

Course MA2C01, Michaelmas Term 2012  
Section 1: The Principle of Mathematical  
Induction

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# 1 The Principle of Mathematical Induction

## 1.1 Integers and Natural Numbers

An *integer* is a whole number. Such numbers are of three types, *positive*, *negative* and *zero*. The *positive integers* (or positive whole numbers) are  $1, 2, 3, 4, \dots$ . Similarly the *negative integers* (or negative whole numbers) are  $-1, -2, -3, -4, \dots$ . There is of course exactly one integer that is zero, namely 0 itself.

The *non-negative integers* are therefore  $0, 1, 2, 3, \dots$ . Similarly the *non-positive integers* are  $0, -1, -2, -3, \dots$ .

It is customary in mathematics to denote the set (or collection) of integers by  $\mathbb{Z}$ . (The word for ‘number’ in German is ‘Zahl’.)

The *natural numbers* are the positive integers  $1, 2, 3, 4, \dots$ . It is customary to denote the set of natural numbers by  $\mathbb{N}$ .

(Note therefore that terms ‘natural number’ and ‘positive integer’ are synonyms, i.e., they refer to the same objects.)

## 1.2 Introduction to the Principle of Mathematical Induction

For each natural number  $n$ , let  $S_n$  denote the sum of the first  $n$  (positive) odd numbers. Calculating  $S_1, S_2, S_3, S_4, S_5$ , we find

$$\begin{array}{rcl} S_1 & = & 1 \\ S_2 & = & 1 + 3 \\ S_3 & = & 1 + 3 + 5 \\ S_4 & = & 1 + 3 + 5 + 7 \\ S_5 & = & 1 + 3 + 5 + 7 + 9 \end{array} \quad \begin{array}{rcl} & = & 1, \\ & = & 4, \\ & = & 9, \\ & = & 16, \\ & = & 25. \end{array}$$

You may notice a pattern beginning to emerge. Does this pattern continue? Suppose that we see whether or not the pattern continues to  $S_6$ . Adding up, we find

$$S_6 = 1 + 3 + 5 + 7 + 9 + 11 = 36.$$

We are thus led to conjecture that

$$S_n = n^2$$

for all natural numbers  $n$ ?

Can we prove it? If so, how?

Merely testing the proposition for a few values of  $n$ , no matter how many, cannot in itself suffice to *prove* that the proposition holds for *all* natural

numbers  $n$ . Moreover propositions may turn out to be true in a very large number of cases, and yet fail for others. Such a proposition is the following:

$$“n < 1,000,000,000”.$$

This proposition holds for a large number of natural numbers  $n$  (indeed for 999,999,999 of them, to be precise), yet it obviously fails to hold for all natural numbers  $n$ .

One might ask what strategies are available for proving that some conjectured result does indeed hold for all natural numbers  $n$ . One such is the *Principle of Mathematical Induction*.

Suppose that, for each natural number  $n$ ,  $P(n)$  denotes some proposition, such as “ $S_n = n^2$ ”. For each value of  $n$ , the proposition  $P(n)$  would be either true or false. Our task is to prove that it is true for all values of  $n$ . The Principle of Mathematical Induction states that this is true provided that (i)  $P(1)$  is true, and (ii) if  $P(m)$  is true for any natural number  $m$  then  $P(m+1)$  is also true.

We can express this more informally as follows. Suppose that we are required to prove that some statement is true for all values of a natural number  $n$ . To do this, it suffices to prove (i) that the statement is true when  $n = 1$ , and (ii) that if the statement is true when  $n = m$  for some natural number  $m$ , then it is also true when  $n = m + 1$  (no matter what the value of  $m$ ).

To understand the justification for the Principle of Mathematical Induction, consider the following. For each natural number  $n$ , let  $P(n)$  denote (as above) a proposition (that is either true or false). We suppose that we have proved that  $P(1)$  is true, and that if  $P(m)$  is true then  $P(m + 1)$  is true. Now

$P(1)$  is true.

If  $P(1)$  is true then  $P(2)$  is true. Moreover  $P(1)$  is true.

Therefore  $P(2)$  is true.

If  $P(2)$  is true then  $P(3)$  is true. Moreover  $P(2)$  is true.

Therefore  $P(3)$  is true.

If  $P(3)$  is true then  $P(4)$  is true. Moreover  $P(3)$  is true.

Therefore  $P(4)$  is true.

$\vdots$

If  $P(n - 2)$  is true then  $P(n - 1)$  is true. Moreover  $P(n - 2)$  is true. Therefore  $P(n - 1)$  is true.

If  $P(n - 1)$  is true then  $P(n)$  is true. Moreover  $P(n - 1)$  is true.

Therefore  $P(n)$  is true.

The pattern exhibited in these statements should convince you that  $P(n)$  is true for any natural number  $n$ , no matter how large.

We now consider how to apply the Principle of Mathematical Induction to prove that  $S_n = n^2$  for all natural numbers  $n$ , where  $S_n$  denotes the sum of the first  $n$  odd numbers. Obviously  $S_1 = 1$ , so that the conjectured result holds when  $n = 1$ . Suppose that  $S_m = m^2$  for some natural number  $m$ . Then

$$S_{m+1} = S_m + (2m + 1) = m^2 + 2m + 1 = (m + 1)^2$$

Thus if the identity  $S_n = n^2$  holds when  $n = m$  then it also holds when  $n = m + 1$ . We conclude from the Principle of Mathematical Induction that  $S_n = n^2$  for all natural numbers  $n$ .

We can write out the argument rather more formally as follows. For each natural number  $n$ , let  $P(n)$  denote the proposition “ $S_n = n^2$ ”. Clearly, for any given natural number  $n$ , such a proposition  $P(n)$  is either true or false. We want to show that  $P(n)$  is true for all natural numbers  $n$ . This however follows on applying the Principle of Mathematical Induction, given that we have noted that  $P(1)$  is true, and have demonstrated that if  $P(m)$  is true for any natural number  $m$  then  $P(m + 1)$  is also true.

### 1.3 Some examples of proofs using the Principle of Mathematical Induction

**Example** We claim that

$$\sum_{i=1}^n i = \frac{1}{2}n(n + 1)$$

for all natural numbers  $n$ , where

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n.$$

We prove this result using the Principle of Mathematical Induction.

For any natural number  $n$  let  $P(n)$  denote the proposition

$$\text{“} \sum_{i=1}^n i = \frac{1}{2}n(n + 1) \text{”}.$$

One can easily see that the proposition  $P(1)$  is true, since both sides of the above identity reduce to the value 1 in this case.

Suppose that  $P(m)$  is true for some natural number  $m$ . Then

$$\sum_{i=1}^m i = \frac{1}{2}m(m+1).$$

But then

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^m i + (m+1) = \frac{1}{2}m(m+1) + (m+1) = \frac{1}{2}(m+1)(m+2),$$

and therefore the proposition  $P(m+1)$  is also true. We can therefore conclude from the Principle of Mathematical Induction that  $P(n)$  is true for all natural numbers, which is the result we set out to prove.

**Example** We prove by induction on  $n$  that

$$\sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all natural numbers  $n$ , where

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2.$$

To achieve this, we have to verify that the formula holds when  $n = 1$ , and that if the formula holds when  $n = m$  for some natural number  $m$ , then the formula holds when  $n = m + 1$ .

The formula does indeed hold when  $n = 1$ , since  $1 = \frac{1}{6} \times 1 \times 2 \times 3$ .

Suppose that the formula holds when  $n = m$ . Then

$$\sum_{i=1}^m i^2 = \frac{1}{6}m(m+1)(2m+1).$$

But then

$$\begin{aligned} \sum_{i=1}^{m+1} i^2 &= \sum_{i=1}^m i^2 + (m+1)^2 \\ &= \frac{1}{6}m(m+1)(2m+1) + (m+1)^2 \\ &= \frac{1}{6}(m+1)(m(2m+1) + 6(m+1)) = \frac{1}{6}(m+1)(2m^2 + 7m + 6) \\ &= \frac{1}{6}(m+1)(m+2)(2m+3), \end{aligned}$$

and therefore the formula holds when  $n = m+1$ . The required result therefore follows using the Principle of Mathematical Induction.

**Example** We prove by induction on  $n$  that

$$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n+3) = \frac{1}{3}n(n+1)(n+5).$$

for all natural numbers  $n$ . The left hand side of the above identity may be written as  $\sum_{i=1}^n i(i+3)$ .

The required identity

$$\sum_{i=1}^n i(i+3) = \frac{1}{3}n(n+1)(n+5)$$

holds when  $n = 1$ , since both sides are then equal to 4. Suppose that this identity holds when  $n$  is equal to some natural number  $m$ , so that

$$\sum_{i=1}^m i(i+3) = \frac{1}{3}m(m+1)(m+5).$$

Then

$$\begin{aligned} \sum_{i=1}^{m+1} i(i+3) &= \sum_{i=1}^m i(i+3) + (m+1)(m+4) \\ &= \frac{1}{3}m(m+1)(m+5) + (m+1)(m+4) \\ &= \frac{1}{3}(m+1)(m(m+5) + 3(m+4)) \\ &= \frac{1}{3}(m+1)(m^2 + 8m + 12) \\ &= \frac{1}{3}(m+1)(m+2)(m+6), \end{aligned}$$

and therefore the required identity  $\sum_{i=1}^n i(i+3) = \frac{1}{3}n(n+1)(n+5)$  holds when  $n = m+1$ . It now follows from the Principle of Mathematical Induction that this identity holds for all natural numbers  $m$ .

**Example** We can use the Principle of Mathematical Induction to prove that

$$\sum_{k=1}^n 5^k k = \frac{5}{16} \left( (4n-1)5^n + 1 \right).$$

for all natural numbers  $n$ . This equality holds when  $n = 1$ , since both sides are then equal to 5. Suppose that the equality holds when  $n = m$  for some natural number  $m$ , so that

$$\sum_{k=1}^m 5^k k = \frac{5}{16} \left( (4m-1)5^m + 1 \right).$$

Then

$$\begin{aligned}
\sum_{k=1}^{m+1} 5^k k &= \sum_{k=1}^m 5^k k + 5^{m+1}(m+1) \\
&= \frac{5}{16} \left( (4m-1)5^m + 1 \right) + 5^{m+1}(m+1) \\
&= \frac{5}{16} \left( (4m-1)5^m + 1 + 16(m+1)5^m \right) \\
&= \frac{5}{16} \left( (20m+15)5^m + 1 \right) = \frac{5}{16} \left( (4m+3)5^{m+1} + 1 \right) \\
&= \frac{5}{16} \left( (4(m+1)-1)5^{m+1} + 1 \right).
\end{aligned}$$

and thus the equality holds when  $n = m + 1$ . It follows from the Principle of Mathematical Induction that the equality holds for all natural numbers  $n$ .

**Example** We now use Principle of Mathematical Induction to prove that  $6^n - 1$  is divisible by 5 for all natural numbers  $n$ . The result is clearly true when  $n = 1$ . Suppose that the result is true when  $n = m$  for some natural number  $m$ . Then  $6^m - 1$  is divisible by 5. But then

$$6^{m+1} - 1 = 6^{m+1} - 6^m + (6^m - 1) = 5 \times 6^m + (6^m - 1),$$

and therefore  $6^{m+1} - 1$  is also divisible by 5. It therefore follows that  $6^n - 1$  is divisible by 5 for all natural numbers  $n$ .

**Example** We can use the Principle of Mathematical Induction to prove that  $(2n)! < 4^n(n!)^2$  for all natural numbers  $n$ . This inequality holds when  $n = 1$ , since in that case  $(2n)! = 2! = 2$  and  $4^n(n!)^2 = 4$ . Suppose that the inequality holds when  $n = m$  for some natural number  $m$ . Then  $(2m)! < 4^m(m!)^2$ . Now

$$(2(m+1))! = (2m+2)! = (2m)!(2m+1)(2m+2).$$

Also

$$4^{m+1}((m+1)!)^2 = 4(4^m(m!)^2)(m+1)^2.$$

Moreover

$$(2m+1)(2m+2) < (2m+2)^2 = 4(m+1)^2.$$

On multiplying together the two inequalities

$$(2m)! < 4^m(m!)^2 \quad \text{and} \quad (2m+1)(2m+2) < 4(m+1)^2$$

(which we are allowed to do since the quantities on both sides of these inequalities are strictly positive), we find that

$$(2m)!(2m+1)(2m+2) < 4(4^m(m!)^2)(m+1)^2.$$

Thus if the inequality  $(2n)! < 4^n(n!)^2$  holds when  $n = m$  then it also holds when  $n = m+1$ . We conclude from the Principle of Mathematical Induction that it must hold for all natural numbers  $n$ .

**Example** We can use the Principle of Mathematical Induction to prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 > \frac{1}{4}(n^4 + 2n^3)$$

for all natural numbers  $n$ . This inequality holds when  $n = 1$ , since the left hand side is then equal to 1, and the right hand side is equal to  $\frac{3}{4}$ . Suppose that the inequality holds when  $n = m$  for some natural number  $m$ , so that

$$\sum_{i=1}^m i^3 > \frac{1}{4}(m^4 + 2m^3).$$

Then

$$\begin{aligned} \sum_{i=1}^{m+1} i^3 &= \sum_{i=1}^m i^3 + (m+1)^3 \\ &> \frac{1}{4}(m^4 + 2m^3) + (m+1)^3 \\ &= \frac{1}{4}(m^4 + 2m^3 + 4(m+1)^3) \\ &= \frac{1}{4}(m^4 + 6m^3 + 12m^2 + 12m + 4) \end{aligned}$$

Now

$$\begin{aligned} (m+1)^4 + 2(m+1)^3 &= (m^4 + 4m^3 + 6m^2 + 4m + 1) \\ &\quad + (2m^3 + 6m^2 + 6m + 2) \\ &= m^4 + 6m^3 + 12m^2 + 10m + 3 \end{aligned}$$

But  $12m + 4 > 10m + 3$  (since  $m > 0$ ), and therefore

$$m^4 + 6m^3 + 12m^2 + 12m + 4 > (m+1)^4 + 2(m+1)^3.$$

It follows that

$$\sum_{i=1}^{m+1} i^3 > \frac{1}{4}(m^4 + 6m^3 + 12m^2 + 12m + 4) > \frac{1}{4}((m+1)^4 + 2(m+1)^3).$$



Thus if the inequality

$$\sum_{i=1}^n i^3 > \frac{1}{4}(n^4 + 2n^3)$$

holds when  $n = m$  for some natural number  $m$ , then it also holds when  $n = m + 1$ . It follows from the Principle of Mathematical Induction that this identity holds for all natural numbers  $n$ .

## Problems

1. Prove by induction on  $n$  that the product  $1 \times 3 \times \cdots \times (2n - 1)$  of the first  $n$  odd natural numbers is equal to  $\frac{(2n)!}{2^n n!}$ .
2. Prove by induction on  $n$  that  $(3n)! > 2^{6n-4}$  for all natural numbers  $n$ .
3. Prove by induction on  $n$  that

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1},$$

for all natural numbers  $n$ , where

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

4. Prove by induction on  $n$  that  $n! > 3^{n-2}$  for all natural numbers  $n$  satisfying  $n \geq 3$  (where  $n!$  denotes the product of all natural numbers from 1 to  $n$  inclusive).
5. Prove by induction on  $n$  that

$$\sum_{i=1}^n 4^{i-1} i(i+1) = \frac{1}{27}((9n^2 + 3n + 2)4^n - 2)$$

for all natural numbers  $n$ .

6. Prove by induction on  $n$  that  $(n!)^2 \geq 2^{2n-2}$  for all natural numbers  $n$  (where  $n!$  denotes the product of all natural numbers from 1 to  $n$  inclusive).

7. Prove by induction on  $n$  that

$$\sum_{i=1}^n \frac{2i+1}{i^2(i+1)^2} = \frac{n^2+2n}{(n+1)^2}.$$

8. Prove by induction on  $n$  that  $(3n)! \geq \frac{1}{20} \times 120^n$  for all natural numbers  $n$  (where  $n!$  denotes the product of all natural numbers from 1 to  $n$  inclusive).

9. Prove by induction on  $n$  that

$$\sum_{i=1}^n (i^3 + i) > \frac{1}{4}(n^4 + n)$$

for all natural numbers  $n$ .

10. Use the Method of Mathematical Induction to prove that

$$\sum_{k=1}^n \frac{1}{(k+2)(k+3)(k+4)} = \frac{1}{24} - \frac{1}{2(n+3)(n+4)}.$$

for all positive integers  $n$ .