[0, +\infty) = \{x \in \mathbb{R} : 0 \leq x < +\infty\}, \quad (0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}.$$

(Thus $[0, +\infty)$ is the set consisting of all non-negative real numbers.) Determine whether or not the function f is injective, whether or not it is surjective, and whether or not it is invertible.