MA232A—Euclidean and Non-Euclidean Geometry
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PR

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Definition

We define an *order specifier* on a set X to be a subset Ω of X^3 with the following properties:

- **(BS1)** for all $x, y, z \in X$, if $(x, y, z) \in \Omega$ then x, y and z are distinct;
- **(BS2)** for all $x, y, z \in X$, if $(x, y, z) \in \Omega$ then $(z, y, x) \in \Omega$;
- **(BS3)** for all $x, y, z \in X$, if $(x, y, z) \in \Omega$ then $(y, z, x) \notin \Omega$;
- **(BS4)** for all $w, x, y, z \in X$, if $(w, x, y) \in \Omega$ and $(x, y, z) \in \Omega$ then $(w, y, z) \in \Omega$;
- **(BS5)** for all $w, x, y, z \in X$, if $(w, x, z) \in \Omega$ and $(w, y, z) \in \Omega$ then either $(w, x, y) \in \Omega$ and $(y, x, z) \notin \Omega$ or else $(w, y, x) \in \Omega$ and $(x, y, z) \notin \Omega$.

Example

Let Ω be the subset of $\mathbb{R}^3$ defined so that

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x < y < z \text{ or } x > y > z\}.$$

Then Ω is an order specifier on $\mathbb{R}$.

Lemma 1.1

Let Ω be an order specifier on a set X, and let x, y and z be distinct elements of X for which $(x, y, z) \in \Omega$. Then $(z, y, x) \in \Omega$ $(y, z, x) \notin \Omega$, $(x, z, y) \notin \Omega$, $(z, x, y) \notin \Omega$, $(y, x, z) \notin \Omega$.

Proof

It follows from property (BS2) that $(z, y, x) \in \Omega$, and it follows from property (BS3) that $(y, z, x) \notin \Omega$. It then follows an application of (BS2) that $(x, z, y) \notin \Omega$ (for if it were the case that $(x, z, y) \in \Omega$ then $(y, z, x) \in \Omega$, contradicting the result obtained above on applying property (BS3)). Next, applying property (BS3) with x, y and z replaced by z, y and x respectively, we conclude that $(y, x, z) \notin \Omega$. It then follows from an application of (BS2) that $(z, x, y) \notin \Omega$. The result follows. ∎

Let Ω be an order specifier on a set X and let x, y and z be distinct elements of X. It follows from Lemma 1.1 that exactly two of the six triples (x, y, z), (z, y, x), (y, z, x), (x, z, y), (z, x, y) and (y, x, z) belong to Ω, and moreover the two triples that belong to Ω are determined by their second component. Indeed exactly one of the distinct elements x, y, z occurs as the second component of the two triples from the above list that belong to Ω.

Lemma 1.2

Let Ω be an order specifier on a set X, let v be an element of X and, let $\prec_v$ denote the binary relation on X defined as follows: elements x and y of X satisfy $x \prec_v y$ if and only if $(v, x, y) \in \Omega$. Then

- **(i)** *if x and y are elements of X then at most one of the relations $x \prec_v y$, $x = y$ and $y \prec_v x$ holds for x and y;*
- **(ii)** *if x, y and z are elements of X, and if $x \prec_v y$ and $y \prec_v z$ then $x \prec_v z$;*

Proof

If x and y are elements of X satisfying $x \prec_v y$ then $(v, x, y) \in \Omega$. It then follows from property (BS1) in the definition of order specifiers that v, x and y are distinct and therefore $x \neq y$. Also $(v, y, x) \notin \Omega$ (see Lemma 1.1), and therefore the relation $y \prec_v x$ does not hold for x and y. Next if x and y satisfy $x = y$ then it follows that $(v, y, x) \notin \Omega$ and therefore the relation $y \prec_v x$ does not hold for x and y. We have thus shown that if the first of the three relations $x \prec_v y$, $x = y$ and $y \prec_v x$ holds for x and y then neither the second nor the third hold for x and y, and also that if the second of these relations holds for x and y then the third does not hold for x and y. Therefore at most one of these three relations holds for x and y. This proves (i).

Definition

Let X be a set, and let Ω be an order specifier on X. Then, given distinct elements v and w of X, we define the *ray* from v through w to be the subset $R_{v,w}$ of X, where

$$R_{v,w} = \{x \in X : x = v \text{ or } x = w \text{ or } (v, x, w) \in \Omega \text{ or } (v, w, x) \in \Omega\}.$$