Möbius Transformations and Stereographic Projection

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1.1. Stereographic Projection

Let a sphere in three-dimensional Euclidean space be given. A geometric construction known as *stereographic projection* gives rise to a one-to-one correspondence between the complement of a chosen point A on the sphere and the points of the plane Z through the centre C of that sphere perpendicular to the line AC. Specifically each point P on the sphere is mapped under stereographic projection to the point where the line PA intersects the plane Z.

Remark

The ancient Greek mathematician Ptolemy wrote a work, the *Planisphere*, or *Planisphærium*, that describes stereographic projection and investigates its properties. No Greek text survives, but the work was translated into Arabic, and the work has survived through the medium of this Arabic translation.

For more information on Ptolemy's *Planiphere*, see the Wikipedia article on the *Planisphærium* at the following location:

> *https://en.wikipedia.org/wiki/Planisphaerium*

A recent translation is the following:

> *Nathan Sidoli and J.L. Berggren, The Arabic version of Ptolemy's Planisphere or Flattening the Surface of the Sphere: Text, Translation, Commentary, SCIAMVS 8 (2007), 37-139*
> *http://individual.utoronto.ca/acephalous/Sidoli_Berggren_2007.pdf*

Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let (u, v, w) be a point of the unit sphere S^2 distinct from $(0, 0, -1)$. Then the unique line passing through the points (u, v, w) and $(0, 0, -1)$ intersects the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ at the point (x, y) at which

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

It follows that stereographic projection from the point $(0, 0, -1)$ sends each point (u, v, w) of S^2 distinct from the point $(0, 0, -1)$ to the point $\psi(u, v, w)$ of $\mathbb{R}^2$, where $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ is the mapping from $S^2 \setminus \{(0, 0, -1)\}$ to $\mathbb{R}^2$ defined so that

$$\psi(u, v, w) = \left(\frac{u}{w+1}, \frac{v}{w+1}\right).$$

for all $(u, v, w) \in S^2 \setminus \{(0, 0, -1)\}$.

Proposition 1.1

Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ be the sphereographic projection mapping defined such that

$$\psi(u, v, w) = \left(\frac{u}{w+1}, \frac{v}{w+1}\right)$$

for all points (u, v, w) of S^2. Then ψ is a bijective mapping whose inverse maps each point (x, y) of $\mathbb{R}^2$ to the corresponding point (u, v, w) of $S^2 \setminus \{(0, 0, -1)\}$ determined by the equations

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

Proof

Let $\lambda\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping defined so that

$$\lambda(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right)$$

for all points (x,y) of $\mathbb{R}^2$. Let (x,y) be an arbitrary point of $\mathbb{R}^2$. Then

$$(1-x^2-y^2)^2 = 1 + x^4 + y^2 + 2x^2y^2 - 2x^2 - 2y^2$$

and

$$\begin{aligned} (1+x^2+y^2)^2 &= 1 + x^4 + y^2 + 2x^2y^2 + 2x^2 + 2y^2 \\ &= 4x^2 + 4y^2 + (1-x^2-y^2)^2. \end{aligned}$$

It follows that if $(u, v, w) = \lambda(x, y)$ then

$$u^2 + v^2 + w^2 = \frac{4x^2 + 4y^2 + (1 - x^2 - y^2)^2}{1 + x^2 + y^2} = 1$$

for all real numbers x and y. Also if $u = 0$ and $v = 0$ then $x = 0$, $y = 0$ and $w = 1$. It follows that $\lambda\colon \mathbb{R}^2 \to \mathbb{R}^3$ maps $\mathbb{R}^2$ into $S^2 \setminus \{(0, 0, -1)\}$.

Moreover

$$w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2} = \frac{2}{1 + x^2 + y^2} - 1,$$

and therefore

$$u = \frac{2x}{1 + x^2 + y^2} = (w + 1)x$$

and

$$v = \frac{2y}{1 + x^2 + y^2} = (w + 1)y$$

It follows that $(x, y) = \psi(u, v, w)$. Thus the $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ is surjective.

Now let (u, v, w) be an element of S^2 distinct from $(0, 0, -1)$ Then u, v and w are real numbers for which $w \neq -1$ and $u^2 + v^2 + w^2 = 1$. Let $(x, y) = \psi(u, v, w)$, where ψ is the map from $S^2 \setminus \{(0, 0, -1)\}$ to $\mathbb{R}^2$ defined by stereographic projection from the point $(0, 0, -1)$. Then

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1},$$

and therefore

$$\begin{aligned} 1 + x^2 + y^2 &= \frac{(w+1)^2 + u^2 + v^2}{(w+1)^2} = \frac{u^2 + v^2 + w^2 + 2w + 1}{(w+1)^2} \\ &= \frac{2w+2)}{(w+1)^2} = \frac{2}{w+1}, \end{aligned}$$

It follows that

$$w + 1 = \frac{2}{1 + x^2 + y^2},$$

and therefore

$$\begin{aligned} u &= (w+1)x = \frac{2x}{1 + x^2 + y^2}, \\ v &= (w+1)y = \frac{2y}{1 + x^2 + y^2}, \\ w &= \frac{2}{1 + x^2 + y^2} - 1 = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}. \end{aligned}$$

Thus $(u, v, w) = \lambda(x, y)$. We conclude therefore that $(u, v, w) = \lambda(\psi(u, v, w))$ for all $(u, v, w) \in S^2 \setminus \{(0, 0, -1)\}$. It follows directly from that that the mapping $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ is injective.

We have now shown that the mapping $\psi\colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ is both surjective and injective. It is therefore a bijective mapping establishing a one-to-one correspondence between points of $S^2 \setminus \{(0,0,-1)\}$ and points of $\mathbb{R}^2$. We have also shown that, for each point (u,v,w) of $S^2 \setminus \{(0,0,-1)\}$, if $(x,y) = \psi(u,v,w)$ then $(u,v,w) = \lambda(x,y)$ and therefore

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2} \quad \text{and} \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

The result follows. ∎

1.2. Images of Circles under Sphereographic Projection

Let $(\ell.m.n)$ and (u, v, w) be points of the unit sphere S^2 in $\mathbb{R}^3$, where $\ell^2 + m^2 + n^2 = 1$ and $u^2 + v^2 + w^2 = 1$. Then

$$\ell u + mv + nw = \cos\theta,$$

where θ is the angle, at the centre of the sphere, between the line segments joining the centre to the given points. It follows that a subset C of S^2 is a circle on the sphere if and only if it takes the form

$$C = \{(u, v, w) \in S^2 : \ell u + mv + nw = c\},$$

where c, l, m and n are constants for which $\ell^2 + m^2 + n^2 = 1$ and $-1 < c < 1$.

Let $\psi \colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ be the stereographic projection mapping that projects the complement of the point $(0,0,-1)$ onto the plane. It follows from Proposition 1.1 that $\psi(u,v,w) = (x,y)$ for all $(u,v,w) \in S^2$, where

$$x = \frac{u}{w+1}, \quad y = \frac{v}{w+1}.$$

Moreover

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2} \quad \text{and} \quad v = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

Proposition 1.2

Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ be the stereographic projection mapping that projects the complement of the point $(0, 0, -1)$ onto the plane. Then the circles on S^2 that pass through the point $(0, 0, -1)$ are in one-to-one correspondence under this stereographic projection mapping with straight lines in the plane. Specifically let ℓ, m and n be real constants satisfying the conditions $\ell^2 + m^2 + n^2 = 1$ and $-1 < n < 1$. Then the circle on the unit sphere consisting of those points of the sphere whose Cartesian coordinates u, v and w satisfy the equation

$$\ell u + mv + nw = -n$$

corresponds under stereographic projection to the line in $\mathbb{R}^2$ consisting of those points of the plane whose Cartesian coordinates x and y satisfy the equation $px + qy = k$, where

$$p = \frac{\ell}{\sqrt{\ell^2 + m^2}}, \quad q = \frac{m}{\sqrt{\ell^2 + m^2}} \quad \text{and} \quad k = \sqrt{\frac{1}{\ell^2 + m^2} - 1}.$$

Also, given real constants p, q and k, where $p^2 + q^2 = 1$, let

$$\ell = \frac{p}{\sqrt{k^2 + 1}}, \quad m = \frac{q}{\sqrt{k^2 + 1}} \quad \text{and} \quad n = -\frac{k}{\sqrt{k^2 + 1}}.$$

Then the line in $\mathbb{R}^2$ expressed in Cartesian coordinates x and y by the equation $px + qy = k$ is the image under stereographic projection of the circle on the unit sphere where that sphere intersects the plane $\ell u + mv + nw = -n$.

Proof

Let C be a circle on S^2 that passes through the point $(0, 0, -1)$ Then

$$C = \{(u, v, w) \in S^2 : \ell u + mv + nw = -n\},$$

where ℓ, m and n are real constants satisfying the condition $\ell^2 + m^2 + n^2 = 1$ and $-1 < n < 1$. Let (x, y) be the image of a point (u, v, w) on the circle C under stereographic projection from the point $(0, 0, -1)$. Then

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}$$

(see Proposition 1.1), The equation $\ell u + mv + nw = -n$ satisfied by u, v and w then ensures that

$$\ell x + my = -n = -\sqrt{1 - \ell^2 - m^2}.$$

Moreover every point on the line in $\mathbb{R}^2$ determined by this equation is the image under stereographic projection of some point on the circle C. Also the requirements that $\ell^2 + m^2 + n^2 = 1$ and $-1 < n < 1$ together ensure that $0 < \ell^2 + m^2 \leq 1$.

Setting $p = \ell/\sqrt{\ell^2 + m^2}$ and $q = m/\sqrt{\ell^2 + m^2}$, we see that the equation of the line can be written in the form

$$px + qy = k,$$

where $p^2 + q^2 = 1$ and

$$k = \sqrt{\frac{1}{\ell^2 + m^2} - 1}.$$

Now, given any line in the plane $\mathbb{R}^2$, there exist real numbers p, q and k, where $p^2 + q^2 = 1$, for which the equation of the line takes the form

$$px + qy = k.$$

Let

$$\ell = \frac{p}{\sqrt{k^2+1}}, \quad m = \frac{q}{\sqrt{k^2+1}} \quad \text{and} \quad n = \frac{-k}{\sqrt{k^2+1}}.$$

Then $\ell^2 + m^2 + n^2 = 1$ and $n < 1$. The line $px + qy = k$ is then the image under stereographic projection of the circle consisting of points on the unit sphere whose displacement vector from the centre of the sphere makes an angle θ the direction of the vector (l, m, n), where $\cos\theta = -n$. The result follows. ∎

Proposition 1.3

Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ be the stereographic projection mapping that projects the complement of the point $(0, 0, -1)$ onto the plane. Then Then the circles on S^2 that do not pass through the point $(0, 0, -1)$ are in one-to-one correspondence under this stereographic projection mapping with circles in the Euclidean plane. Specifically the circle on the unit sphere consisting of those points of the sphere whose Cartesian coordinates u, v and w satisfy the equation $\ell u + mv + nw = c$, where $\ell^2 + m^2 + n^2 = 1$, $-1 < c < 1$ and $c \neq -n$ corresponds under stereographic projection to the circle in $\mathbb{R}^2$ consisting of those points of the plane whose Cartesian coordinates x and y satisfy the equation $(x - a)^2 + (y - b)^2 = r^2$, where

$$a = \frac{\ell}{c+n}, \quad b = \frac{m}{c+n} \quad \text{and} \quad r = \frac{\sqrt{1-c^2}}{|c+n|}.$$

Conversely, given real constants a, b and r, where $r > 0$, the circle in $\mathbb{R}^2$ of radius r centred on the point (a, b) is the image under stereographic projection of the circle on the unit sphere where that sphere intersects the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation

$$2au + 2bv + (1 + r^2 - a^2 - b^2)w = 1 - r^2 + a^2 + b^2.$$

Proof

Let C be a circle on the unit sphere S^2 in $\mathbb{R}^3$ that does not pass through the point $(0, 0, 1)$. Then there exist real numbers ℓ, m, n and c satisfying the conditions $\ell^2 + m^2 + n^2 = 1$, $-1 < c < 1$ and $c \neq -n$ such that

$$C = \{(u, v, w) \in S^2 : \ell u + mv + nw = c\}.$$

Let (x, y) be the image of a point (u, v, w) on the circle C under stereographic projection from the point $(0, 0, -1)$. Then

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}$$

(see Proposition 1.1), and therefore

$$2\ell x + 2my + n(1 - x^2 - y^2) = c(1 + x^2 + y^2).$$

Moreover every point on the curve in $\mathbb{R}^2$ determined by this equation is the image under stereographic projection of some point on the circle C.

Now $c + n \neq 0$. It follows that point of the plane lies on the curve

$$2\ell x + 2my + n(1 - x^2 - y^2) = c(1 + x^2 + y^2)$$

if and only if

$$x^2 + y^2 - 2ax - 2by + s = 0,$$

where

$$a = \frac{\ell}{c+n}, \quad b = \frac{m}{c+n} \quad \text{and} \quad s = \frac{c-n}{c+n}.$$

The equation

$$x^2 + y^2 - 2ax - 2by + s = 0$$

may be expressed in the form

$$(x - a)^2 + (y - b)^2 = r^2,$$

where

$$\begin{aligned} r^2 &= a^2 + b^2 - s = \frac{\ell^2 + m^2 + n^2 - c^2}{(c + n)^2} \\ &= \frac{1 - c^2}{(c + n)^2}. \end{aligned}$$

(We have used here the condition that $\ell^2 + m^2 + n^2 = 1$.)

It follows that, under stereographic projection from the point $(0, 0, -1)$ the image of the circle on the unit sphere along which the unit sphere intersects the plane

$$\ell u + mv + nw = c$$

(where $\ell^2 + m^2 + n^2 = 1$ and $-1 < c < 1$) is the circle of radius r about the point (a, b) of $\mathbb{R}^2$, where

$$a = \frac{\ell}{c+n}, \quad b = \frac{m}{c+n} \quad \text{and} \quad r = \frac{\sqrt{1-c^2}}{|c+n|}.$$

Now let a, b and r be real numbers, where $r > 0$. We determine which points (u, v, w) of the unit sphere $u^2 + v^2 + w^2 = 1$ are mapped by stereographic projection onto the circle of radius r centred on the point (a, b) of the Euclidean plane. Such points must satisfy the equation

$$\left(\frac{u}{w+1} - a\right)^2 + \left(\frac{v}{w+1} - b\right)^2 = r^2.$$

Expanding out, we find that

$$\frac{u^2+v^2}{(w+1)^2} - \frac{2au+2bv}{w+1} + a^2 + b^2 = r^2.$$

But $u^2 + v^2 = 1 - w^2 = (w+1)(1-w)$. It follows that

$$\frac{1-w-2au-2bv}{w+1} = r^2 - a^2 - b^2,$$

and therefore

$$2au + 2bv + (1 + r^2 - a^2 - b^2)w = 1 - r^2 + a^2 + b^2.$$

Now

$$(2a)^2 + (2b)^2 + (1 + r^2 - a^2 - b^2)^2 = K^2,$$

where

$$K = \sqrt{(1 + a^2 + b^2)^2 + 2(1 - a^2 - b^2)r^2 + r^4}.$$

The equation satisfied by the points on the unit sphere that map under projection to the circle of radius r about a point (a, b) of $\mathbb{R}^2$ therefore takes the form

$$\ell u + mv + nw = c,$$

where $\ell^2 + m^2 + n^2 = 1$, provided we take

$$\ell = \frac{2a}{K}, \quad m = \frac{2b}{K}, \quad n = \frac{1 + r^2 - a^2 - b^2}{K},$$

and

$$c = \frac{1 + a^2 + b^2 - r^2}{K}.$$

Moreover

$$c^2K^2 = (1 + a^2 + b^2)^2 - 2(1 + a^2 + b^2)r^2 + r^4 = K^2 - 4r^2,$$

and therefore $c^2 < 1$. Thus $-1 < c < 1$. The result follows. $\blacksquare$

2.1. The Riemann Sphere

The *Riemann sphere* $\mathbb{P}^1$ may be defined as the set $\mathbb{C} \cup \{\infty\}$ obtained by augmenting the system $\mathbb{C}$ of complex numbers with an additional element, denoted by ∞, where ∞ is not itself a complex number, but is an additional element added to the set, with the additional conventions that

$$z + \infty = \infty, \quad \infty \times \infty = \infty \quad \text{and} \quad \frac{z}{\infty} = 0$$

for all complex numbers z, and

$$z \times \infty = \infty \quad \text{and} \quad \frac{z}{0} = \infty$$

for all non-zero complex numbers z. The symbol ∞ cannot be added to, or subtracted from, itself. Also 0 and ∞ cannot be divided by themselves.

Note that, because the sum of two elements of $\mathbb{P}^1$ is not defined for every single pair of elements of $\mathbb{P}^1$, this set cannot be regarded as constituting a group under the operation of addition. Similarly its non-zero elements cannot be regarded as constituting a group under multiplication. In particular, the Riemann sphere cannot be regarded as constituting a field.

There is a natural one-to-one correspondence between elements of the Riemann sphere $\mathbb{P}^1$ and one-dimensional complex subspaces of the complex vector space $\mathbb{C}^2$ of ordered pairs of complex numbers. Indeed there is a well-defined mapping

$$\rho\colon \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1,$$

from the set $\mathbb{C}^2 \setminus \{(0,0)\}$ of non-zero elements of the complex vector space $\mathbb{C}^2$ to the Riemann sphere $\mathbb{P}^1$, where, given any complex numbers z_1 and z_2 that are not both zero, their image $\rho(z_1, z_2)$ under this mapping is defined such that

$$\rho(z_1, z_2) = \begin{cases} \dfrac{z_1}{z_2} & \text{if } z_2 \neq 0; \\ \infty & \text{if } z_1 \neq 0 \text{ and } z_2 = 0. \end{cases}$$

Proposition 2.1

Let $\rho\colon \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ be the mapping defined in accordance with the following requirements: $\rho(z_1, z_2) = z_1/z_2$ for all complex numbers z_1; z_2 for which $z_2 \neq 0$ and $\rho(z_1, 0) = \infty$ for all non-zero complex numbers z_1. Then the mapping ρ is surjective, and moreover, given complex numbers z_1, z_2, z_3, z_4, where z_1 and z_2 are not both zero and also z_3 and z_4 are not both zero, those complex numbers satisfy the equation

$$\rho(z_1, z_2) = \rho(z_3, z_4)$$

if and only if there exists some non-zero complex number w for which $z_3 = wz_1$ and $z_4 = wz_2$.

Proof

Note that $z = \rho(z, 1)$ for all complex numbers z, and $\infty = \rho(1, 0)$. It follows that the mapping ρ is surjective.

Let z_1, z_2, z_3 and z_4 be complex numbers, where z_1 and z_2 are not both zero, z_3 and z_4 are not both zero, and

$$\rho(z_1, z_2) = \rho(z_3, z_4).$$

We must prove the existence of a non-zero complex number w for which $z_3 = wz_1$ and $z_4 = wz_2$.

First suppose that $z_2 = 0$. Then $z_1 \neq 0$ and $\rho(z_1, z_2) = \infty = \rho(z_3, z_4)$ and therefore $z_4 = 0$ and $z_3 \neq 0$. If we then take $w = z_3/z_1$ then $w \neq 0$, $z_3 = wz_1$ and $z_4 = wz_2$. Thus the existence of the required non-zero complex number w follows in the case where $z_2 = 0$.

Next suppose that $z_2 \neq 0$. Then

$$\rho(z_1, z_2) = \frac{z_1}{z_2} = \rho(z_3, z_4),$$

and therefore $\rho(z_3, z_4) \in \mathbb{C}$. It follows that $z_4 \neq 0$ and

$$\frac{z_3}{z_4} = \rho(z_3, z_4) = \rho(z_1, z_2) = \frac{z_1}{z_2}.$$

Let $w = z_4/z_2$. Then $w \neq 0$, $z_3 = wz_1$ and $z_4 = z_2$. Thus, in all cases where

$$\rho(z_1, z_2) = \rho(z_3, z_4),$$

there exists some non-zero complex number w for which $z_3 = wz_1$ and $z_4 = wz_2$. The result follows. $\blacksquare$

The set $\mathbb{C}^2$ of all ordered pairs of complex numbers is a vector space over the field $\mathbb{C}$ of complex numbers. We now discuss how the Riemann sphere $\mathbb{P}^1$ parameterizes the one-dimensional complex vector subspaces this the two-dimensional complex vector space. Let H be the particular one-dimensional complex vector subspace of $\mathbb{C}^2$ defined so that

$$H = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\},$$

and let L be a one-dimensional complex vector subspace of $\mathbb{C}^2$ distinct from H. Then L contains an element (z_1, z_2) of $\mathbb{C}^2$ for which $z_2 \neq 0$. Then, because L is a vector subspace of $\mathbb{C}^2$, L must also contain $(z, 1)$, where $z = z_1/z_2$. Thus, for each one-dimensional complex vector subspace L of $\mathbb{C}^2$ distinct from H, there exists some complex number z for which $(z, 1) \in L$.

Now L, considered as a complex vector subspace of $\mathbb{C}^2$, is one-dimensional, and, because L is distinct from H, there exists a complex number z for which $(z, 1) \in L$. It then follows from the one-dimensionality of L as a complex vector space that

$$L = \{(wz, w) : w \in \mathbb{C}\},$$

and therefore the complex number z is the unique complex number for which $(z, 1) \in L$. This complex number is the element of the Riemann sphere $\mathbb{P}^1$ that parameterizes the complex vector subspace L.

Thus to each one-dimensional complex vector subspace L of $\mathbb{C}^2$ distinct from H there corresponds the unique complex number z for which $(z, 1) \in H$; and, in the other direction, to each complex number z there corresponds the unique one-dimensional complex vector subspace L distinct from H that contains $(z, 1)$. Thus the one-dimensional complex vector subspaces of $\mathbb{C}^2$ distinct from the particular one-dimensional subspace H are parameterized by elements of the complex plane in the manner just described.

Now each complex number z is an element of the Riemann sphere $\mathbb{P}^1$. The Riemann sphere also has exactly one element ∞ that is not a complex number. We regard this extra element of $\mathbb{P}^1$ as representing the one-dimenionsional complex vector subspace H of $\mathbb{C}^2$ consisting of those elements (z_1, z_2) of $\mathbb{C}^2$ for which $z_2 = 0$. Then, to each one-dimensional complex vector subspace of $\mathbb{C}^2$ there corresponds exactly one element of the Riemann sphere $\mathbb{P}^1$ that parameterizes it; and, in the other direction, to each element of the Riemann sphere, there exists exactly one one-dimensional complex vector subspace of $\mathbb{C}^2$ that is parameterized by it.

Moreover each element (z_1, z_2) of $\mathbb{C}^2$ distinct from $(0, 0)$ is contained in exactly one one-dimensional complex vector subspace of $\mathbb{C}^2$. This subspace is that which corresponds, under the parameterization described above, to the element $\rho((z_1, z_2))$ of the Riemann sphere, where

$$\rho(z_1, z_2) = \begin{cases} \dfrac{z_1}{z_2} & \text{if } z_1, z_2 \in \mathbb{C} \text{ and } z_2 \neq 0; \\ \infty & \text{if } z_1 \in \mathbb{C} \text{ and } z_2 = 0. \end{cases}$$

The conclusions just arrived at may be formally stated as follows.

Corollary 2.2

Let $\rho\colon \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ be the mapping defined in accordance with the following requirements: $\rho(z_1, z_2) = z_1/z_2$ for all complex numbers z_1; z_2 for which $z_2 \neq 0$ and $\rho(z_1, 0) = \infty$ for all non-zero complex numbers z_1. Then, given any one-dimensional complex vector subspace L of $\mathbb{C}^2$, there exists a unique elmement ω of $\mathbb{P}^1$ with the property that $\rho(z_1, z_2) = \omega$ for all non-zero elements (z_1, z_2) of L. In the other direction, given any element ω of $\mathbb{P}^1$, there exists a unique one-dimensional complex vector subspace of $\mathbb{C}^2$ whose non-zero elements are those ordered pairs (z_1, z_2) of complex numbers whose components are not both zero and satisfy the equation $\rho(z_1, z_2) = \omega$. Thus ρ induces a one-to-one correspondence between one-dimensional complex subspaces of $\mathbb{C}^2$ and points of the Riemann sphere $\mathbb{P}^1$.

2.2. Stereographic Projection of the Riemann Sphere

Proposition 2.3

Let $\sigma\colon \mathbb{P}^1 \to \mathbb{R}^3$ *be the mapping from the Riemann sphere* $\mathbb{P}^1$ *to* $\mathbb{R}^3$ *defined such that* $\sigma(\infty) = (0, 0, -1)$ *and*

$$\sigma(x + y\sqrt{-1}) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)$$

for all real numbers x *and* y*. Then* σ *maps* $\mathbb{P}^1$ *injectively and surjectively onto the unit sphere* S^2 *in* $\mathbb{R}^3$*. Moreover if* (u, v, w) *is a point of* S^2 *distinct from* $(0, 0, -1)$ *then* $(u, v, w) = \sigma(x + y\sqrt{-1})$*, where*

$$x = \frac{u}{w + 1} \quad \textit{and} \quad y = \frac{v}{w + 1}.$$

2.3. Möbius Transformations

Definition

Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$. The *Möbius transformation* $\mu_{a,b,c,d} \colon \mathbb{P}^1 \to \mathbb{P}^1$ with coefficients a, b, c and d is defined to be the function from the Riemann sphere $\mathbb{P}^1$ to itself determined by the following properties:

$$\mu_{a,b,c,d}(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$;
$\mu_{a,b,c,d}(-d/c) = \infty$ and $\mu_{a,b,c,d}(\infty) = a/c$ if $c \neq 0$;
$\mu_{a,b,c,d}(\infty) = \infty$ if $c = 0$.

Note that the requirement in the above definition of a Möbius transformation that its coefficients a, b, c and d satisfy the condition $ad - bc \neq 0$ ensures that there is no complex number for which $az + b$ and $cz + d$ are both zero.

Let A be a non-singular 2×2 matrix whose coefficients are complex numbers, and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We denote by μ_A the Möbius transformation $\mu_{a,b,c,d}$ with coefficients a, b, c, d, defined so that

$$\begin{aligned} \mu_A(z) &= \begin{cases} \dfrac{az+b}{cz+d} & \text{if } cz + d \neq 0; \\ \infty & \text{if } c \neq 0 \text{ and } z = -d/c; \end{cases} \\ \mu_A(\infty) &= \begin{cases} \dfrac{a}{c} & \text{if } c \neq 0; \\ \infty & \text{if } c = 0. \end{cases} \end{aligned}$$

The following result exemplifies the reason for representing the coefficients of a Möbius transformation in the form of a matrix.

Proposition 2.4

The composition of two Möbius transformations is a Möbius transformation. Specifically let A and B be non-singular 2×2 matrices with complex coefficients, and let μ_A and μ_B be the corresponding Möbius transformations of the Riemann sphere. Then the composition $\mu_A \circ \mu_B$ of these Möbius transformations is the Möbius transformation μ_{AB} of the Riemann sphere determined by the product AB of the matrices A and B.

Proof

Let

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

and let

$$AB = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Then

$$a_3 = a_1 a_2 + b_1 c_2, \quad b_3 = a_1 b_2 + b_1 d_2,$$

$$c_3 = c_1 a_2 + d_1 c_2 \quad \text{and} \quad d_3 = c_1 b_2 + d_1 d_2.$$

The definitions of Möbius transformations determined by non-singular 2×2 matrices ensure that

$$\mu_A(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$$

whenever $c_1 z + d_1 \neq 0$ and

$$\mu_B(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

whenever $c_2 z + d_2 \neq 0$.

First suppose that z is a complex number for which $c_2 z + d_2 \neq 0$. Then

$$\begin{aligned}
(a_1 \mu_B(z) + b_1)(c_2 z + d_2) &= a_1(a_2 z + b_2) + b_1(c_2 z + d_2) \\
&= a_3 z + b_3, \\
(c_1 \mu_B(z) + d_1)(c_2 z + d_2) &= c_1(a_2 z + b_2) + d_1(c_2 z + d_2) \\
&= c_3 z + d_3.
\end{aligned}$$

It follows that if $c_2 z + d_2 \neq 0$ and $c_1 \mu_B(z) + d_1 \neq 0$ then

$$\mu_A(\mu_B(z)) = \frac{a_1 \mu_B(z) + b_1}{c_1 \mu_B(z) + d_1} = \frac{a_3 z + b_3}{c_3 z + d_3} = \mu_{AB}(z).$$

If $c_2 z + d_2 \neq 0$ but $c_1 \mu_B(z) + d_1 = 0$ then $c_3 z + d_3 = 0$ and

$$\mu_A(\mu_B(z)) = \infty = \mu_{AB}(z).$$

We conclude that $\mu_A(\mu_B(z)) = \mu_{AB}(z)$ for all complex numbers z satisfying $c_2 z + d_2 \neq 0$.

Next suppose that z is a complex number for which $c_2 z + d_2 = 0$. Now the definition of Möbius transformations requires that $a_2 d_2 - b_2 c_2 \neq 0$. It follows that c_2 and d_2 cannot both be equal to zero. Thus if $c_2 z + d_2 = 0$ then either $z = d_2 = 0$ and $c_2 \neq 0$ or else z, c_2 and d_2 are all non-zero. Thus, in all cases where $c_2 z + d_2 = 0$, the coefficient c_2 of the Möbius transformation is non-zero and $z = -d_2/c_2$. Also the equations $a_2 z + b_2 = 0$ and $c_2 z + d_2 = 0$ cannot both be satisfied, because $a_2 d_2 - b_2 c_2 \neq 0$, and therefore $a_2 z + b_2 \neq 0$.

Now the equations determining a_3, b_3, c_3 and d_3 ensure that if $c_2 z + d_2 = 0$ then

$$
\begin{aligned}
c_2(a_3 z + b_3) &= -d_2 a_3 + c_2 b_3 \\
&= c_2(a_1 b_2 + b_1 d_2) - d_2(a_1 a_2 + b_1 c_2) \\
&= a_1(b_2 c_2 - a_2 d_2) \\
&= a_1 c_2(a_2 z + b_2) \\
c_2(c_3 z + d_3) &= -d_2 c_3 + c_2 d_3 \\
&= c_2(c_1 b_2 + d_1 d_2) - d_2(c_1 a_2 + d_1 c_2) \\
&= c_1(b_2 c_2 - a_2 d_2) \\
&= c_1 c_2(a_2 z + b_2),
\end{aligned}
$$

and therefore

$$a_3 z + b_3 = a_1(a_2 z + b_2) \quad \text{and} \quad c_3 z + d_3 = c_1(a_2 z + b_2),$$

Thus if $c_2 z + d_2 = 0$ and $c_1 \neq 0$ then $c_3 z + d_3 \neq 0$ and

$$\mu_{AB}(z) = \frac{a_3 z + b_3}{c_3 z + d_3} = \frac{a_1}{c_1} = \mu_A(\infty) = \mu_A(\mu_B(z)).$$

And if $c_2 z + d_2 = 0$ and $c_1 = 0$ then $c_3 z + d_3 = 0$ and

$$\mu_{AB}(z) = \infty = \mu_A(\infty) = \mu_A(\mu_B(z)).$$

Thus $\mu_{AB}(z) = \mu_A(\mu_B(z))$ in all cases for which $c_2 z + d_2 = 0$.

It remains to show that $\mu_{AB}(\infty) = \mu_A(\mu_B(\infty))$. If $c_2 \neq 0$ (so that $\mu_B(\infty) = a_2/c_2$) and $c_1\mu_B(\infty) + d_2 \neq 0$ then

$$\mu_A(\mu_B(\infty)) = \frac{a_1\mu_B(\infty) + b_1}{c_1\mu_B(\infty) + d_1} = \frac{a_1a_2 + b_1c_2}{c_1a_2 + d_1c_2} = \frac{a_3}{c_3} = \mu_{AB}(\infty).$$

If $c_2 \neq 0$ and $c_1\mu_B(\infty) + d_2 = 0$ then $c_3 = c_1a_2 + d_1c_2 = 0$, because $\mu_B(\infty) = a_2/c_2$, and therefore

$$\mu_A(\mu_B(\infty)) = \infty = \mu_{AB}(\infty).$$

If $c_1 = c_2 = 0$ then $\mu_B(\infty) = \infty$ and therefore

$$\mu_A(\mu_B(\infty)) = \mu_A(\infty) = \infty = \mu_{AB}(\infty).$$

If $c_2 = 0$ and $c_1 \neq 0$ then $a_3 = a_1a_2$, $c_3 = c_1a_2$ and $a_2 \neq 0$ (because $a_2d_2 - b_2c_2 \neq 0$), and therefore

$$\mu_A(\mu_B(\infty)) = \mu_A(\infty) = \frac{a_1}{c_1} = \frac{a_3}{c_3} = \mu_{AB}(\infty).$$

We conclude that $\mu_A(\mu_B(\infty)) = \mu_{AB}(\infty)$ in all cases. This completes the proof. $\blacksquare$

Corollary 2.5

Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$, and let $\mu_{a,b,c,d}\colon \mathbb{P}^1 \to \mathbb{P}^1$ denote the Möbius transformation of the Riemann sphere $\mathbb{P}^1$ defined such that $\mu_{a,b,c,d}(z) = \dfrac{az+b}{cz+d}$ if $z \in \mathbb{C}$ and $cz + d \neq 0$, $\mu_{a,b,c,d}(-d/c) = \infty$ and $\mu_{a,b,c,d}(\infty) = a/c$ if $c \neq 0$, and $\mu_{a,b,c,d}(\infty) = \infty$ if $c = 0$. Then the mapping $\mu_{a,b,c,d}\colon \mathbb{P}^1 \to \mathbb{P}^1$ is invertible, and its inverse is the Möbius transformation $\mu_{d,-b,-c,a}\colon \mathbb{P}^1 \to \mathbb{P}^1$, where $\mu_{d,-b,-c,a}(z) = \dfrac{dz-b}{a-cz}$ if $z \in \mathbb{C}$ and $a - cz \neq 0$, $\mu_{d,-b,-c,a}(a/c) = \infty$ and $\mu_{d,-b,-c,a}(\infty) = -d/c$ if $c \neq 0$, and $\mu_{d,-b,-c,a}(\infty) = \infty$ if $c = 0$.

Proof

If the coefficients a, b, c and d of a Möbius transformation are all multiplied by a non-zero complex number then this does not change the Möbius transformation represented by those coefficients. It follows that we may assume, without loss of generality, that $ad - bc = 1$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $ad - bc = 1$. Then

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The result therefore follows directly on applying Proposition 2.4. ∎

2.4. Inversions of the Riemann Sphere

Let S^2 denote the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let us refer to the points $(0, 0, 1)$ and $(0, 0, -1)$ as the *North Pole* and *South Pole* respectively. Let E denote the *Equatorial Plane* in $\mathbb{R}^3$, consisting of those points whose Cartesian coordinates are of the form $(x, y, 0)$, where x and y are real numbers.

Stereographic projection from the South Pole maps each point (u, v, w) of the unit sphere S^2 distinct from the South Pole to the point $(x, y, 0)$ of the equatorial plane E for which

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Moreover a point $(x, y, 0)$ of the Equatorial Plane E is the image under stereographic projection from the South Pole of the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2}, \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

We can also stereographically project from the North Pole. Note that, given a point in the Equatorial Plane, reflection in that Equatorial Plane will interchange the points of the sphere corresponding to it under stereographic projection from the North and South Poles. Thus a point (u, v, w) of the unit sphere S^2 distinct from the North Pole corresponds under stereographic projection to the point $(x, y, 0)$ of the Equatorial Plane E for which

$$x = \frac{u}{1-w} \quad \text{and} \quad y = \frac{v}{1-w}.$$

In the other direction, a point $(x, y, 0)$ of the Equatorial Plane E corresponds under stereographic projection from the North Pole to the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2}, \quad w = \frac{x^2+y^2-1}{1+x^2+y^2}.$$

Proposition 2.6

Let O denote the origin $(0, 0, 0)$ of the Equatorial Plane E, where

$$E = \{(x, y, z) \in \mathbb{R}^3 : z = 0\},$$

and let A be a point $(x, y, 0)$ of E distinct from the origin O. Let C be the point on the unit sphere S^2 that corresponds to A under stereographic projection from the North Pole $(0, 0, 1)$, and let B be the point of the Equatorial Plane E that corresponds to C under stereographic projection from the South Pole. Then $B = (p, q, 0)$, where

$$p = \frac{x}{x^2 + y^2} \quad \text{and} \quad q = \frac{y}{x^2 + y^2}.$$

Thus the points O, A and B are collinear, and the points A and B lie on the same side of the origin O. Also the distances $|OA|$ and $|OB|$ of the points A and B from the origin satisfy $|OA| \times |OB| = 1$.

Proof

Let $(x, y, 0)$ be a point of the Equatorial plane E distinct from the origin. This point is the image, under stereographic projection from the North Pole $(0, 0, 1)$ of the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2}, \quad w = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}.$$

This point then gets mapped under stereographic projection from the South Pole to the point $(p, q, 0)$ of the Equatorial Plane E for which

$$p = \frac{u}{w + 1} \quad \text{and} \quad q = \frac{v}{w + 1}.$$

2. Möbius Transformations of the Riemann Sphere (continued)

Now

$$w + 1 = \frac{2(x^2 + y^2)}{1 + x^2 + w^2}.$$

It follows that

$$p = \frac{x}{x^2 + y^2} \quad \text{and} \quad q = \frac{y}{x^2 + y^2}.$$

Finally we note that O, A and B are collinear, where $0 = (0, 0, 0)$, $A = (x, y, 0)$ and $B = (p, q, 0)$, and the points A and B lie on the same side of the origin O. Also

$$|OA| = \sqrt{x^2 + y^2}, \quad \text{and} \quad |OB| = \frac{1}{x^2 + y^2},$$

and therefore $|OA| \times |OB| = 1$, as required. $\blacksquare$

2.5. Möbius Transformations representing Rotations

Let a and b be complex numbers satisfying the equation $|a|^2 + |b|^2 = 1$, and let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius Transformation of the Riemann sphere defined such that

$$\mu(z) = \frac{az + b}{-\overline{b}z + \overline{a}}$$

when $z \neq \overline{a}/\overline{b}$, $\mu(\overline{a}/\overline{b}) = \infty$ and $\mu(\infty) = -a/\overline{b}$. (Here $\overline{a}$ and $\overline{b}$ denote the complex conjugates of the complex numbers a and b respectively.) We denote the complex number $\sqrt{-1}$ by i, as is customary.

Let u_0, v_0 and w_0 are real numbers satisfying $u_0^2 + v_0^2 + w_0^2 = 1$ and $w_0 \neq -1$. Then the point (u_0, v_0, w_0) of the unit sphere in $\mathbb{R}^3$ corresponds, under stereographic projection from $(0, 0, -1)$, to the complex number z_0 for which

$$z_0 = \frac{u_0 + iv_0}{w_0 + 1}.$$

Let $z_1 = \mu(z_0)$. Then

$$z_1 = \frac{a(u_0 + iv_0) + b(w_0 + 1)}{-\overline{b}(u_0 + iv_0) + \overline{a}(w_0 + 1)}.$$

Now there exists a point (u_1, v_1, w_1) of the unit sphere in $\mathbb{R}^3$ which corresponds under stereographic projection from the point $(0, 0, -1)$ to the complex number z_1. The Cartesian coordinates u_1, v_1 and w_1 of this point satisfy the equation $u_1^2 + v_1^2 + w_1^2 = 1$ and their relationship to the complex number z_1 is expressed by the following equations:

$$z_1 = \frac{u_1 + iv_1}{w_1 + 1},$$

$$u_1 = \frac{2\mathrm{Re}[z_1]}{|z_1|^2 + 1}, \quad v_1 = \frac{2\mathrm{Im}[z_1]}{|z_1|^2 + 1}, \quad w_1 = \frac{2}{|z_1|^2 + 1} - 1.$$

We seek to express the values of u_1, v_1 and w_1 in terms of u_0, v_0 and w_0.

Now

$$\begin{aligned} &|a(u_0 + iv_0) + b(w_0 + 1)|^2 \\ &\quad = \quad (a(u_0 + iv_0) + b(w_0 + 1))(\overline{a}(u_0 - iv_0) + \overline{b}(w_0 + 1)) \\ &\quad = \quad |a|^2(u_0^2 + v_0^2) + |b|^2(w_0 + 1)^2 + 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]\,(w_0 + 1). \end{aligned}$$

But $u_0^2 + v_0^2 = 1 - w_0^2 = (w_0 + 1)(1 - w_0)$. It follows that

$$\begin{aligned} &|a(u_0 + iv_0) + b(w_0 + 1)|^2 \\ &\quad = \quad \left(|a|^2(1 - w_0) + |b|^2(w_0 + 1) + 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]\right)(w_0 + 1) \\ &\quad = \quad \left(1 - (|a|^2 - |b|^2)w_0 + 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]\right)(w_0 + 1). \end{aligned}$$

Similarly

$$\begin{aligned} & |-\overline{b}(u_0 + iv_0) + \overline{a}(w_0 + 1)|^2 \\ & \qquad = \quad \left(1 + (|a|^2 - |b|^2)w_0 - 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]\right)(w_0 + 1). \end{aligned}$$

It follows from these identities that

$$|z_1|^2 \quad = \quad \frac{1 - (|a|^2 - |b|^2)w_0 + 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]}{1 + (|a|^2 - |b|^2)w_0 - 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]},$$

and thus

$$|z_1|^2 + 1 = \frac{2}{1 + (|a|^2 - |b|^2)w_0 - 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]}.$$

Also

$$
\begin{aligned}
z_1 &= \frac{a(u_0 + iv_0) + b(w_0 + 1)}{-\overline{b}(u_0 + iv_0) + \overline{a}(w_0 + 1)} \\
&= \frac{(a(u_0 + iv_0) + b(w_0 + 1))(-b(u_0 - iv_0) + a(w_0 + 1))}{|-\overline{b}(u_0 + iv_0) + \overline{a}(w_0 + 1)|^2}
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&(a(u_0 + iv_0) + b(w_0 + 1))(-b(u_0 - iv_0) + a(w_0 + 1)) \\
&\quad = (a^2(u_0 + iv_0) - b^2(u_0 - iv_0))(w_0 + 1) \\
&\qquad\quad + ab((w_0 + 1)^2 - u_0^2 - v_0^2) \\
&\quad = ((a^2 - b^2)u_0 + i(a^2 + b^2)v_0 + 2abw_0)(w_0 + 1)
\end{aligned}
$$

It follows that

$$z_1 = \frac{(a^2 - b^2)u_0 + i(a^2 + b^2)v_0 + 2abw_0}{1 + (|a|^2 - |b|^2)w_0 - 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)]},$$

and thus

$$\begin{aligned} u_1 + iv_1 &= \frac{2z_1}{|z_1|^2 + 1} \\ &= (a^2 - b^2)u_0 + i(a^2 + b^2)v_0 + 2abw_0 \end{aligned}$$

and

$$\begin{aligned} w_1 &= \frac{2}{|z_1|^2 + 1} - 1 \\ &= (|a|^2 - |b|^2)w_0 - 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)] \end{aligned}$$

Now $|a|^2 + |b|^2 = 1$ and $b \neq 0$. It follows that $|a| < 1$, and thus $\mathrm{Re}[a] = \cos \frac{1}{2}\theta$ for some real number θ satisfying $0 < \theta < 2\pi$. Let ℓ, m and n be real numbers determined so that

$$a = \cos \tfrac{1}{2}\theta + in \sin \tfrac{1}{2}\theta \quad \text{and} \quad b = (m - i\ell) \sin \tfrac{1}{2}\theta.$$

Then

$$1 = |a|^2 + |b|^2 = \cos^2 \tfrac{1}{2}\theta + (\ell^2 + m^2 + n^2) \sin^2 \tfrac{1}{2}\theta,$$

and therefore

$$\ell^2 + m^2 + n^2 = 1.$$

Then

$$
\begin{aligned}
|a|^2 - |b|^2 &= \cos^2 \tfrac{1}{2}\theta + (n^2 - \ell^2 - m^2) \sin^2 \tfrac{1}{2}\theta \\
&= \cos\theta + n^2(1 - \cos\theta), \\
a^2 - b^2 &= \cos^2 \tfrac{1}{2}\theta + (\ell^2 - m^2 - n^2 + 2i\ell m) \sin^2 \tfrac{1}{2}\theta \\
&\qquad + 2in \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta \\
&= \cos\theta + (\ell^2 + i\ell m)(1 - \cos\theta) + in \sin\theta \\
a^2 + b^2 &= \cos^2 \tfrac{1}{2}\theta + (m^2 - \ell^2 - n^2 - 2i\ell m) \sin^2 \tfrac{1}{2}\theta \\
&\qquad + 2in \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta \\
&= \cos\theta + (m^2 - i\ell m)(1 - \cos\theta) + in \sin\theta \\
2a\overline{b} &= 2(m + i\ell) \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta - 2(\ell - im) n \sin^2 \tfrac{1}{2}\theta \\
&= (m + i\ell) \sin\theta - (\ell - im) n (1 - \cos\theta) \\
2ab &= 2(m - i\ell) \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta + 2(\ell + im) n \sin^2 \tfrac{1}{2}\theta \\
&= (m - i\ell) \sin\theta + (\ell + im) n (1 - \cos\theta)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
u_1 &= \mathrm{Re}[a^2 - b^2]u_0 - \mathrm{Im}[a^2 + b^2]v_0 + 2\mathrm{Re}[ab]w_0 \\
&= (mw_0 - nv_0)\sin\theta + u_0\cos\theta \\
&\qquad + (\ell u_0 + mv_0 + nw_0)\ell(1-\cos\theta) \\
v_1 &= \mathrm{Im}[a^2 - b^2]u_0 + \mathrm{Re}[a^2 + b^2]v_0 + 2\mathrm{Im}[ab]w_0 \\
&= (nu_0 - \ell w_0)\sin\theta + v_0\cos\theta \\
&\qquad + (\ell u_0 + mv_0 + nw_0)m(1-\cos\theta) \\
w_1 &= (|a|^2 - |b|^2)w_0 - 2\mathrm{Re}[a\overline{b}(u_0 + iv_0)] \\
&= (\ell v_0 - mu_0)\sin\theta + w_0\cos\theta \\
&\qquad + (\ell u_0 + mv_0 + nw_0)n(1-\cos\theta).
\end{aligned}
$$

Let

$$\mathbf{r}_0 = (u_0, v_0, w_0), \quad \mathbf{r}_1 = (u_1, v_1, w_1) \quad \text{and} \quad \mathbf{L} = (\ell, m, n).$$

Then the vectors $\mathbf{r}_0$, $\mathbf{r}_1$ and $\mathbf{L}$ are of unit length, and

$$\begin{aligned}
\mathbf{r}_1 &= \cos\theta\, \mathbf{r}_0 + (\mathbf{r}_0 \,.\, \mathbf{L})(1 - \cos\theta)\, \mathbf{L} + \sin\theta \mathbf{L} \times \mathbf{r}_0 \\
&= (\mathbf{r}_0 \,.\, \mathbf{L})\mathbf{L} + \cos\theta\, (\mathbf{r}_0 - (\mathbf{r}_0 \,.\, \mathbf{L})\mathbf{L} + \sin\theta\, \mathbf{L} \times \mathbf{r}_0.
\end{aligned}$$

Interpreting this formula geometrically, we see that the point (u_1, v_1, w_1) of the unit sphere in $\mathbb{R}^3$ is the image of the point (u_0, v_0, w_0) under a rotation through an angle θ about the axis passing through the origin in the direction of the vector $\mathbf{L}$.

Proposition 2.7

Let a, b, c and d be complex numbers satisfying $ad - bc = 1$, and let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{when } cz + d \neq 0,.$$

$\mu(-d/c) = \infty$ and $\mu(\infty) = a/c$ in the case $c \neq 0$ and $\mu(\infty) = \infty$ in the case $c = 0$. Then the Möbius transformation represents a rotation of the unit sphere in $\mathbb{R}^3$ if and only if $d = \overline{a}$ and $c = -\overline{b}$.

Proof

The calculations undertaken in this subsection show that if $b \neq 0$, $d = \overline{a}$ and $c = -\overline{b}$ then the Möbius transformation $\mu \colon \mathbb{P}^1 \to \mathbb{P}^1$ corresponds to a rotation of the unit sphere in $\mathbb{R}^3$. The same is true in the case when $b = 0$. Indeed in that case the conditions $d = \overline{a}$ and $c = -\overline{b}$ ensure that $b = c = 0$, $|a| = 1$, and these conditions ensure that the Möbius μ implements a rotation of the unit sphere about the direction $(0, 0, 1)$ through an angle θ, where

$$a = \cos \tfrac{1}{2}\theta + \sqrt{-1} \sin \tfrac{1}{2}\theta.$$

Thus, in all cases where $d = \overline{a}$, $c = -\overline{b}$ and $|a|^2 + |b|^2 = 1$, the Möbius transformation μ of the Riemann sphere corresponds to a rotation of the unit sphere in $\mathbb{R}^3$.

Now every rotation about the origin in $\mathbb{R}^3$ is a rotation about a fixed axis through a given angle. It follows that, given any rotation of the sphere, there are values of the complex numbers a and b for which the Möbius transformation μ implements the given rotation with $d = \overline{a}$ and $c = -\overline{b}$. Moreover a Möbius transformation with coefficients a', b', c', d' implements the same transformation of the sphere as the Möbius transformation μ if and only if either a', b', c' and d' are respectively equal to a, b, c and d or else a', b', c' and d' are respectively equal to $-a$, $-b$, $-c$ and $-d$. The result follows. ∎

Let

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}.$$

Then the 2×2 matrices belonging to the set $SU(2)$ constitute a group with respect to the operation of matrix multiplication. Morever a 2×2 matrix with complex coefficients belongs to the group $\mathrm{SU}(2)$ if and only if it is a unitary matrix whose determinant is equal to one. Proposition 2.7 ensures that every 2×2 unitary matrix with determinant equal to one determines a corresponding rotation of three-dimensional space $\mathbb{R}^3$. We obtain in this way a two-to-one homomorphism from the group $\mathrm{SU}(2)$ to the rotation group $\mathrm{SO}(3)$ of 3-dimensional space.

Remark

The homomorphism from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$ whose existence is guaranteed by Proposition 2.7 can also be described using properties of quaternions. Independently of one another, Arthur Cayley and Sir William Rowan Hamilton discovered how to represent rotations of three-dimensional space using quaternions. (Cayley's account appeared in print in 1845 before Hamilton's account, read at a meeting of the Royal Irish Academy in 1844 but published in 1847.)

The homomorphism between these matrix groups gives rise to the fundamental properties of spin in quantum mechanics, where the traditional account is expressed in terms of *Pauli matrices*.

2.6. The Action of Möbius Transformations on the Riemann Sphere

Proposition 2.8

Let $\zeta_1, \zeta_2, \zeta_3$ be distict points of the Riemann sphere $\mathbb{P}^1$, and let $\omega_1, \omega_2, \omega_3$ also be distinct points of $\mathbb{P}^1$. Then there exists a Möebius transformation $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(\zeta_j) = \omega_j$ for $j = 1, 2, 3$.

Proof

The composition of two Möbius transformations of the Riemann sphere $\mathbb{P}^1$ is itself a Möebius transformation of $\mathbb{P}^1$ (Proposition 2.4). Also the inverse of any Möbius transformation of the Riemann sphere is itself a Möbius transformation (Corollary 2.5). It follows that the Möbius transformations of the Riemann sphere constitute a group under the operation of composition of transformations.

Next we note that permution of the elements 0, 1 and ∞ of the Riemann sphere can be effected by a suitable Möbius transformation. Indeed the Möbius transformation $z \mapsto 1 - z$ transposes 0 and 1 whilst fixing ∞, and the Möbius transformation $z \mapsto -1/(z-1)$ cyclicly permutes 0, 1 and ∞. It follows that any permutation of 0, 1 and ∞ may be effected by the action of some Möbius transformation.

Next we show that there exists a Möbius transformation $\mu_1 \colon \mathbb{P}^1 \to \mathbb{P}_2$ with the property that $\mu_1(\zeta_1) = 0$, $\mu_1(\zeta_2) = 1$ and $\mu_1(\zeta_3) = \infty$. Suppose first that at least one of the distinct points $\zeta_1, \zeta_2, \zeta_3$ of $\mathbf{P}^1$ is the point ∞. Because we have shown that there exist Möbius transformations permuting 0, 1 and ∞ amongst themselves, we may assume in this case, without loss of generality, that $\zeta_3 = \infty$. Let $\zeta_0 = z_0$ and $\zeta_1 = z_1$, where z_0 and z_1 are complex numbers, and let

$$\mu_1(z) = \frac{z - z_0}{z_1 - z_0}.$$

Then $\mu_1(\zeta_0) = \mu_1(z_0) = 0$, $\mu_1(\zeta_1) = \mu_1(z_1) = 1$ and $\mu_1(\infty) = \infty$. The existence of the Möbius transformation μ_1 has thus been verified in the case where at least one of $\zeta_1, \zeta_2, \zeta_3$ is the point ∞ of the Riemann sphere.

Next we consider the case where $\zeta_j = z_j$ for $j = 1, 2, 3$, where z_1, z_2, z_3 are complex numbers. In this case let μ_1 be the Möbius transformation defined so that

$$\mu_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

for all complex numbers z. Then $\mu_1(\zeta_1) = \mu_1(z_1) = 0$, $\mu_1(\zeta_2) = \mu_1(z_2) = 1$ and $\mu_1(\zeta_3) = \mu_1(z_3) = \infty$. We conclude therefore that, given any distinct points $\zeta_1, \zeta_2, \zeta_3$ of the Riemann sphere, there exists a Möbius transformation μ_1 of the Riemann sphere for which $\mu_1(\zeta_1) = 0$, $\mu_1(\zeta_2) = 1$ and $\mu_1(\zeta_3) = \infty$.

Let $\omega_1, \omega_2, \omega_3$ also be distinct points of the Riemann sphere. Then there exists a Möbius transformation μ_2 with the property that $\mu_2(\omega_1) = 0$, $\mu_2(\omega_2) = 1$ and $\mu_2(\omega_3) = \infty$. Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere that is the composition $\mu_2^{-1} \circ \mu_1$ of μ_1 followed by the inverse of μ_2. Then $\mu(\zeta_j) = \omega_j$ for $j = 1, 2, 3$, as required. $\blacksquare$

Proposition 2.9

Let $\rho\colon \mathbb{P}^1 \to \mathbf{P}^1$ denote the Möbius transformation defined so that $\rho(z) = 1/z$ for all non-zero complex numbers z. Then the mapping ρ preserves the angles between circles and straight lines. contained in the set $\mathbb{C} \setminus \{0\}$.

Proof

Let z be a non-zero complex number, and let h be a non-zero complex number satisfying $|h| < |z|$, and let t be a real number satisfying $-1 \leq t \leq 1$. Then $z + th \neq 0$, and

$$\left(1 + \frac{th}{z}\right)\left(1 - \frac{th}{z}\right) = 1 - \frac{t^2 h^2}{z^2}.$$

It follows that

$$\frac{1}{z} - \frac{th}{z^2} = \frac{1}{z+th} - \frac{t^2h^2}{z^2(z+th)},$$

and thus

$$\frac{1}{z+th} = \frac{1}{z} - \frac{th}{z^2} + \frac{t^2h^2}{z^2(z+th)}$$

Let θ be a real number, let $k = h(cos\theta + i\sin\theta)$. Then the directions represented by the complex numbers h and k are at an angle θ to each other. Let $\alpha\colon (-1,1) \to \mathbb{C}$ and $\beta\colon (-1,1) \to \mathbb{C}$ be the curves defined such that

$$\alpha(t) = \rho(z+th) = \frac{1}{z+th}$$

and

$$\beta(t) = \rho(z+tk) = \frac{1}{z+tk}$$

for all real numbers t satisfying $-1 < t < 1$.

Then the tangent directions to the curve $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ at $t = 0$ are in the directions determined by the complex numbers

$$-\frac{h}{z^2} \quad \text{and} \quad -\frac{k}{z^2}.$$

Moreover

$$-\frac{k}{z^2} = -\frac{h}{z^2}(\cos\theta + i\sin\theta)$$

and therefore the tangent directions to the curves $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ at $t = 0$ are also at an angle θ to each other. The result follows. ∎

Proposition 2.10

Any Möbius transformation of the Riemann sphere maps straight lines and circles to straight lines and circles, and also preserves angles between lines and circles.

Proof

Proposition 2.9 ensures that the Möbius transformation that sends z to $1/z$ for all non-zero complex numbers z is angle-preserving.

The equation of a line or circle in the complex plane can be expressed in the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

where g and h are real numbers, and b is a complex number. Moreover a locus of points in the complex plane satisfying an equation of this form is a circle if $g \neq 0$ and is a line if $g = 0$.

Let g and h be real constants, let b be a complex constant, and let $z = 1/w$, where $w \neq 0$ and w satisfies the equation

$$g|w|^2 + 2\mathrm{Re}[\overline{b}w] + h = 0,$$

Then

$$g|w|^2 + \overline{b}w + b\overline{w} + h = 0,$$

Then

$$\begin{aligned} g + \mathrm{Re}[bz] + h|z|^2 &= g + \overline{b}\overline{z} + bz + h|z|^2 \\ &= \frac{1}{|w|^2}\left(g|w|^2 + \overline{b}w + b\overline{w} + h\right) = 0. \end{aligned}$$

We deduce from this that the Möbius transformation that sends z to $1/z$ for all non-zero complex numbers z maps lines and circles to lines and circles.

Let $\mu \colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere. Then there exist complex numbers a, b, c and d satisfying $ad - bc \neq 0$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. The result is immediate when $c = 0$. We therefore suppose that $c \neq 0$. Then

$$\mu(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \times \frac{1}{cz+d}$$

when $cz + d \neq 0$. The Möbius transformation μ is thus the composition of three maps that each send circles and straight lines to circles and straight lines and preserve angles between lines and circles, namely the maps

$$z \mapsto cz + d, \quad z \mapsto \frac{1}{z} \quad \text{and} \quad z \mapsto \frac{a}{c} - \frac{(ad-bc)z}{c}.$$

Thus the Möbius transformation μ must itself map circles and straight line to circles and straight lines and also preserve angles between lines and circles, as required. ∎