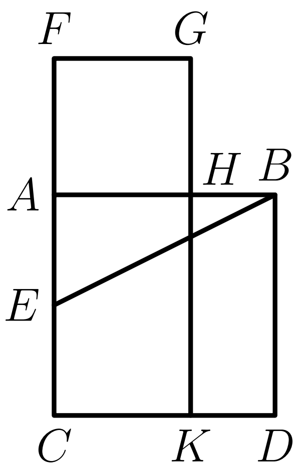
Gloss on Euclid, *Elements*, Book II, Proposition 11



Let the square $ABCD$ be given. The task is to construct a point H for which the square $AFGH$ is equal to the rectangle $HBDK$.

Bisect AC at E, and join BE. Then produce EA beyond A to a point F for which $EF = BE$. Then complete the square $AFGH$ as shown.

Let $AE = x$ and $AF = y$. Then $CF = 2x + y$ and $FG = y$, and therefore $CF \times FG = 2xy + y^2$.

It follows (*Euclid*, Book II, Proposition 6) that $CF \times FG + AE^2 = EF^2 = EB^2$. (Indeed

$$CF \times FG + AE^2 = (2x + y)y + x^2 = (x + y)^2 = EF^2 = EB^2.$$

But

$$AB^2 + AE^2 = EB^2,$$

by Pythagoras' Theorem (*Euclid*, Book I, Proposition 47). Thus $CF \times FG + AE^2 = AB^2 + AE^2$. Subtracting AE^2 from both sides, we find that $CF \times FG = AB^2$.

But $CF \times FG = AF^2 + CA \times AH$ and $AB^2 = CA \times HB + CA \times AH$. Subtracting $CA \times AH$ from both sides, we find that $AF^2 = CA \times HB = HB \times BD$, as required.

Note that $AF^2 = y^2$, $HB = 2x - y$ and $BD = 2x$. It follows that $y^2 + 2xy = 4x^2$, and thus

$$y = (\sqrt{5} - 1)x.$$