MA232A—Euclidean and Non-Euclidean Geometry
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Vector Algebra and Spherical Trigonometry

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3.1. Vectors in Three-Dimensional Euclidean Space

A 3-dimensional *vector* $\mathbf{v}$ in the vector space $\mathbb{R}^3$ can be represented as a triple (v_1, v_2, v_3) of real numbers. Vectors in $\mathbb{R}^3$ are added together, subtracted from one another, and multiplied by real numbers by the usual rules, so that

$$\begin{aligned}(u_1, u_2, u_3) + (v_1, v_2, v_3) &= (u_1 + v_1, u_2 + v_2, u_3 + v_3),\\ (u_1, u_2, u_3) - (v_1, v_2, v_3) &= (u_1 - v_1, u_2 - v_2, u_3 - v_3),\\ t(u_1, u_2, u_3) &= (tu_1, tu_2, tu_3)\end{aligned}$$

for all vectors (u_1, u_2, u_3) and (v_1, v_2, v_3) in $\mathbb{R}^3$, and for all real numbers t.

The operation of vector addition is commutative and associative. Also $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$, where $\mathbf{0} = (0, 0, 0)$, and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^3$, where $-(v_1, v_2, v_3) = (-v_1, v_2, v_3)$ for all $(v_1, v_2, v_3) \in \mathbb{R}^3$. Moreover

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}), \quad t(\mathbf{u} + \mathbf{v}) = t\mathbf{u} + t\mathbf{v}, \quad (s + t)\mathbf{v} = s\mathbf{v} + t\mathbf{v},$$

$$s(t\mathbf{v}) = (st)\mathbf{v}, \quad 1\mathbf{v} = \mathbf{v}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ and $s, t \in \mathbb{R}$.

The set of all vectors in three-dimensional space, with the usual operations of vector addition and of scalar multiplication constitute a three-dimensional real vector space.

The *Euclidean norm* $|\mathbf{v}|$ of a vector $\mathbf{v}$ is defined so that if $\mathbf{v} = (v_1, v_2, v_3)$ then

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

The *scalar product* $\mathbf{u} \,.\, \mathbf{v}$ and the *vector product* $\mathbf{u} \times \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$ are defined such that

$$\begin{aligned} (u_1, u_2, u_3) \,.\, (v_1, v_2, v_3) &= u_1 v_1 + u_2 v_2 + u_3 v_3, \\ (u_1, u_2, u_3) \times (v_1, v_2, v_3) & \\ &= (u_2 v_3 - u_3 v_2,\; u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \end{aligned}$$

for all vectors (u_1, u_2, u_3) and (v_1, v_2, v_3) in $\mathbb{R}^3$. Then

$$(\mathbf{u}+\mathbf{v})\,.\,\mathbf{w} = \mathbf{u}\,.\,\mathbf{w} + \mathbf{v}\,.\,\mathbf{w}, \quad \mathbf{u}\,.\,(\mathbf{v}+\mathbf{w}) = \mathbf{u}\,.\,\mathbf{v} + \mathbf{u}\,.\,\mathbf{w},$$

$$(\mathbf{u}+\mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w},$$

$$(t\mathbf{u})\,.\,\mathbf{v} = \mathbf{u}\,.\,(t\mathbf{v}) = t(\mathbf{u}\,.\,\mathbf{v}), \quad (t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u}\,.\,\mathbf{v} = \mathbf{v}\,.\,\mathbf{u}, \quad \mathbf{u}\,.\,\mathbf{u} = |\mathbf{u}|^2, \quad \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the *standard basis* of $\mathbb{R}^3$ are defined so that

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Then

$$\mathbf{i} \,.\, \mathbf{i} = \mathbf{j} \,.\, \mathbf{j} = \mathbf{k} \,.\, \mathbf{k} = 1,$$

$$\mathbf{i} \,.\, \mathbf{j} = \mathbf{j} \,.\, \mathbf{i} = \mathbf{j} \,.\, \mathbf{k} = \mathbf{k} \,.\, \mathbf{j} = \mathbf{k} \,.\, \mathbf{i} = \mathbf{i} \,.\, \mathbf{k} = 0,$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \quad \mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}.$$

Let A and B be points in three-dimensional Euclidean space. These points may be represented in Cartesian coordinates so that

$$A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3).$$

The *displacement vector* $\overrightarrow{AB}$ from A to B is defined such that

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3).$$

If A, B and C are points in three-dimensional Euclidean space then

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

Points A, B, C and D of three-dimensional Euclidean space are the vertices of a parallelogram (labelled in clockwise or anticlockwise) order if and only if $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Let the origin O be the point with Cartesian coordinates. The *position vector* of a point A (with respect to the chosen origin) is defined to be the displacement vector $\overrightarrow{OA}$.

3.2. Geometrical Interpretation of the Scalar Product

Let **u** and **v** be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. The scalar product $\mathbf{u} \,.\, \mathbf{v}$ of the vectors **u** and **v** is then given by the formula

$$\mathbf{u} \,.\, \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Proposition 3.1

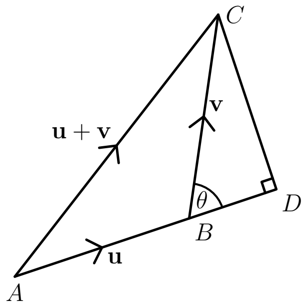
Let $\mathbf{u}$ *and* $\mathbf{v}$ *be non-zero vectors in three-dimensional space. Then their scalar product* $\mathbf{u} \,.\, \mathbf{v}$ *is given by the formula*

$$\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta,$$

where θ *denotes the angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof

Suppose first that the angle θ between the vectors **u** and **v** is an acute angle, so that $0 < \theta < \frac{1}{2}\pi$. Let us consider a triangle ABC, where $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, and thus $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$. Let ADC be the right-angled triangle constructed as depicted in the figure below, so that the line AD extends AB and the angle at D is a right angle.



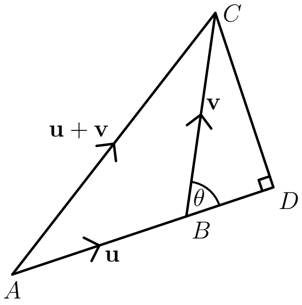
Then the lengths of the line segments AB, BC, AC, BD and CD may be expressed in terms of the lengths $|\mathbf{u}|$, $|\mathbf{v}|$ and $|\mathbf{u}+\mathbf{v}|$ of the displacement vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ and the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ by means of the following equations:

$$AB = |\mathbf{u}|, \quad BC = |\mathbf{v}|, \quad AC = |\mathbf{u}+\mathbf{v}|,$$

$$BD = |\mathbf{v}| \cos\theta \quad \text{and} \quad DC = |\mathbf{v}| \sin\theta.$$

Then

$$AD = AB + BD = |\mathbf{u}| + |\mathbf{v}| \cos\theta.$$



The triangle ADC is a right-angled triangle with hypotenuse AC. It follows from Pythagoras' Theorem that

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= AC^2 = AD^2 + DC^2 = (|\mathbf{u}| + |\mathbf{v}| \cos\theta)^2 + |\mathbf{v}| \sin^2\theta \\ &= |\mathbf{u}|^2 + 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta + |\mathbf{v}| \cos^2\theta + |\mathbf{v}| \sin^2\theta \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta, \end{aligned}$$

because $\cos^2\theta + \sin^2\theta = 1$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

and therefore

$$\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 &= (u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2 \\ &= u_1^2 + 2u_1 v_1 + v_1^2 + u_2^2 + 2u_2 v_2 + v_2^2 \\ &\qquad + u_3^2 + 2u_3 v_3 + v_3^2 \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \,.\, \mathbf{v}. \end{aligned}$$

On comparing the expressions for $|\mathbf{u} + \mathbf{v}|^2$ given by the above equations, we see that $\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$ when $0 < \theta < \frac{1}{2}\pi$.

The identity $\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ clearly holds when $\theta = 0$ and $\theta = \pi$. Pythagoras' Theorem ensures that it also holds when the angle θ is a right angle (so that $\theta = \frac{1}{2}\pi$. Suppose that $\frac{1}{2}\pi < \theta < \pi$, so that the angle θ is obtuse. Then the angle between the vectors $\mathbf{u}$ and $-\mathbf{v}$ is acute, and is equal to $\pi - \theta$. Moreover $\cos(\pi - \theta) = -\cos\theta$ for all angles θ. It follows that

$$\mathbf{u} \,.\, \mathbf{v} = -\mathbf{u} \,.\, (-\mathbf{v}) = -|\mathbf{u}|\,|\mathbf{v}|\,\cos(\pi - \theta) = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$$

when $\frac{1}{2}\pi < \theta < \pi$. We have therefore verified that the identity $\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ holds for all non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, as required. ∎

Corollary 3.2

Two non-zero vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in three-dimensional space are perpendicular if and only if* $\mathbf{u} \,.\, \mathbf{v} = 0$.

Proof

It follows directly from Proposition 3.1 that $\mathbf{u} \,.\, \mathbf{v} = 0$ if and only if $\cos\theta = 0$, where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. This is the case if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

3.3. Geometrical Interpretation of the Vector Product

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in three-dimensional space, with Cartesian components given by the formulae $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. The vector product $\mathbf{a} \times \mathbf{b}$ is then determined by the formula

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

Proposition 3.3

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in three-dimensional space* $\mathbb{R}^3$*. Then their vector product* $\mathbf{a} \times \mathbf{b}$ *is a vector of length* $|\mathbf{a}|\,|\mathbf{b}|\,|\sin\theta|$*, where* θ *denotes the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$*. Moreover the vector* $\mathbf{a} \times \mathbf{b}$ *is perpendicular to the vectors* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Proof

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, and let l denote the length $|\mathbf{a} \times \mathbf{b}|$

of the vector $\mathbf{a} \times \mathbf{b}$. Then

$$\begin{aligned}
l^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= a_2^2 b_3^2 + a_3^2 b_2^2 - 2 a_2 a_3 b_2 b_3 \\
&\quad + a_3^2 b_1^2 + a_1^2 b_3^2 - 2 a_3 a_1 b_3 b_1 \\
&\quad + a_1^2 b_2^2 + a_2^2 b_1^2 - 2 a_1 a_2 b_1 b_2 \\
&= a_1^2 (b_2^2 + b_3^2) + a_2^2 (b_1^2 + b_3^2) + a_3^2 (b_1^2 + b_2^2) \\
&\quad - 2 a_2 a_3 b_2 b_3 - 2 a_3 a_1 b_3 b_1 - 2 a_1 a_2 b_1 b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - a_1^2 b_1^2 - a_2^2 b_2^2 - a_3^2 b_3^2 - 2 a_2 b_2 a_3 b_3 \\
&\quad - 2 a_3 b_3 a_1 b_1 - 2 a_1 b_1 a_2 b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \,.\, \mathbf{b})^2
\end{aligned}$$

since

$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\mathbf{b}|^2 = b_1^2 + b_2^2 + b_3^2, \quad \mathbf{a} . \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

But $\mathbf{a} . \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$ (Proposition 3.1). Therefore

$$l^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta$$

(since $\sin^2\theta + \cos^2\theta = 1$ for all angles θ) and thus
$l = |\mathbf{a}|\,|\mathbf{b}|\,|\sin\theta|$. Also

$$\mathbf{a} . (\mathbf{a} \times \mathbf{b}) = a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) = 0$$

and

$$\mathbf{b} . (\mathbf{a} \times \mathbf{b}) = b_1(a_2 b_3 - a_3 b_2) + b_2(a_3 b_1 - a_1 b_3) + b_3(a_1 b_2 - a_2 b_1) = 0$$

and therefore the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (Corollary 3.2), as required. ∎

3.4. Scalar Triple Products

Given three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in three-dimensional space, we can form the *scalar triple product* $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$. This quantity can be expressed as the determinant of a 3×3 matrix whose rows contain the Cartesian components of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Indeed

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\; v_3 w_1 - v_1 w_3,\; v_1 w_2 - v_2 w_1),$$

and thus

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1).$$

The quantity on the right hand side of this equality defines the determinant of the 3×3 matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

We have therefore obtained the following result.

Proposition 3.4

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Corollary 3.5

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\begin{aligned} \mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \,.\, (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \,.\, (\mathbf{u} \times \mathbf{v}) \\ &= -\mathbf{u} \,.\, (\mathbf{w} \times \mathbf{v}) = -\mathbf{v} \,.\, (\mathbf{u} \times \mathbf{w}) = -\mathbf{w} \,.\, (\mathbf{v} \times \mathbf{u}). \end{aligned}$$

Proof

The basic theory of determinants ensures that 3×3 determinants are unchanged under cyclic permutations of their rows by change sign under transpositions of their rows. These identities therefore follow directly from Proposition 3.4. ∎

3.5. The Vector Triple Product Identity

Proposition 3.6 (Vector Triple Product Identity)

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\, \mathbf{w}$$

and

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\, \mathbf{u}.$$

Proof

Let $\mathbf{q} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$, and $\mathbf{q} = (q_1, q_2, q_3)$. Then

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

and hence $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{q} = (q_1, q_2, q_3)$, where

$$\begin{aligned} q_1 &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= (u_2 w_2 + u_3 w_3) v_1 - (u_2 v_2 + u_3 v_3) w_1 \\ &= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_1 \\ &= (\mathbf{u} \,.\, \mathbf{w}) v_1 - (\mathbf{u} \,.\, \mathbf{v}) w_1 \end{aligned}$$

Similarly

$$q_2 = (\mathbf{u} \,.\, \mathbf{w}) v_2 - (\mathbf{u} \,.\, \mathbf{v}) w_2$$

and

$$q_3 = (\mathbf{u} \,.\, \mathbf{w}) v_3 - (\mathbf{u} \,.\, \mathbf{v}) w_3$$

(In order to verify the formula for q_2 with an minimum of calculation, take the formulae above involving q_1, and cyclicly permute the subcripts 1, 2 and 3, replacing 1 by 2, 2 by 3, and 3 by 1. A further cyclic permutation of these subscripts yields the formula for q_3.) It follows that

$$\mathbf{q} = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w},$$

as required, since we have shown that the Cartesian components of the vectors on either side of this identity are equal. Thus

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w}.$$

On replacing $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ by $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v}$ respectively, we find that

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \,.\, \mathbf{v})\,\mathbf{u} - (\mathbf{w} \,.\, \mathbf{u})\,\mathbf{v}.$$

It follows that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\,\mathbf{u},$$

as required. ∎

Remark

When recalling these identities for use in applications, it is often helpful to check that the summands on the right hand side have the correct sign by substituting, for example, **i**, **j** and **i** for **u**, **v** and **w**, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Thus, for example, $(\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = \mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $(\mathbf{i}.\mathbf{i})\mathbf{j} - (\mathbf{j}.\mathbf{i})\mathbf{i} = \mathbf{j}$. This helps check that the summands on the right hand side of the identity $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\, \mathbf{u}$ have been chosen with the correct sign (assuming that these summands have opposite signs).

We present below a second proof making use of the following standard identity.

Proposition 3.7

Let $\varepsilon_{i,j,k}$ and $\delta_{i,j}$ be defined for $i, j, k \in \{1, 2, 3\}$ such that

$$\varepsilon_{i,j,k} = \begin{cases} 1 & \text{if } (i,j,k) \in \{(1,2,3),\ (2,3,1),\ (3,1,2)\}; \\ -1 & \text{if } (i,j,k) \in \{(1,3,2),\ (2,1,3),\ (3,2,1)\}; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{i=1}^{3} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m}$$

for all $i, j, m \in \{1, 2, 3\}$.

Proof

Suppose that $j = k$. Then $\varepsilon_{i,j,k} = 0$ for $i = 1, 2, 3$ and thus the left hand side is zero. The right hand side is also zero in this case, because

$$\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = \delta_{j,m}\delta_{k,n} - \delta_{k,n}\delta_{j,m} = 0$$

when $j = k$. Thus $\sum_{i=1}^{3} \varepsilon_{i,j,k}\,\varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$ when $j = k$. Similarly $\sum_{i=1}^{3} \varepsilon_{i,j,k}\,\varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$ when $m = n$.

Next suppose that $j \neq k$ and $m \neq n$ but $\{j, k\} \neq \{m, n\}$. In this case the single value of i in $\{1, 2, 3\}$ for which $\varepsilon_{i,j,k} \neq 0$ does not coincide with the single value of i for which $\varepsilon_{i,m,n} \neq 0$, and therefore $\sum_{i=1}^{3} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n} = 0$. Moreover either $j \notin \{m, n\}$, in which case $\delta_{j,m} = \delta_{j,n} = 0$ and thus $\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$, or else $k \notin \{m, n\}$, in which case $\delta_{k,m} = \delta_{k,n} = 0$ and thus $\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$.

It follows from all the cases considered above that $\sum_{i=1}^{3} \varepsilon_{i,j,k}\,\varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$ unless both $j \neq k$ and $\{j,k\} = \{m,n\}$. Suppose then that $j \neq k$ and $\{j,k\} = \{m,n\}$. Then there is a single value of i for which $\varepsilon_{i,j,k} \neq 0$. For this particular value of i we find that

$$\varepsilon_{i,j,k}\,\varepsilon_{i,m,n} = \begin{cases} 1 & \text{if } j \neq k,\ j = m \text{ and } k = n; \\ -1 & \text{if } j \neq k,\ j = n \text{ and } k = m. \end{cases}$$

It follows that, in the cases where $j \neq k$ and $\{j,k\} = \{m,n\}$,

$$\begin{aligned} \sum_{i=1}^{3} \varepsilon_{i,j,k}\,\varepsilon_{i,m,n} &= \begin{cases} 1 & \text{if } j \neq k,\ j = m \text{ and } k = n, \\ -1 & \text{if } j \neq k,\ j = n \text{ and } k = m, \\ 0 & \text{otherwise}, \end{cases} \\ &= \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m}, \end{aligned}$$

as required. ∎

Second Proof of Proposition 3.6

Let $\mathbf{p} = \mathbf{v} \times \mathbf{w}$ and $\mathbf{q} = \mathbf{u} \times \mathbf{p} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and let

$$\mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{v} = (v_1, v_2, v_3), \quad \mathbf{w} = (w_1, w_2, w_3),$$

$$\mathbf{p} = (p_1, p_2, p_3) \quad \text{and} \quad \mathbf{q} = (q_1, q_2, q_3).$$

The definition of the vector product ensures that $p_i = \sum_{j,k=1}^{3} \varepsilon_{i,j,k} v_j w_k$ for $i = 1, 2, 3$, where $\varepsilon_{i,j,k}$ and $\delta_{i,j}$ are defined for $i, j, k \in \{1, 2, 3\}$ as described in the statement of Proposition 3.7. It follows that

$$\begin{aligned}
q_m &= \sum_{n,i=1}^{3} \varepsilon_{m,n,i}\, u_n p_i = \sum_{n,i,j,k=1}^{3} \varepsilon_{m,n,i}\varepsilon_{i,j,k}\, u_n v_j w_k \\
&= \sum_{n,j,k=1}^{3} \sum_{i=1}^{3} \varepsilon_{i,m,n}\varepsilon_{i,j,k}\, u_n v_j w_k \\
&= \sum_{n,j,k=1}^{3} (\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m}) u_n v_j w_k \\
&= \sum_{n,k=1}^{3} \delta_{k,n}\, v_m u_n w_k - \sum_{n,j=1}^{3} \delta_{j,n}\, u_n v_j w_m \\
&= v_m \sum_{k=1}^{3} u_k w_k - w_m \sum_{j=1}^{3} u_j v_j \\
&= (\mathbf{u} \,.\, \mathbf{w}) v_m - (\mathbf{u} \,.\, \mathbf{v}) w_m
\end{aligned}$$

for $m = 1, 2, 3$, and therefore

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{q} = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\, \mathbf{w},$$

as required. ∎

Remark

The identity

$$\alpha \mathrm{S} \, . \, \alpha' \alpha'' - \alpha' \mathrm{S} . \alpha'' \alpha = \mathrm{V}(\mathrm{V} \, . \, \alpha \alpha' \, . \, \alpha'')$$

occurs as equation (12) in article 22 of William Rowan Hamilton's *On Quaternions, or on a new System of Imaginaries in Algebra*, published in the *Philosophical Magazine* in August 1846. Hamilton noted in that paper that this identity "will be found to have extensive applications."

In Hamilton's quaternion algebra, vectors in three-dimensional space are represented as pure imaginary quaternions and are denoted by Greek letters. Thus α, α' and α'' denote (in Hamilton's notation) three arbitrary vectors. The product of two vectors α' and α'' in Hamilton's system is a quaternion which is the sum of a *scalar part* $\mathrm{S}\,.\,\alpha\alpha'$ and a *vector part* $\mathrm{V}.\alpha\alpha'$. (The scalar and vector parts of a quaternion are the analogues, in Hamilton's quaternion algebra, of the real and imaginary parts of a complex number.)

Now a quaternion can be represented in the form $s + u_1 i + u_2 j + u_3 k$ where s, u_1, u_2, u_3 are real numbers. The operations of quaternion addition, quaternion subtraction and scalar multiplication by real numbers are defined so that the space $\mathbb{H}$ of quaternions is a four-dimensional vector space over the real numbers with basis 1, i, j, k. The operation of quaternion multiplication is defined so that quaternion multiplication is distributive over addition and is determined by the identities

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

that Hamilton formulated in 1843. It then transpires that the operation of quaternion multiplication is associative.

Hamilton described his discovery of the quaternion algebra in a letter to P.G. Tait dated October 15, 1858 as follows:—

> *...P.S.—To-morrow will be the 15th birthday of the Quaternions. They started into life, or light, full grown, on [Monday] the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, which my boys have since called the Quaternion Bridge. That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since.*

I pulled out on the spot a pocket-book, which still exists, and made an entry, on which, at the very moment, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a problem to have been at that moment solved—an intellectual want relieved—which had haunted me for at least fifteen years before.

Let quaternions q and r be defined such that $q = s + u_1 i + u_2 j + u_3 k$ and $r = t + v_1 i + v_2 j + v_3 k$, where $s, t, u_1, u_2, u_3, v_1, v_2, v_3$ are real numbers. We can then write $q = s + \alpha$ and $r = t + \beta$, where

$$\alpha = u_1 i + u_2 j + u_3 k, \quad \beta = v_1 i + v_2 j + v_3 k.$$

Hamilton then defined the *scalar part* of the quaternion q to be the real number s, and the *vector part* of the quaternion q to be the quaternion α determined as described above.

The Distributive Law for quaternion multiplication and the identities for the products of i, j and k then ensure that

$$qr = st + \mathrm{S}\,.\,\alpha\beta + s\beta + t\alpha + \mathrm{V}\,.\,\alpha\beta,$$

where

$$\mathrm{S}\,.\,\alpha\beta = -(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

and

$$\mathrm{V}\,.\,\alpha\beta = (u_2 v_3 - u_3 v_2)i + (u_3 v_1 - u_1 v_3)j + (u_1 v_2 - u_2 v_1)k.$$

Thus the *scalar part* $\mathrm{S}\,.\,\alpha'\alpha''$ of the quaternion product $\alpha'\alpha''$ represents the negative of the scalar product of the vectors α' and α'', and the *vector part* $\mathrm{V}\,.\,\alpha'\alpha''$ represents the vector product of the quaternion $\alpha\alpha'$. Thus Hamilton's identity can be represented, using the now customary notation for the scalar and vector products, as follows:—

$$-\alpha(\alpha'\,.\,\alpha'') + \alpha'(\alpha''\,.\,\alpha) = (\alpha \times \alpha') \times \alpha''.$$

Hamilton's identity of 1846 (i.e., equation (12) in article 22 of *On quaternions*) is thus the Vector Triple Product Identity stated in Proposition 3.6.

Corollary 3.8

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^3$. *Then*

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{u}.(\mathbf{v} \times \mathbf{w}))\mathbf{u}.$$

Proof

Using the Vector Triple Product Identity (Proposition 3.6) and basic properties of the scalar triple product Corollary 3.5, we find that

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) &= (\mathbf{u}.(\mathbf{u} \times \mathbf{w}))\mathbf{v} - (\mathbf{v}.(\mathbf{u} \times \mathbf{w}))\mathbf{u} \\ &= (\mathbf{u}.(\mathbf{v} \times \mathbf{w}))\mathbf{u},\end{aligned}$$

as required. ∎

3.6. Lagrange's Quadruple Product Identity

Proposition 3.9 (Lagrange's Quadruple Product Identity)

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{z}$ *be vectors in* $\mathbb{R}^3$. *Then*

$$(\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \,.\, \mathbf{w})(\mathbf{v} \,.\, \mathbf{z}) - (\mathbf{u} \,.\, \mathbf{z})(\mathbf{v} \,.\, \mathbf{w}).$$

Proof

Using the Vector Triple Product Identity (Proposition 3.6) and basic properties of the scalar triple product Corollary 3.5, we find that

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{w} \times \mathbf{z}) &= \mathbf{z}.((\mathbf{u} \times \mathbf{v}) \times \mathbf{w}) \\
&= \mathbf{z}.((\mathbf{u} \,.\, \mathbf{w})\mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\mathbf{u}) \\
&= (\mathbf{u} \,.\, \mathbf{w})(\mathbf{v} \,.\, \mathbf{z}) - (\mathbf{u} \,.\, \mathbf{z})(\mathbf{v} \,.\, \mathbf{w}),
\end{aligned}$$

as required. ∎

Remark

Substituting **i**, **j**, **i** and **j** for **u**, **v**, **w** and **z** respectively, where

$$\mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \mathbf{k} = (0,0,1),$$

we find that $(\mathbf{i} \times \mathbf{j}) \,.\, (\mathbf{i} \times \mathbf{j}) = \mathbf{k}.\mathbf{k} = 1$ and $(\mathbf{i} \,.\, \mathbf{i})(\mathbf{j} \,.\, \mathbf{j}) - (\mathbf{i} \,.\, \mathbf{j})(\mathbf{j} \,.\, \mathbf{i}) = 1 - 0 = 1$. This helps check that the summands on the right hand side have been allocated the correct sign.

Second Proof of Proposition 3.9

Let

$$\mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{v} = (v_1, v_2, v_3),$$

$$\mathbf{w} = (w_1, w_2, w_3), \quad \mathbf{z} = (z_1, z_2, z_3),$$

and let $\varepsilon_{i,j,k}$ and $\delta_{i,j}$ be defined for $i, j, k \in \{1, 2, 3\}$ as described in the statement of Proposition 3.7. Then the components of $\mathbf{u} \times \mathbf{v}$ are the values of $\displaystyle\sum_{j,k=1}^{3} \varepsilon_{i,j,k} u_j v_k$ for $i = 1, 2, 3$. It follows from Proposition 3.7 that

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{w} \times \mathbf{z}) &= \sum_{i,j,k,m,n} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n}\, u_j v_k w_m z_n \\
&= \sum_{j,k,m,n} (\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m})\, u_j v_k w_m z_n \\
&= \sum_{j,k} (u_j v_k w_j z_k - u_j v_k w_k z_j) \\
&= (\mathbf{u} \,.\, \mathbf{w})(\mathbf{v} \,.\, \mathbf{z}) - (\mathbf{u} \,.\, \mathbf{z})(\mathbf{v} \,.\, \mathbf{w}),
\end{aligned}$$

as required. ∎

3.7. Some Applications of Vector Algebra to Spherical Trigometry

Let S^2 be the unit sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

in three-dimensional Euclidean space $\mathbb{R}^3$. Each point of S^2 may be represented in the form

$$(\sin\theta \, \cos\varphi, \sin\theta \, \sin\varphi, \cos\theta).$$

Let I, J and K denote the points of S^2 defined such that

$$I = (1, 0, 0), \quad J = (0, 1, 0), \quad K = (0, 0, 1).$$

We take the origin O of Cartesian coordinates to be located at the centre of the sphere. The position vectors of the points I, J and K are then the standard unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

It may be helpful to regard the point K as representing the "north pole" of the sphere. The "equator" is then the great circle consisting of those points (x, y, z) of S^2 for which $z = 0$. Every point P of S^2 is the pole of a great circle on S^2 consisting of those points of S^2 whose position vectors are orthogonal to the position vector $\mathbf{p}$ of the point P.

Let A and B be distinct points of S^2 with position vectors $\mathbf{u}$ and $\mathbf{v}$ respectively. We denote by $\sin AB$ and $\cos AB$ the sine and cosine of the angles between the lines joining the centre of the sphere to the points A and B.

Lemma 3.10

Let A and B be points on the unit sphere S^2 in $\mathbb{R}^3$, and let $\mathbf{u}$ and $\mathbf{v}$ denote the displacement vectors of those points from the centre of the sphere. Then

$$\mathbf{u} \,.\, \mathbf{v} = \cos AB$$

and

$$\mathbf{u} \times \mathbf{v} = \sin AB \, \mathbf{n}_{A,B},$$

where $\mathbf{n}_{A,B}$ is a unit vector orthogonal to the plane through the centre of the sphere that contains the points A and B.

Proof

The displacement vectors $\mathbf{u}$ and $\mathbf{v}$ of the points A and B from the centre of the sphere satisfy $|\mathbf{u}| = 1$ and $|\mathbf{v}| = 1$ (because the sphere has unit radius). The required identities therefore follows from basic properties of the scalar and vector products stated in Proposition 3.1 and Proposition 3.3. ∎

Lemma 3.11

Let V and W be planes in $\mathbb{R}^3$ that are not parallel, and let $\mathbf{n}_V$ and $\mathbf{v}_W$ be the unit vectors orthogonal to the planes V and W, and let α be the angle between those planes. Then

$$\mathbf{n}_V \,.\, \mathbf{n}_W = \cos\alpha,$$

and

$$\mathbf{n}_V \times \mathbf{n}_W = \sin\alpha\,\mathbf{u},$$

where $\mathbf{u}$ is a unit vector in the direction of the line of intersection of the planes V and W.

Proof

The vectors $\mathbf{n}_V$ and $\mathbf{n}_W$ are not parallel, because the planes are not parallel, and therefore $\mathbf{n}_V \times \mathbf{n}_W$ is a non-zero vector. Let $t = |\mathbf{n}_V \times \mathbf{n}_W|$. Then $\mathbf{n}_V \times \mathbf{n}_W = t\mathbf{u}$, where $\mathbf{u}$ is a unit vector orthogonal to both $\mathbf{n}_V$ and $\mathbf{n}_W$. This vector $\mathbf{u}$ must be parallel to both V and W, and must therefore be parallel to the line of intersection of these two planes. Let $\mathbf{v} = \mathbf{u} \times \mathbf{n}_V$ and $\mathbf{w} = \mathbf{u} \times \mathbf{n}_W$. Then the vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel to the planes V and W respectively, and both vectors are orthogonal to the line of intersection of these planes. It follows that angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ is the angle α between the planes V and W.

Now the vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{n}_V$ and $\mathbf{n}_W$ are all parallel to the plane that is orthogonal to $\mathbf{u}$, the angle between the vectors $\mathbf{v}$ and $\mathbf{n}_V$ is a right angle, and the angle between the vectors $\mathbf{w}$ and $\mathbf{n}_W$ is also a right angle. It follows that the angle between the vectors $\mathbf{n}_V$ and $\mathbf{n}_W$ is equal to the angle α between the vectors $\mathbf{v}$ and $\mathbf{w}$, and therefore

$$\begin{aligned} \mathbf{n}_V \,.\, \mathbf{n}_W &= \mathbf{v} \,.\, \mathbf{w} = \cos\alpha, \\ \mathbf{n}_V \times \mathbf{n}_W &= \mathbf{v} \times \mathbf{w} = \sin\alpha\, \mathbf{u}. \end{aligned}$$

These identities can also be verified by vector algebra. Indeed, using Lagrange's Quadruple Product Identity, we see that

$$\begin{aligned} \mathbf{v} \,.\, \mathbf{w} &= (\mathbf{n}_V \times \mathbf{u}) \,.\, (\mathbf{n}_W \times \mathbf{u}) \\ &= (\mathbf{n}_V \,.\, \mathbf{n}_W)(\mathbf{u} \,.\, \mathbf{u}) - (\mathbf{n}_V \,.\, \mathbf{u})(\mathbf{u} \,.\, \mathbf{n}_W) \\ &= \mathbf{n}_V \,.\, \mathbf{n}_W, \end{aligned}$$

because $\mathbf{u} \,.\, \mathbf{u} = |\mathbf{u}|^2 = 1$ $\mathbf{n}_V \,.\, \mathbf{u} = 0$ and $\mathbf{n}_W \,.\, \mathbf{u} = 0$. Thus $\mathbf{n}_V \,.\, \mathbf{n}_W = \cos\alpha$. Also $\mathbf{n}_V \times \mathbf{n}_W$ is parallel to the unit vector $\mathbf{u}$, and therefore

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= (\mathbf{n}_V \times \mathbf{u}) \times (\mathbf{n}_W \times \mathbf{u}) = (\mathbf{u} \times \mathbf{n}_V) \times (\mathbf{u} \times \mathbf{n}_W) \\ &= (\mathbf{u}.(\mathbf{n}_V \times \mathbf{n}_W))\mathbf{u} = \mathbf{n}_V \times \mathbf{n}_W.\end{aligned}$$

(see Corollary 3.8). It follows that

$$|\mathbf{n}_V \times \mathbf{n}_W| = |\mathbf{v} \times \mathbf{w}| = \sin\alpha,$$

and therefore

$$\mathbf{n}_V \times \mathbf{n}_W = \sin\alpha\, \mathbf{u},$$

as required. $\blacksquare$

Proposition 3.12 (Cosine Rule of Spherical Trigonometry)

Let A, B and C be distinct points on the unit sphere in $\mathbb{R}^3$, let α be the angle at A between the great circle through A and B and the great circle through A and C. Then

$$\cos BC = \cos AB \, \cos AC + \sin AB \, \sin AC \, \cos \alpha.$$

Proof

The angle α at A between the great circle AB and the great circle AC is equal to the angle between the planes through the origin that intersect the unit sphere in those great circles, and this angle is in turn equal to the angle between the normal vectors $\mathbf{n}_{A,B}$ and $\mathbf{n}_{A,C}$ to those planes, and therefore $\mathbf{n}_{A,B} \cdot \mathbf{n}_{A,C} = \cos \alpha$ (see Lemma 3.11). Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ denote the displacement vectors of the points A, B and C respectively from the centre of the sphere. Then

$$\mathbf{u} \times \mathbf{v} = \sin AB \, \mathbf{n}_{A,B}, \quad \mathbf{u} \times \mathbf{w} = \sin AC \, \mathbf{n}_{A,C}.$$

It follows that

$$(\mathbf{u} \times \mathbf{v}).(\mathbf{u} \times \mathbf{w}) = \sin AB \, \sin AC \, \cos \alpha.$$

But it follows from Lagrange's Quadruple Product Identity that Proposition 3.9 that

$$(\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{u} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{u})(\mathbf{v} \,.\, \mathbf{w}) - (\mathbf{u} \,.\, \mathbf{w})(\mathbf{v} \,.\, \mathbf{u}).$$

But $\mathbf{u}.\mathbf{u} = |\mathbf{u}|^2 = 1$, because the point $\mathbf{u}$ lies on the unit sphere. Therefore

$$(\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \,.\, \mathbf{w}) - (\mathbf{u} \,.\, \mathbf{v})(\mathbf{u} \,.\, \mathbf{w}) = \cos BC - \cos AB \cos AC.$$

Equating the two formulae for $(\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{u} \times \mathbf{w})$, we find that

$$\cos BC = \cos AB \, \cos AC + \sin AB \, \sin AC \, \cos \alpha,$$

as required. ∎

Second Proof

Let **u**, **v** and **w** denote the displacement vectors of the points A, B and C respectively from the centre O of the sphere. Without loss of generality, we may assume that the Cartesian coordinate system with origin at the centre O of the sphere has been oriented so that

$$\begin{aligned} \mathbf{u} &= (0, 0, 1), \\ \mathbf{v} &= (\sin AB, 0, \cos AB), \\ \mathbf{w} &= (\sin AC\ \cos\alpha, \sin AC\ \sin\alpha, \cos AC). \end{aligned}$$

Then $|\mathbf{u}| = 1$ and $|\mathbf{v}| = 1$. It follows that

$$\cos BC = \mathbf{v} \,.\, \mathbf{w} = \cos AB\ \cos AC + \sin AB\ \sin AC\ \cos\alpha,$$

as required. ∎

Proposition 3.13 (Gauss)

If A, B, C and D denote four points on the sphere, and η the angle which the arcs AB, CD make at their point of intersection, then we shall have

$$\cos AC \cos BD - \cos AD \cos BC = \sin AB \sin CD \cos \eta.$$

Proof

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$ denote the displacement vectors of the points A, B, C and D from the centre of the sphere. It follows from Lagrange's Quadruple Product Identity (Proposition 3.9) that

$$(\mathbf{u} \,.\, \mathbf{w})(\mathbf{v} \,.\, \mathbf{z}) - (\mathbf{u} \,.\, \mathbf{z})(\mathbf{v} \,.\, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{w} \times \mathbf{z}).$$

Now it follows from the standard properties of the scalar and vector products recorded in the statement of Lemma 3.10 that $\mathbf{u} \,.\, \mathbf{w} = \cos AC$ etc., $\mathbf{u} \times \mathbf{v} = \sin AB\, \mathbf{n}_{A,B}$ and $\mathbf{w} \times \mathbf{z} = \sin CD\, \mathbf{n}_{C,D}$, where $\mathbf{n}_{A,B}$ is a unit vector orthogonal to the plane through the origin containing the points A and B, and $\mathbf{n}_{C,D}$ is a unit vector orthogonal to the plane through the origin containing the points C and D. Now $\mathbf{n}_{A,B} \,.\, \mathbf{n}_{C,D} = \cos \eta$, where $\cos \eta$ is the cosine of the angle η between these two planes (see Lemma 3.11).

This angle is also the angle, at the points of intersection, between the great circles on the sphere that represent the intersection of those planes with the sphere. It follows that

$$\begin{aligned} & \cos AC\ \cos BD - \cos AD\ \cos BC \\ = \; & (\mathbf{u} \,.\, \mathbf{w})(\mathbf{v} \,.\, \mathbf{z}) - (\mathbf{u} \,.\, \mathbf{z})(\mathbf{v} \,.\, \mathbf{w}) \\ = \; & (\mathbf{u} \times \mathbf{v}) \,.\, (\mathbf{w} \times \mathbf{z}) \\ = \; & \sin AB\ \sin CD\, (\mathbf{n}_{A,B} \,.\, \mathbf{n}_{C,D}) \\ = \; & \sin AB\ \sin CD\ \cos \eta, \end{aligned}$$

as required. ∎

Second Proof

(This proof follows fairly closely the proof given by Gauss in the *Disquisitiones Generales circa Superficies Curvas*, published in 1828.) Let the point O be the centre of the sphere, and let P be the point where the great circle passing through AB intersects the great circle passing through CD. The angle η is then the angle between these great circles at the point P. Let the angles between the line OP and the lines OA, OB, OC and OD be denoted by α, β, γ, δ respectively (so that $\cos PA = \cos\alpha$ etc.).

It then follows from the Cosine Rule of Spherical Trigonometry (Proposition 3.12) that

$$\begin{aligned}
\cos AC &= \cos\alpha\,\cos\gamma + \sin\alpha\,\sin\gamma\,\cos\eta,\\
\cos AD &= \cos\alpha\,\cos\delta + \sin\alpha\,\sin\delta\,\cos\eta,\\
\cos BC &= \cos\beta\,\cos\gamma + \sin\beta\,\sin\gamma\,\cos\eta,\\
\cos BD &= \cos\beta\,\cos\delta + \sin\beta\,\sin\delta\,\cos\eta.
\end{aligned}$$

From these equations it follows that

$$\begin{aligned}
&\cos AC\,\cos BD - \cos AD\,\cos BC\\
&= \cos\eta\,\bigl(\cos\alpha\,\cos\gamma\,\sin\beta\,\sin\delta + \cos\beta\,\cos\delta\,\sin\alpha\,\sin\gamma\\
&\qquad - \cos\alpha\,\cos\delta\,\sin\beta\,\sin\gamma - \cos\beta\,\cos\gamma\,\sin\alpha\,\sin\delta\bigr)\\
&= \cos\eta\,(\cos\alpha\,\sin\beta - \sin\alpha\cos\beta)(\cos\gamma\,\sin\delta - \sin\gamma\cos\delta)\\
&= \cos\eta\,\sin(\beta-\alpha)\,\sin(\delta-\gamma)\\
&= \cos\eta\,\sin AB\,\sin CD,
\end{aligned}$$

as required. ∎

Remark

In his *Disquisitiones Generales circa Superficies Curvas*, published in 1828, Gauss proved Proposition 3.13, using the method of the second of the proofs of that theorem given above.

Proposition 3.14 (Gauss)

Let A, B and C be three distinct points on the unit sphere that do not all lie on any one great circle of the sphere, and let p be the angle which the line from the centre of the sphere to the point C makes with the plane through the centre of the sphere that contains the points A and B. Then

$$\sin p = \sin A \sin AC = \sin B \sin BC,$$

where $\sin A$ denotes the sine of the angle between the arcs AB and AC at A and $\sin B$ denotes the sine of the angle between the arcs BC and AB at B.

Proof

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ denote the displacement vectors of the points A, B and C from the centre of the sphere. A straightforward application of the Vector Triple Product Identity shows that

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{u}.(\mathbf{v} \times \mathbf{w}))\mathbf{u}.$$

(see Corollary 3.8). Now $\mathbf{u} \times \mathbf{v} = \sin AB\, \mathbf{n}_{A,B}$, where $\mathbf{n}_{A,B}$ is a unit vector orthogonal to the plane spanned by A and B. Similarly $\mathbf{u} \times \mathbf{w} = \sin AC\, \mathbf{n}_{A,C}$, where $\mathbf{n}_{A,C}$ is a unit vector orthogonal to the plane spanned by A and B. Moreover the vector $\mathbf{n}_{A,B} \times \mathbf{n}_{A,C}$ is orthogonal to the vectors $\mathbf{n}_{A,B}$ and $\mathbf{n}_{A,C}$, and therefore is parallel to the line of intersection of the plane through the centre of the sphere containing A and B and the plane through the centre of the sphere containing A and C. Moreover the magnitude of this vector is the sine of the angle between them. It follows that $\mathbf{n}_{A,B} \times \mathbf{n}_{A,C} = \pm \sin A\, \mathbf{u}$.

We note also that $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = \mathbf{w}.(\mathbf{u} \times \mathbf{v})$. (see Corollary 3.5.) Putting these identities together, we see that we see that

$$\sin AB \, \sin AC \, \sin A = \pm\mathbf{u}.(\mathbf{v}\times\mathbf{w}) = \pm\mathbf{w}.(\mathbf{u}\times\mathbf{v}) = \pm \sin AB \, \mathbf{w}.\mathbf{n}_{A,B}.$$

Now the cosine of the angle between the unit vector $\mathbf{v}$ and the unit vector $\mathbf{n}_{A,C}$ is the sine $\sin p$ of the angle between the vector $\mathbf{w}$ and the plane through the centre of the sphere that contains the points A and B. It follows that $\mathbf{w} \,.\, \mathbf{n}_{A,B} = \sin p$, and therefore

$$\sin AB \, \sin AC \, \sin A = \pm \sin AB \, \sin p.$$

Now the angles concerned are all between 0 and π, and therefore their sines are non-negative. Also $\sin AB \neq 0$, because A and B are distinct and are not antipodal points on opposite sides of the sphere. Dividing by $\sin AB$, we find that

$$\sin A \sin AC = \sin p.$$

Interchanging A and B, we find that

$$\sin B \sin BC = \sin p,$$

as required. ∎

Corollary 3.15 (Sine Rule of Spherical Trigonometry)

Let A, B and C be three distinct points on the unit sphere that do not all lie on any one great circle of the sphere. Then

$$\frac{\sin BC}{\sin A} = \frac{\sin AC}{\sin B} = \frac{\sin AB}{\sin C},$$

where $\sin A$ denotes the sine of the angle between the arcs AB and AC at A and $\sin B$ denotes the sine of the angle between the arcs BC and AB at B.

Proposition 3.16 (Gauss)

Let A, B, C be points on the unit sphere in $\mathbb{R}^3$, and let the point O be at the centre of that sphere. Then the volume V of the tetrahedron with apex O and base ABC satisfies

$$V = \frac{1}{6} \sin A \sin AB \sin AC,$$

where $\sin AB$, $\sin AC$ and $\sin BC$ are the sines of the angles between the lines joining the indicated points to the centre of the sphere, and where $\sin A$, $\sin B$ and $\sin C$ are the sines of angles of the geodesic triangle ABC whose vertices are A and B and C and whose sides are the arcs of great circles joining its vertices.

Proof

This tetrahedron may be described as the tetrahedron with base OAB and apex C. Now the area of the base of the tetrahedron is $\frac{1}{2}\sin AB$, and the height is $\sin p$, where $\sin p$ is the perpendicular distance from the point C to the plane passing through the centre of the sphere that contains the points A and B. The volume V of the tetrahedron is one sixth of the area of the base of the tetrahedron multiplied by the height of the tetrahedron. On applying Proposition 3.14 we see that

$$V = \frac{1}{6}\sin p \sin AB = \frac{1}{6}\sin A \sin AB \sin AC. \qquad \blacksquare$$

Proposition 3.17

Let Π_1, Π_2 and Π_3 be planes in $\mathbb{R}^3$ that intersect at a single point, let $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ be vectors of unit length normal to Π_1, Π_2 and Π_3 respectively, let φ_1 denote the angle between the planes Π_1 and Π_3, let φ_2 denote the angle between the planes Π_2 and Π_3, and let θ denote the angle between the lines along which the plane Π_3 intersects the planes Π_1 and Π_2. Then

$$\pm \sin\varphi_1 \sin\varphi_2 \cos\theta = \mathbf{n}_1 \,.\, \mathbf{n}_2 - (\mathbf{n}_3.\mathbf{n}_1)(\mathbf{n}_3 \,.\, \mathbf{n}_2).$$

Proof

The vector $\mathbf{n}_3 \times \mathbf{n}_2$ is of length $\sin\varphi_2$ and is orthogonal to both $\mathbf{n}_2$ and $\mathbf{n}_3$, and therefore

$$\mathbf{n}_3 \times \mathbf{n}_2 = \sin\varphi_2\, \mathbf{m}_1.$$

where $\mathbf{m}_1$ is a vector of unit length parallel to the line of intersection of the planes Π_2 and Π_3. Similarly

$$\mathbf{n}_3 \times \mathbf{n}_1 = \sin\varphi_1\, \mathbf{m}_2.$$

where $\mathbf{m}_2$ is a vector of unit length parallel to the line of intersection of the planes Π_1 and Π_3. Now $\cos\theta = \pm\mathbf{m}_1 \,.\, \mathbf{m}_2$ (see Proposition 3.1). Applying Lagrange's Quadruple Product Identity (Proposition 3.9), we find that

$$\begin{aligned}
\pm \sin\varphi_1 \, \sin\varphi_2 \, \cos\theta &= (\mathbf{n}_3 \times \mathbf{n}_2) \, . \, (\mathbf{n}_3 \times \mathbf{n}_1) \\
&= (\mathbf{n}_3 \, . \, \mathbf{n}_3)(\mathbf{n}_1 \, . \, \mathbf{n}_2) - (\mathbf{n}_3 \, . \, \mathbf{n}_1)(\mathbf{n}_3 \, . \, \mathbf{n}_2). \\
&= (\mathbf{n}_1 \, . \, \mathbf{n}_2) - (\mathbf{n}_3 \, . \, \mathbf{n}_1)(\mathbf{n}_3 \, . \, \mathbf{n}_2),
\end{aligned}$$

as required. ∎

3.8. A Vector Treatment of Stereographic Projection

Let S^2 denote the unit sphere in $\mathbb{R}^3$, defined so that

$$\{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r}| = 1\}.$$

let $\mathbf{Q}$ be a fixed element of S^2, let

$$\Pi_{\mathbf{Q}} = \{\mathbf{r} \in \mathbb{R}^3 : \mathbf{Q} \,.\, \mathbf{r} = 0\},$$

and let $T_{\mathbf{Q}}\colon \mathbb{R}^3 \to \Pi_{\mathbf{Q}}$ be the linear transformation characterized by the requirement that $T_{\mathbf{Q}}(\mathbf{p} + \lambda\mathbf{Q}) = \mathbf{p}$ for all $\mathbf{p} \in \Pi_{\mathbf{Q}}$ and $\lambda \in \mathbb{R}$. Then the point $\mathbf{Q}$ determines a stereographic projection mapping

$$\psi_{\mathbf{Q}}\colon S^2 \setminus \{\mathbf{Q}\} \to \Pi_{\mathbf{Q}},$$

where

$$\psi_{\mathbf{Q}}(\mathbf{r}) = \frac{1}{1 - \mathbf{Q} \,.\, \mathbf{r}}\,\mathbf{r} - \frac{\mathbf{Q} \,.\, \mathbf{r}}{1 - \mathbf{Q} \,.\, \mathbf{r}}\,\mathbf{Q}.$$

Note that, for all elements $\mathbf{r}$ of S^2 distinct from $\mathbf{Q}$, the point $\psi_{\mathbf{Q}}(\mathbf{r})$ lies on the line passing through the points $\mathbf{r}$ and $\mathbf{Q}$. Moreover

$$\mathbf{Q} \,.\, \psi_{\mathbf{Q}}(\mathbf{r}) = \frac{1}{1 - \mathbf{Q} \,.\, \mathbf{r}}\, \mathbf{Q} \,.\, \mathbf{r} - \frac{\mathbf{Q} \,.\, \mathbf{r}}{1 - \mathbf{Q} \,.\, \mathbf{r}}\, \mathbf{Q} \,.\, \mathbf{Q} = 0,$$

because $\mathbf{Q} \,.\, \mathbf{Q} = 1$. It follows that $\psi_{\mathbf{Q}}(\mathbf{r}) \in \Pi_{\mathbf{Q}}$ for all $\mathbf{r} \in S^2 \setminus \{\mathbf{Q}\}$.

Now

$$\begin{aligned} |\psi_{\mathbf{Q}}(\mathbf{r})|^2 &= \frac{1}{(1 - \mathbf{Q} \,.\, \mathbf{r})^2} (\mathbf{r} - (\mathbf{Q} \,.\, \mathbf{r})\mathbf{Q}) \,.\, (\mathbf{r} - (\mathbf{Q} \,.\, \mathbf{r})\mathbf{Q}) \\ &= \frac{1}{(1 - \mathbf{Q} \,.\, \mathbf{r})^2} (1 - (\mathbf{Q} \,.\, \mathbf{r})^2) \\ &= \frac{1 + \mathbf{Q} \,.\, \mathbf{r}}{1 - \mathbf{Q} \,.\, \mathbf{r}} \end{aligned}$$

for all $\mathbf{r} \in S^2 \setminus \{\mathbf{Q}\}$, because $\mathbf{Q} \,.\, \mathbf{Q} = 1$ and $\mathbf{r} \,.\, \mathbf{r} = 1$. Then

$$|\psi_{\mathbf{Q}}(\mathbf{r})|^2 - 1 = \mathbf{Q} \,.\, \mathbf{r}(1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2),$$

and therefore

$$\mathbf{Q} \,.\, \mathbf{r} = \frac{|\psi_{\mathbf{Q}}(\mathbf{r})|^2 - 1}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2},$$

Thus

$$1 - \mathbf{Q} \,.\, \mathbf{r} = \frac{2}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2}.$$

It follows that

$$\begin{aligned}
\mathbf{r} &= (1 - \mathbf{Q} \,.\, \mathbf{r})\, \psi_{\mathbf{Q}}(\mathbf{r}) + (\mathbf{Q} \,.\, \mathbf{r})\, \mathbf{Q} \\
&= \frac{2}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2}\, \psi_{\mathbf{Q}}(\mathbf{r}) + \frac{|\psi_{\mathbf{Q}}(\mathbf{r})|^2 - 1}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2}\, \mathbf{Q}.
\end{aligned}$$

Thus $\mathbf{r} = \lambda_{\mathbf{Q}}(\psi_{\mathbf{Q}}(\mathbf{r}))$ for all $\mathbf{r} \in S^2 \setminus \{\mathbf{Q}\}$, where

$$\lambda_{\mathbf{Q}} \colon \Pi_{\mathbf{Q}} \to \mathbb{R}^3$$

is the mapping from the plane $\Pi_{\mathbf{Q}}$ through the origin perpendicular to the vector $\mathbf{Q}$ to $\mathbb{R}^3$ defined such that

$$\lambda_{\mathbf{Q}}(\mathbf{p}) = \frac{2}{1 + |\mathbf{p}|^2}\, \mathbf{p} + \frac{|\mathbf{p}|^2 - 1}{1 + |\mathbf{p}|^2}\, \mathbf{Q}$$

for all $\mathbf{p} \in \Pi_{\mathbf{Q}}$.

Let S^2 denote the sphere of unit radius in $\mathbb{R}^3$ centred on the origin O, let Q be a point of S^2, and let Π_Q denote the plane through the origin O perpendicular to the line OQ. For each point P of the sphere S^2 that is distinct from Q, let $\psi(P)$ denote the image of P under stereographic projection from Q. Then, for each point P of S^2 distinct from the fixed point Q, $\psi(P)$ is the unique point of the plane Π_Q at which that plane intersects the line passing through both P and Q.

For each point P of the sphere S^2 let T_P denote the tangent plane to S^2 at the point P. Then, for each point P of S^2 the tangent plane T_P is the union of all lines passing through the point P that are perpendicular to the radius vector OP.

Proposition 3.18

Let S^2 be the unit sphere centered at the origin O, let Q be a fixed point of S^2, let Π_Q be the plane passing through the origin O that is perpendicular to the line OQ, and let $\psi\colon S^2 \setminus \{0\} \to \Pi_Q$ be the mapping implementing stereographic projection from the point Q onto the plane Π_Q. Then the mapping ψ is an angle-preserving mapping from $S_2 \setminus \{0\}$ to Π_Q.

Proof

Let Q be a fixed point of the unit sphere S^2, let P be a point of S^2 distinct from the point Q, let Π_Q denote the plane through the origin perpendicular to the radius vector OQ, and let $\psi(P)$ be the image of P under stereographic projection from the point Q, so that $\psi(P)$ is the unique point of the plane Π_Q at which that plane intersects the line passing through both P and Q. Let L_1 and L_2 be distinct lines contained in the tangent plane T_P to the unit sphere S^2 at the point P, that intersect at the point P. Then $L_1 \subset T_P$, $L_2 \subset T_P$ and $L_1 \cap L_2 = \{P\}$. Let Π_1 denote the unique plane in $\mathbb{R}^3$ that contains both the line L_1 and the point Q, and let Π_2 denote the unique plane in $\mathbb{R}^3$ that contains both the line L_2 and the point Q. Let M_1 and M_2 denote the distinct lines in the tangent plane T_Q to the unit sphere at the point Q along which which the tangent plane T_Q intersects the planes Π_1 and Π_2. Then $M_1 = \Pi_1 \cap T_Q$, $M_2 = \Pi_2 \cap T_Q$ and $M_1 \cap M_2 = \{Q\}$.

Let Λ denote the plane in $\mathbb{R}^3$ consisting of all points of $\mathbb{R}^3$ that are equidistant from the points P and Q, and let $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the reflection of the space $\mathbb{R}^3$ in the plane Λ. Then the plane Λ contains the centre O of the unit sphere S^2, located at the origin. Also the line segment PQ from P to Q is perpendicular to the plane Λ and is bisected by Λ. It follows that $\tau(P) = Q$. Also $\tau(\Pi_1) = \Pi_1$, because the plane contains a line, namely the line PQ, which is perpendicular to the plane Λ, and similarly $\tau(\Pi_2) = \Pi_2$. Also the mapping $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ preserves lengths and angles, and therefore $\tau(T_P) = T_Q$. It follows that

$$\tau(L_1) = \tau(T_P \cap \Pi_1) = \tau(T_P) \cap \tau(\Pi_1) = T_Q \cap \Pi_1 = M_1,$$

and similarly $\tau(L_2) = M_2$.

The angle-preserving property of the reflection τ therefore ensures that the angle between the lines L_1 and L_2 at their point P of intersection is equal to the angle between the lines M_1 and M_2 at their point Q of intersection.

Let N_1 and N_2 denote the lines along which the plane Π_Q through the origin O perpendicular to OQ cuts the planes Π_1 and Π_2 respectively. Then $N_1 = \Pi_Q \cap \Pi_1$ and $N_2 = \Pi_Q \cap \Pi_2$. The plane Π_Q is parallel to the tangent plane T_Q at the point Q. It follows that the lines N_1 and N_2 are parallel to the lines M_1 and M_2. Therefore the angle between the lines N_1 and N_2 at their point $\psi(P)$ of intersection is equal to the angle between the lines M_1 and M_2 at their point Q of intersection, and is therefore equal to the angle between the lines L_1 and L_2 at their point P of intersection.

Let C_1 and C_2 denote the circles on the unit sphere S^2 along which the unit sphere cuts the planes Π_1 and Pi_2 respectively. Then the line L_1 is tangent to C_1 at the point P, and similarly the line L_2 is tangent to C_2 at the point P. Now the point Q belongs to the both the planes Π_1 and Π_2. It follows from the definition of stereographic projection that $\psi(C_1) \subset \Pi_1$ and $\psi(C_2) \subset \Pi_2$.

But $\psi(C_1) \subset \Pi_Q$, $\psi(C_2) \subset \Pi_Q$, $\Pi_Q \cap \Pi_1 = N_1$ and $\Pi_Q \cap \Pi_2 = N_2$. It follows that $\psi(C_1) \subset N_1$ and $\psi(C_2) \subset N_2$. Moreover all points of the line N_1 are the images of points of $S^2 \cap \Pi_1$ under stereographic projection. It follows that $\psi(C_1) = N_1$. Similarly $\psi(C_2) = N_2$.

Now the angle between the circles C_1 and C_2 at the point P is equal to the angle between their tangent lines L_1 and L_2 at the point P. This angle has been shown to be equal to the angle between the lines N_1 and N_2. Therefore the stereographic projection mapping $\psi\colon S^2 \setminus \{Q\} \to \Pi_Q$ is angle-preserving, as required. $\blacksquare$